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New Integral Inequality on Time Scales

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Abstract. In this paper we establish some new integral inequalities related to a certain inequality arising in the theory of dynamic equations on time scales.

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1. Introduction

The theory of dynamic equations on time scales (aka measure chains) was introduced by Hilger [5] with the motivation of providing a unified approach to continuous and discrete analysis. The generalized derivative or Hilger derivative $f^{\triangle}(t)$ of a function $f:\mathbb{T}\longrightarrow\mathbb{R}$, where $\mathbb{T}$ is a so-called "time scale"(an arbitrary closed non-empty subset of $\mathbb{R}$) becomes the usual derivative when $\mathbb{T}=\mathbb{R}$, that is $f^{\triangle}(t)=f^{'}(t)$. On the other hand, if $\mathbb{T}=\mathbb{Z}$, then $f^{\triangle}(t)$ reduces to the usual forward difference, that is $f^{\triangle}(t)=\triangle f(t)$. This theory not only brought equations leading to new applications. Also, this theory allows one to get some insight into and better understanding of the subtle differences between discrete and continuous systems [1, 3]. Some basic dynamic inequalities are given as established in the paper by Agarwal, Bohner, and Peterson [2].

In this paper we establish new integral inequality related to a certain inequality arising in the theory of dynamic equations on time scales.

Here, first we mention several foundational definitions without proof and results from the calculus on time scales in an excellent introductory text by Bohner and Peterson [3, 4].

2. General Definitions

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. We define the *forward jump operator* σ on $\mathbb{T}$ by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\} \in \mathbb{T} \text{ for all } t \in \mathbb{T}.$$

In this definition we put $\sigma(\emptyset) = \sup \mathbb{T}$. where $\emptyset$ is the empty set. if $\sigma(t) > t$, then we say that t is *right-scattered*. If $\sigma(t) = t$ and $t < \sup \mathbb{T}$, then we say that t is *right-dense.* The backward jump operator and left-scattered and left-dense point are defined in a similar way. *The graininess* $\mu : \mathbb{T} \longrightarrow [0, \infty)$ is defined by $\mu(t) = \sigma(t) - t$. The set $\mathbb{T}^k$ is derived from $\mathbb{T}$ as follows: If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T} - \{m\}$; otherwise, $\mathbb{T}^k = \mathbb{T}$. For $f : \mathbb{T} \longrightarrow \mathbb{R}$ and $t \in \mathbb{T}^k$, we define $f^{\triangle}(t)$ to be the number (provided it exists) such that given any $\varepsilon > 0$, there is neighborhood U of t with

$$\left| f^{\sigma}(t) - f(s) - f^{\triangle}(t)[\sigma(t) - s] \right| \leq \varepsilon |\sigma(t) - s| \text{ for all } s \in U.$$

We call $f^{\triangle}(t)$ the delta derivative of f at t, and $f^{\triangle}$ is the usual derivative f' if $\mathbb{T} = \mathbb{R}$ and the usual forward difference $\triangle f$ (defined by $\triangle f = f(t+1) - f(t)$) if $\mathbb{T} = \mathbb{Z}$.

Theorem 1. *Assume $f, g : \mathbb{T} \longrightarrow \mathbb{R}$ and let $t \in \mathbb{T}^k$. Then we have the following:*

(i) If f is differentiable at t, then f is continuous at t.

(ii) If f is continuous at t and t is right-scattered, then f is differentiable at t with

$$f^{\triangle}(t) = \frac{f^{\sigma}(t) - f(t)}{\sigma(t) - t}.$$

(iii) If f is differentiable at t and t is right-dense, then

$$f^{\triangle}(t) = \lim_{t \to s} \frac{f(t) - f(s)}{t - s}.$$

(iv) If f is differentiable at t, then

$$f^{\sigma}(t) = f(t) + \mu(t) f^{\triangle}(t), \text{ where } f^{\sigma} := f \circ \sigma.$$

(*v*) *If* f *and* g *are differentiable at* t, *then so is* fg *with*

$$(fg)^{\triangle}(t) = f^{\triangle}(t)\,g(t) + f^{\sigma}(t)\,g^{\triangle}(t).$$

We say that $f:\mathbb{T}\to\mathbb{R}$ is *rd-continuous* provied f is continuous at each right-dense point of $\mathbb{T}$ and has a finite left-sided limit at each left-dense point of $\mathbb{T}$. The set of rd-continuous functions will be denoted in this paper by C_{rd}, and the set of functions that are differentiable and whose derivative is rd-continuous is denoted by C^1_{rd}. A function $F:\mathbb{T}\to\mathbb{R}$ is called an *antiderivative* of $f:\mathbb{T}\to\mathbb{R}$ provided $F^{\triangle}(t) = f(t)$ holds for all $\mathbb{T}^k$. In this case we define the integral of f by

$$\int_s^t f(\tau)\triangle\tau = F(t) - F(s), \text{ for } s,t\in\mathbb{T}.$$

We say that $p:\mathbb{T}\to\mathbb{R}$ is *regressive* provided $1+\mu(t)\,p(t)\neq 0$ for all $t\in\mathbb{T}$, we denote by $\mathfrak{R}$ the set of all regressive and rd-continuous functions. We define the set of all positively regressive functions by $\mathfrak{R}^+ = \{p\in\mathfrak{R} : 1+\mu(t)\,p(t) > 0 \text{ for all } t\in\mathbb{T}\}$. If $p,q\in\mathfrak{R}$, then we define

$$p\oplus q = p+q+\mu pq,\ \ \ominus q = -\frac{q}{1+\mu q},\ \text{and } p\ominus q = p\oplus(\ominus q).$$

If $p:\mathbb{T}\to\mathbb{R}$ is rd-continouos and regressivei then the *exponential function* $e_p(.,t_0)$ is for each fixed $t_0\in\mathbb{T}$ the unique solution of the initial value problem

$$x^{\triangle} = p(t)\,x,\ x(t_0) = 1 \text{ on } \mathbb{T}.$$

We use following four theorems which are proved in Bohner and Peterson [3].

Theorem 2. *If* $p,q\in\mathfrak{R}$, *then*
(*i*) $e_p(t,t)\equiv 1$ *and* $e_0(t,s)\equiv 1$;
(*ii*) $e_p(\sigma(t),s) = (1+\mu(t)\,p(t))\,e_p(t,s)$;
(*iii*) $\frac{1}{e_p(t,s)} = e_{\ominus p}(t,s) = e_p(s,t)$;
(*iv*) $\frac{e_p(t,s)}{e_p(s,t)} = e_{p\ominus q}(t,s)$;
(*v*) $e_p(t,s)\,e_q(t,s) = e_{p\oplus q}(t,s)$;
(*vi*) *if* $p\in\mathfrak{R}^+$, *then* $e_p(t,t_0) > 0$ *for all* $t\in\mathbb{T}$.

Remark 1. *It is easy to see that, if* $\mathbb{T}=\mathbb{R}$, *the exponential function is given by*

$$e_p(t,s) = e^{\int_s^t p(\tau)d\tau},\ e_\alpha(t,s) = e^{\alpha(t-s)},\ e_\alpha(t,0) = e^{\alpha t}$$

for $s,t\in\mathbb{R}$, *where* $\alpha\in\mathbb{R}$ *is a constant and* $p:\mathbb{R}\to\mathbb{R}$ *is a continuous function; if* $\mathbb{T}=\mathbb{Z}$, *the exponential function is given by*

$$e_p(t,s)=\prod_{r=s}^{t-1}[1+p(\tau)],\ e_\alpha(t,s)=(1+\alpha)^{t-s},\ e_\alpha(t,0)=(1+\alpha)^t$$

for $s,t\in\mathbb{Z}$ *with* $s<t$, *where* $\alpha\neq-1$ *is a constant and* $p:\mathbb{Z}\to\mathbb{R}$ *is a sequence satisfying* $p(t)\neq-1$ *for all* $t\in\mathbb{Z}$.

Theorem 3. *If* $p\in\mathfrak{R}$ *and* $a,b,c\in\mathbb{T}$, *then*

$$\int_a^b p(t)\,e_p(c,\sigma(t))\,\triangle t=e_p(c,a)-e_p(c,b).$$

Theorem 4. *If* $a,b,c\in\mathbb{T}$ *and* $f\in C_{rd}$ *such that* $f(t)\geq0$ *for all* $a\leq t\leq b$, *then*

$$\int_a^b f(t)\,\triangle t\geq0$$

Theorem 5. *Let* $t_0\in\mathbb{T}^k$ *and assume* $k:\mathbb{T}\times\mathbb{T}\to\mathbb{R}$ *is continuous at* (t,t), *where* $t\in\mathbb{T}^k$ *with* $t>t_0$. *Also assume that* $k(t,.)$ *is rd-continuous on* $[t_0,\sigma(t)]$. *Suppose that for each* $\varepsilon>0$ *there exists a neighborhood* U *of* t, *independent of* $\tau\in[t_0,\sigma(t)]$, *such that*

$$\left|k(\sigma(t),\tau)-k(s,\tau)-k^{\triangle}(t,\tau)[\sigma(t)-s]\right|\leq\varepsilon\left|\sigma(t)-s\right|\ \textit{for all } s\in U.$$

where $k^{\triangle}$ *denotes the derivative of* k *with respect to the first variable. Then*

$$g(t):=\int_{t_0}^{t}k(t,\tau)\,\triangle\tau\ \textit{implies}\ g^{\triangle}(t)=\int_{t_0}^{t}k^{\triangle}(t,\tau)\,\triangle\tau+k(\sigma(t),t).$$

The next four results are proved by Agarval, Bohner and Peterson [2]. For convenience of notation we let throughout

$$t_0\in\mathbb{T},\mathbb{T}_0=[t_0,\infty)\cap\mathbb{T},\ \text{and}\ \mathbb{T}_0^-=(-\infty,t_o]\cap\mathbb{T}$$

Also, for a function $b\in C_{rd}$ we write

$$b\geq0\ \text{if}\ b(t)\geq0\ \text{for all}\ t\in\mathbb{T}.$$

Theorem 6. Theorem 7. $(Comparison\ Theorem)$. *Suppose* $u,b\in C_{rd}$ *and* $a\in\mathfrak{R}^+$. *Then*

$$u^{\triangle}(t)\leq a(t)\,u(t)+b(t)\ \textit{for all}\ t\in\mathbb{T}_0$$

implies

$$u(t) = u(t_0)\, e_a(t, t_0) + \int_{t_0}^{t} e_a(t, \sigma(\tau))\, b(\tau) \triangle\tau \text{ for all } t \in \mathbb{T}_0.$$

Remark 2. **Theorem 8.** $(Gronwall's\ Inequality)$. *Suppose* $u, a, b \in C_{rd}$ *and* $b \geq 0$. *Then*

$$u(t) \leq a(t) + \int_{t_0}^{t} b(\tau)\, u(\tau) \triangle\tau \text{ for all } t \in \mathbb{T}_0$$

implies

$$u(t) = a(t) + \int_{t_0}^{t} a(\tau)\, b(\tau)\, e_b(t, \sigma(\tau)) \triangle\tau \text{ for all } t \in \mathbb{T}_0.$$

3. Main Results

Theorem 9. *Let* u, a, b, g *be real-valued nonnegative rd-continuous functions defined on* $\mathbb{T}_0$ *and* $p > 1$ *be real constant.*

(*i*) *Let* $f : \mathbb{T}_0 \times \mathbb{R}_+ \to \mathbb{R}_+$ *be a rd-continuous function such that*

$$0 \leq f(t, x) - f(t, y) \leq m(t, y)(x - y), \tag{3.1}$$

for $t \in \mathbb{T}_0$ *and* $x \geq y \geq 0$, *where* $m : \mathbb{T}_0 \times \mathbb{R}_+ \to \mathbb{R}_+$ *is a rd-continuous function. If*

$$u^p(t) \leq a(t) + b(t) \int_{t_0}^{t} f(s, u(s)) \triangle s, \tag{3.2}$$

for $t \in \mathbb{T}_0$, *then*

$$u(t) \leq \left\{ a(t) + b(t) \int_{t_0}^{t} f\left(s, \frac{p-1}{p} + \frac{a(s)}{p}\right) \times e_{m\left(t, \frac{p-1}{p} + \frac{a(t)}{p}\right)\frac{b(t)}{p}}(t, \sigma(s)) \triangle s \right\}^{\frac{1}{p}}, \tag{3.3}$$

for $t \in \mathbb{T}_0$.

(*ii*) *Let* $f : \mathbb{T}_0 \times \mathbb{R}_+ \to \mathbb{R}_+$ *be a rd-continuous function and* $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ *be a rd-continuous and strictly increasing function with* $\phi(0) = 0$ *such that*

$$0 \leq f(t, x) - f(t, y) \leq m(t, y)\, \phi^{-1}(x - y), \tag{3.4}$$

for $t \in T_0$ *and* $x \geq y \geq 0$, *where* $m : \mathbb{T}_0 \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ *is a rd-continuous function and* ϕ^{-1} *is inverse function of* ϕ *and*

$$\phi^{-1}(xy) \leq \phi^{-1}(x)\,\phi^{-1}(y) \tag{3.5}$$

for $x, y \in \mathbb{R}_+$. *If*

$$u^p(t) \leq a(t) + b(t)\,\phi\left(\int_{t_0}^{t} f(s, u(s))\,\triangle s\right). \tag{3.6}$$

for $t \in \mathbb{T}_0$, *then*

$$\begin{aligned} u(t) \quad &\leq \quad \left\{ a(t) + b(t)\,\phi\left(\int_{t_0}^{t} f\left(s, \frac{p-1}{p} + \frac{a(s)}{p}\right)\right)\right. \\ &\qquad \left. \times e_{m\left(t, \frac{p-1}{p} + \frac{a(t)}{p}\right)\phi^{-1}\left(\frac{b(t)}{p}\right)}(t, \sigma(s))\,\triangle s \right\}^{\frac{1}{p}}. \end{aligned} \tag{3.7}$$

for $t \in \mathbb{T}_0$.

Proof. (i) Obviously, if $t = t_0$, then the inequality 3.3 holds. Thus, in the next proof, we always assume that $t > t_0, t \in \mathbb{T}_0$.

Define a function $z(t)$ by

$$z(t) = \int_{t_0}^{t} f(s, u(s))\,\triangle s. \tag{3.8}$$

Then $z(t_0) = 0$ and (3.2) can be written as

$$u^p(t) \leq a(t) + b(t)\,z(t) \tag{3.9}$$

From (3.9) and using the elementary inequality (See [6,p,30])

$$u^{\frac{1}{p}} v^{\frac{1}{q}} \leq \frac{u}{p} + \frac{v}{q},$$

where $u \geq 0, v \geq 0$, and $\frac{1}{p} + \frac{1}{q} = 1$ with $p > 1$, we observe that

$$\begin{aligned} u(t) \quad &\leq \quad [\alpha(t) + b(t)\,z(t)]^{\frac{1}{p}} \\ &\leq \quad \frac{p-1}{p} + \frac{a(t)}{p} + \frac{b(t)}{p} z(t). \end{aligned} \tag{3.10}$$

From (3.8),(3.10), and the condition (3.1) it follows that

$$
\begin{aligned}
(3.11)\quad z^{\triangle}(t) &= f(t,u(t)) \\
&\le f\left(t,\frac{p-1}{p}+\frac{a(t)}{p}+\frac{b(t)}{p}z(t)\right)-f\left(t,\frac{p-1}{p}+\frac{a(t)}{p}\right) \\
&\quad +f\left(t,\frac{p-1}{p}+\frac{a(t)}{p}\right) \\
&\le m\left(t,\frac{p-1}{p}+\frac{a(t)}{p}\right)\frac{b(t)}{p}z(t)+f\left(t,\frac{p-1}{p}+\frac{a(t)}{p}\right).
\end{aligned}
$$

The inequality (3.11) implies the estimate

$$
(3.12)\qquad z(t)\le\int_{t_0}^{t} e_{m\left(t,\frac{p-1}{p}+\frac{a(t)}{p}\right)\frac{b(t)}{p}}(t,\sigma(s))\,f\left(s,\frac{p-1}{p}+\frac{a(s)}{p}\right)\triangle s.
$$

From (3.12) and (3.9) the desired inequality in (3.3) follows.

(ii) Obviously, if $t=t_0$, then the inequality (3.7) holds. Thus, in the next proof, we always assume that $t>t_0$, $t\in\mathbb{T}_0$.

Define a function $z(t)$ by

$$
z(t)=\int_{t_0}^{t} f(s,u(s))\,\triangle s
$$

Then $z(t_0)=0$ and (3.2) can be written as

$$
(3.13)\qquad u^{p}(t)\le a(t)+b(t)\,\phi(z(t))
$$

and

$$
(3.14)\qquad u(t)\le\frac{p-1}{p}+\frac{a(t)}{p}+\frac{b(t)}{p}\phi(z(t)).
$$

From (3.8), (3.14), and the condition (3.4), (3.5) it follows that

$$
\begin{aligned}
(3.15)\quad z^{\triangle}(t) &= f(t,u(t)) \\
&\le f\left(t,\frac{p-1}{p}+\frac{a(t)}{p}+\frac{b(t)}{p}\phi(z(t))\right)-f\left(t,\frac{p-1}{p}+\frac{a(t)}{p}\right) \\
&\quad +f\left(t,\frac{p-1}{p}+\frac{a(t)}{p}\right) \\
&\le m\left(t,\frac{p-1}{p}+\frac{a(t)}{p}\right)\phi^{-1}\left(\frac{b(t)}{p}\phi(z(t))\right)+f\left(t,\frac{p-1}{p}+\frac{a(t)}{p}\right) \\
&\le m\left(t,\frac{p-1}{p}+\frac{a(t)}{p}\right)\phi^{-1}\left(\frac{b(t)}{p}\right)z(t)+f\left(t,\frac{p-1}{p}+\frac{a(t)}{p}\right).
\end{aligned}
$$

The inequality (3.15) implies the estimate

$$z(t) \leq \int_{t_0}^{t} e_{m\left(t,\frac{p-1}{p}+\frac{a(t)}{p}\right)\phi^{-1}\left(\frac{b(t)}{p}\right)}(t,\sigma(s))\, f\left(s,\frac{p-1}{p}+\frac{a(s)}{p}\right)\triangle s, \tag{3.16}$$

The required inequlity (3.7) follows from (3.13) and (3.16). ∎

For the cases $\mathbb{T}=\mathbb{R}$ and $\mathbb{T}=\mathbb{Z}$, see [7,Theorem 2] and [7,Theorem 4], respectively.

Corollary 1. *Let $\mathbb{T}=\mathbb{R}$ and assume that u,a,b,g be real-valued nonnegative continuous functions defined on $\mathbb{R}_+$ and $p>1$ be real constant.*

(i) Let $f:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+$ be a continuous function such that

$$0\leq f(t,x)-f(t,y)\leq m(t,y)(x-y),$$

for $t\in\mathbb{R}_+$ and $x\geq y\geq 0$, where $m:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+$ is a continuous function. If

$$u^p(t)\leq a(t)+b(t)\int_0^t f(s,u(s))\,ds,$$

for $t\in\mathbb{R}_+$, then

$$\begin{aligned} u(t) \;\leq\; & \left\{a(t)+b(t)\int_0^t f\left(s,\frac{p-1}{p}+\frac{a(s)}{p}\right)\right. \\ & \left.\times\exp\left(\int_s^t m\left(\sigma,\frac{p-1}{p}+\frac{a(\sigma)}{p}\right)\frac{b(\sigma)}{p}\right)ds\right\}^{\frac{1}{p}}, \end{aligned}$$

for $t\in\mathbb{R}_+$.

(ii) Let $f:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+$ be a continuous function and $\phi:\mathbb{R}_+\to\mathbb{R}_+$ be a continuous and strictly increasing function with $\phi(0)=0$ such that

$$0\leq f(t,x)-f(t,y)\leq m(t,y)\,\phi^{-1}(x-y),$$

for $t\in\mathbb{R}_+$ and $x\geq y\geq 0$, where $m:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+$ is a continuous function and ϕ^{-1} is the inverse function of ϕ and

$$\phi^{-1}(xy)\leq\phi^{-1}(x)\,\phi^{-1}(y)$$

for $x,y\in\mathbb{R}_+$. If

$$u^p(t)\leq a(t)+b(t)\,\phi\left(\int_0^t f(s,u(s))\,ds\right),$$

for $t \in \mathbb{R}_+$, *then*

$$u(t) \leq \left\{ a(t) + b(t)\phi\left(\int_{t_0}^{t} f\left(s, \frac{p-1}{p} + \frac{a(s)}{p}\right) \right) \times \exp\left(\int_{s}^{t} m\left(\sigma, \frac{p-1}{p} + \frac{a(\sigma)}{p}\right) \phi^{-1}\left(\frac{b(\sigma)}{p}\right) d\sigma \right) ds \right\}^{\frac{1}{p}}.$$

for $t \in \mathbb{R}_+$.

Corollary 2. *Let* $T = Z$ *and assume that* u, a, b, g *be real-valued nonnegative functions defined on* $\mathbb{N}_0$ *and* $p > 1$ *be real constant.*

(i) *Let* $L : \mathbb{N}_0 \times \mathbb{R}_+ \to \mathbb{R}_+$ *be a function such that*

$$0 \leq L(n, x) - L(n, y) \leq M(n, y)(x - y),$$

for $n \in \mathbb{N}_0$ *and* $x \geq y \geq 0$, *where* $M(n, y)$ *is a real-valued nonnegative function defined for* $n \in \mathbb{N}_0$, $y \in R_+$. *If*

$$u^p(n) \leq a(n) + b(n) \sum_{s=0}^{n-1} L(s, u(s)),$$

for $n \in \mathbb{N}_0$, *then*

$$u(n) \leq \left\{ a(n) + b(n) \sum_{s=0}^{n-1} L\left(s, \frac{p-1}{p} + \frac{a(s)}{p}\right) \times \prod_{\sigma=s+1}^{n-1} \left[1 + M\left(\sigma, \frac{p-1}{p} + \frac{a(\sigma)}{p}\right) \frac{b(\sigma)}{p} \right] \right\}^{\frac{1}{p}},$$

for $n \in \mathbb{N}_0$.

(ii) *Let* $L : \mathbb{N}_0 \times \mathbb{R}_+ \to \mathbb{R}_+$ *be a function which satisfies the condition*

$$0 \leq L(n, x) - L(n, y) \leq M(n, y)\psi^{-1}(x - y),$$

for $n \in \mathbb{N}_0$, $x \geq y \geq 0$, *where* $M(n, y)$ *is as defined in* (i), $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ *is a continuous and strictly increasing function with* $\psi(0) = 0$, ψ^{-1} *is the inverse function of* ψ *and*

$$\psi^{-1}(xy) \leq \psi^{-1}(x)\psi^{-1}(y)$$

for $x, y \in \mathbb{R}_+$. *If*

$$u^p(n) \leq a(n) + b(n)\psi\left(\sum_{s=0}^{n-1} L(s, u(s))\right),$$

for $n \in \mathbb{N}_0,$ *then*

$$u(n) \leq \left\{ a(n) + b(n)\,\psi\left(\sum_{s=0}^{n-1} L\left(s, \frac{p-1}{p} + \frac{a(s)}{p} \right) \times \prod_{\sigma=s+1}^{n-1} \left[1 + M\left(\sigma, \frac{p-1}{p} + \frac{a(\sigma)}{p} \right) \psi^{-1}\left(\frac{b(\sigma)}{p} \right) \right] \right) \right\}^{\frac{1}{p}}.$$

for $n \in \mathbb{N}_0$.

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