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On the Explicit Powers and the Inverse of One Type of Band Toeplitz Matrices

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Abstract. We derive the general expression of the powers of one type of banded Toeplitz matrices.

Keywords: Toeplitz matrices, Chebyshev polynomials, Eigenvalues, Eigenvectors

1. Introduction

Toeplitz matrices emerge in plenty of applications and have been extensively studied for about a century. The literature on them is immense and ranges from thousands of articles in periodicals to huge monographs [1]. In many applications, the problem of computing integer powers of such matrices arise. In this paper we derive the general expression of the mth power($m \in \mathbb{N}$) for one type of banded Toeplitz matrices. Similar expressions for tridigaonal [2] and pentadiagonal matrice [4, 5] are obtained as a special case. Moreover the inverse of such matrices is also derived provided that the matrix is nonsingular.

2. The eigenvalues and the eigenvectors of the matrix T

Consider the symmetric $k-$*band Toeplitz* matrix $T = (t_{i,j})_{1\leq i,j\leq n}$

$$T=\begin{pmatrix} c & 0 & \dots & 1 & & \\ 0 & c & 0 & & \ddots & \\ \vdots & 0 & \ddots & & & 1 \\ 1 & & & & \ddots & \\ & \ddots & & \ddots & & 0 \\ & & 1 & & 0 & c \end{pmatrix},$$

such that:

$$t_{i,i}=c,\ t_{i,i+k}=1=t_{i+k,i} \quad \text{and} \quad t_{i,j}=0 \quad \text{if} \quad |i-j|\neq k,$$

where $c\in\mathbb{C}$ and n assumed to be of the form $n=ak+b$ for some $a,b\in\mathbb{N}$, $b\leqslant k-1$.

We denote by U_m the mth degre *Chebyshev polynomial* of the second kind which satisfies the term recurrence relations

$$2xU_m(x)=U_{m+1}(x)+U_{m-1}(x), \quad m=1,2,..., \tag{2.1}$$

with intitial conditions $U_0(x)=1$ and $U_1(x)=2x$.

U_m is of degree m and it satisfies

$$U_m(\cos\theta)=\frac{\sin(m+1)\theta}{\sin\theta} \quad \text{for} \quad 0<\theta<\pi. \tag{2.2}$$

By consequent the zeros of the polynomial U_m are $x_{m,j}=\cos\frac{j\pi}{m+1}$, $j=1,...,m$.

Let us denotes $\lambda_j=c+2\cos\frac{j\pi}{a+2}$, $j=1,...,a+1$, and $\mu_s=c+2\cos\frac{s\pi}{a+1}$, $s=1,...,a$. Namely $\{\lambda_j\}$ and $\{\mu_s\}$ are respectivly the zeros of the polynomial $U_{a+1}\left(\frac{x-c}{2}\right)$ and $U_a\left(\frac{x-c}{2}\right)$.

For $p=0,...,b-1$ and $r=1,...,a$ we denote by $V_{p,r}$ the column vector

$$V_{p,r}=\left[\delta_{p,i \bmod k}U_{\left[\frac{i}{k}\right]}\left(\cos\frac{r\pi}{a+2}\right)\right]^T_{0\leq i\leq n-1}, \tag{2.3}$$

where δ is the *Kronecker's delta* symbol, $i \bmod k$ is the remainder of the euclidean division of i by k and $\left[\frac{i}{k}\right]$ is the greatest integer less or equal to $\frac{i}{k}$.

Similarly if $q=b,...,k-1$ and $s=1,...,a+1$ we denote by $W_{q,s}$ the column vector

$$(2.4) \qquad W_{q,s} = \left[\delta_{q,i \bmod k} U_{\left[\frac{i}{k}\right]}\left(\cos\frac{s\pi}{a+1}\right)\right]^T_{0\le i\le n-1}.$$

Theorem 1. *We have for p, q, r, s:*

1. λ_r is an eigenvalue of the matrix T and $V_{p,r}$ is a corresponding eigenvector.

2. μ_s is an eigenvalue of the matrix T and $W_{q,s}$ is a corresponding eigenvector.

*Proof. 1.*The ith entry of the vector $TV_{p,r}$ is:

$$\begin{aligned}
[TV_{p,r}]_i &= \sum_{j=1}^{n} t_{i.j}\,[V_{p,r}]_j \\
&= t_{i,i-k}\delta_{p,i-1 \bmod k} U_{\left[\frac{i-1}{k}-1\right]}\left(\cos\frac{r\pi}{a+2}\right) + \\
&\quad ct_{i,i-k}\delta_{p,i-1 \bmod k} U_{\left[\frac{i-1}{k}\right]}\left(\cos\frac{r\pi}{a+2}\right) + \\
&\quad t_{i,i+k}\delta_{p,i-1 \bmod k} U_{\left[\frac{i-1}{k}+1\right]}\left(\cos\frac{r\pi}{a+2}\right),
\end{aligned}$$

with $t_{i,i-k} = 0$ if $i \le k$ and $t_{i,i+k} = 0$ if $n < i+k$.
The result follows from the relations 2.1 by putting $x = \cos\frac{r\pi}{a+2}$ and $m = \left[\frac{i-1}{k}\right]$.
*2.*The proof is similar. ∎

Theorem 2. *The caracteristic polynomial of the matrix T is*

$$\det(xI_n - T) = \left[U_{a+1}\left(\frac{x-c}{2}\right)\right]^b \left[U_a\left(\frac{x-c}{2}\right)\right]^{k-b}.$$

Proof. $W_{q,s}$, $q = b, ..., k-1$ are $k-b$ linearly independant eigenvectors of the matrix T. Hence μ_s is an eigenvalue of the matrix T of multiplicity at least $k-b$. Similarly $V_{p,r}$, $p = 0, ..., b-1$ are b linearly independant eigenvectors of the matrix T, hence λ_r is an eigenvalue of the matrix T of multiplicity at least b. Obiouvsly $\lambda_s \ne \mu_r$ for all r, s. The result follows from the fact that $n = ak+b = a(k-b) + (a+1)b$ and that the polynomials $U_{a+1}\left(\frac{x-c}{2}\right), U_a\left(\frac{x-c}{2}\right)$ are monics. ∎

Example 1. [3, 7]
Let the pentadiagonal Toeplitz matrix of order n

$$H_n(\alpha,\beta)=\begin{pmatrix} \alpha & 0 & \beta & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \\ \beta & & & & & 0 \\ 0 & & & & & \beta \\ \vdots & \ddots & & & & 0 \\ 0 & \dots & 0 & \beta & 0 & \alpha \end{pmatrix}, \ \alpha,\beta\in\mathbb{C},\beta\neq 0.$$

If $n=2p$, $p\geq 1$ *then*

$$\det(H_n(\alpha,\beta))=\beta^n\left[U_p\left(\frac{\alpha}{2\beta}\right)\right]^2.$$

If $n=2p+1$, $p\geq 1$ *then*

$$\det(H_n(a,\beta))=\beta^n U_p\left(\frac{\alpha}{2\beta}\right)U_{p+1}\left(\frac{\alpha}{2\beta}\right).$$

3. An eigen decomposition of the matrix T

It follows from the proof of Theorem 2. that the matrix T is diagonalizable of the form $T=PJP^{-1}$, where J is the Jordan's form of T and P is the transforming matrix. Matrices J and P can be found provided eigenvalues and eigenvectors of the matrix T are known. Roughly speaking, J is the block diagonal matrix:

$$J=\begin{pmatrix} \lambda_1 I_b & & & & & \\ & \ddots & & & & \\ & & \lambda_{a+1}I_b & & & \\ & & & \mu_1 I_{k-b} & & \\ & & & & \ddots & \\ & & & & & \mu_a I_{k-b} \end{pmatrix},$$

where I_N denotes the identity matrix of size N, and P is defined by his columns $C_1,C_2,..,C_n$ obtained from the vectors $V_{p,r}$, $W_{q,s}$ following the order $\preceq$ such that:

i) $V_{p,r}\preceq W_{q,s}$.
ii) $V_{p,r}\preceq V_{p',r'} \Leftrightarrow r<r'$ or $r=r'$ and $p\leqslant p'$.
iii) $W_{q,s}\preceq W_{q',s'}\Leftrightarrow s<s'$ or $q=q'$ and $s\leqslant s'$.

Remark 1. *Suppose that* $P = (p_{i,j})_{1 \le i,j \le n}$. *One can show that:*
If $j \le (a+1)\,b$:
Then we can write $j-1 = (r-1)\,b + p$, *where* $1 \le r \le a+1$ *and* $0 \le p < b$.
Hence $C_j = V_{p,r}$, *and by consequent :*

$$p_{i,j} = [V_{p,r}]_{i-1} = \delta_{p,i-1 \bmod k} U_{\left[\frac{i-1}{k}\right]} \left(\cos \frac{r\pi}{a+2}\right).$$

If $(a+1)\,b < j$:
Then we can write $j-1-(a+1)\,b = (s-1)\,(k-b)+r$, *where* $1 \le s \le a$ *and* $b \le q = r+b < k$. *Hence* $C_j = W_{q,s}$ *and by consequent:*

$$p_{i,j} = [W_{q,s}]_{i-1} = \delta_{q,i-1 \bmod k} U_{\left[\frac{i-1}{k}\right]} \left(\cos \frac{s\pi}{a+1}\right).$$

We shall now give the expression of the inverse P^{-1}. We need the following lemma:

Lemma 1 (*Christoffel-Darboux* formula)**.** *For all* $m \geqslant 1$ *we have:*

1. $\sum_{i=0}^{m-1} U_i(x)\, U_i(y) = \dfrac{U_m(y)\, U_{m-1}(x) - U_m(x)\, U_{m-1}(y)}{2\,(y-x)}$ for $x \neq y$.
2. $\sum_{i=0}^{m-1} U_i(x)\, U_i(x) = \frac{1}{2}\left[U'_m(x)\, U_{m-1}(x) - U_m(x)\, U'_{m-1}(x)\right]$.

Proof. See [6]. ∎

Lemma 2. *We have:*

1. $V_{p,r}^T.W_{q,s} = 0$.
2. $V_{p,r}^T.V_{p',r'} = \dfrac{\delta_{p,p'}\delta_{r,r'}}{2} U'_{a+1}\left(\cos\frac{r\pi}{a+2}\right) U_a\left(\cos\frac{r\pi}{a+2}\right)$.
3. $W_{q,s}^T.W_{q',s'} = \dfrac{\delta_{q,q'}\delta_{s,s'}}{2} U'_{a}\left(\cos\frac{s\pi}{a+1}\right) U_{a-1}\left(\cos\frac{s\pi}{a+1}\right)$.

Proof. 1. $V_{p,r}^T.W_{q,s} = \sum_{i=0}^{n-1} \delta_{p,i \bmod k}\delta_{q,i \bmod k} U_{\left[\frac{i}{k}\right]}\left(\cos\frac{r\pi}{a+2}\right) U_{\left[\frac{i}{k}\right]}\left(\cos\frac{s\pi}{a+1}\right)$.
Since $p < b$ and $q \ge b$ then $\delta_{p,i \bmod k}\delta_{q,i \bmod k} = 0$ for all i.
2. $V_{p,r}^T.V_{p',r'} = \sum_{i=0}^{n-1} \delta_{p,i \bmod k}\delta_{p',i \bmod k} U_{\left[\frac{i}{k}\right]}\left(\cos\frac{r\pi}{a+2}\right) U_{\left[\frac{i}{k}\right]}\left(\cos\frac{r'\pi}{a+2}\right)$.
If $p \neq p'$ then $\delta_{p,i \bmod k}\delta_{p',i \bmod k} = 0$ for all i.
On the other hand, by Lemma 1:

$V_{p,r}^T.V_{p,r'} = \sum_{j=0}^{a} U_j \left(\cos \frac{r\pi}{a+2}\right) U_j \left(\cos \frac{r'\pi}{a+2}\right) = \dfrac{\delta_{r,r'}}{2} U'_{a+1} \left(\cos \frac{r\pi}{a+2}\right) U_a \left(\cos \frac{r\pi}{a+2}\right).$

(Since $p < b: \quad i = jk + p \leq n-1 \Leftrightarrow j \leq a$)

3. $W_{q,s}^T.W_{q',s'} = \sum_{i=0}^{n-1} \delta_{q,i \bmod k} \delta_{q',i \bmod k} U_{\left[\frac{i}{k}\right]} \left(\cos \frac{s\pi}{a+1}\right) U_{\left[\frac{i}{k}\right]} \left(\cos \frac{s'\pi}{a+1}\right).$

If $q \neq q'$ then $\delta_{q,i \bmod k} \delta_{q',i \bmod k} = 0$.

On the other hand, by Lemma 1:

$$W_{q,s}^T.W_{q,s'} = \sum_{j=0}^{a-1} U_j \left(\cos \frac{s\pi}{a+1}\right) U_j \left(\cos \frac{s'\pi}{a+1}\right) = \frac{\delta_{s,s'}}{2} U'_a \left(\cos \frac{s\pi}{a+1}\right) U_{a-1} \left(\cos \frac{s\pi}{a+1}\right).$$

(Since $q \geq b: \quad i = jk + q \leq n-1 \Leftrightarrow j \leq a-1$)

∎

Corollary 1. *Let $C_1, C_2, .., C_n$ the columns of the matrix P, then:*

1. $C_i^T.C_j = 0$,
   for all i, j such that $i \neq j$.
2. $C_j^T.C_j = \frac{1}{2} U'_{a+1} \left(\cos \frac{r\pi}{a+2}\right) U_a \left(\cos \frac{r\pi}{a+2}\right)$,
   for $j - 1 = b(r-1) + p \leq (a+1)b$, where $1 \leq r \leq a+1$ and $0 \leq p < b$.
3. $C_j^T.C_j = \frac{1}{2} U'_a \left(\cos \frac{s\pi}{a+1}\right) U_{a-1} \left(\cos \frac{s\pi}{a+1}\right)$,
   for $j > (a+1)b$ such that $j - 1 - (a+1)b = (s-1)(k-b) + r$, where $1 \leq s \leq a$ and $b \leq q = r + b < k$.

To determine the explicit expression of the inverse of the matrix P we must calculate $U'_{a+1} \left(\cos \frac{r\pi}{a+2}\right)$, $U_a \cos \left(\frac{r\pi}{a+2}\right)$, $U'_a \left(\cos \frac{s\pi}{a+1}\right)$ and $U_{a-1} \left(\cos \frac{s\pi}{a+1}\right)$.

First by the relation 2.2

$$U_a \left(\cos \frac{r\pi}{a+2}\right) = \frac{\sin \frac{r(a+1)\pi}{a+2}}{\sin \frac{r\pi}{a+2}} = (-1)^{r-1}.$$

and

$$U_{a-1} \left(\cos \frac{s\pi}{a+1}\right) = (-1)^{s-1}.$$

Differentiating the relation 2.2 with respect to θ we get:

$$\cos\theta U_i(\cos\theta) - \sin^2\theta U'_i(\cos\theta) = (i+1)\cos(i+1)\theta.$$

Hence

$$U'_a \left(\cos \frac{s\pi}{a+1}\right) = \frac{(-1)^{s+1}(a+1)}{\sin^2 \frac{s\pi}{a+1}},$$

and

$$U'_{a+1}\left(\cos\frac{r\pi}{a+2}\right)=\frac{(-1)^{r+1}(a+2)}{\sin^2\frac{r\pi}{a+2}}.$$

Theorem 3. *Let* $P^{-1}=(d_{i,j})_{1\le i,j\le n}$ *then* $d_{i,j}=h_i p_{j,i}$*, where*

$$h_i=\frac{2}{a+2}\sin^2\frac{r\pi}{a+2},$$

for $i-1=b(r-1)+p<(a+1)b,\ 1\le r\le a+1$ *and* $0\le p<b$.
and

$$h_i=\frac{2}{a+1}\sin^2\frac{s\pi}{a+1},$$

if $i>(a+1)b$ *such that* $i-1-(a+1)b=(s-1)(k-b)+r,\ 1\le s\le a$ *and* $b\le q=r+b<k$.

Proof. According to Corollary 1., the jth row of the matrix P^{-1} is $\gamma_j^{-1}C_j^T$, where

$$\gamma_j=\frac{1}{2}U'_{a+1}\left(\cos\frac{r\pi}{a+2}\right)U_a\left(\cos\frac{r\pi}{a+2}\right),$$

for $j-1=b(r-1)+p<(a+1)b,\ 1\le r\le a+1$ and $0\le p<b$,
and

$$\gamma_j=\frac{1}{2}U'_a\left(\cos\frac{s\pi}{a+1}\right)U_{a-1}\left(\cos\frac{s\pi}{a+1}\right),$$

for $j>(a+1)b$ such that $j-1-(a+1)b=(s-1)(k-b)+r,\ 1\le s\le a$ and $b\le q=r+b<k$.
Hence:
If $j-1=b(r-1)+p<(a+1)b,\ 1\le r\le a+1$ and $0\le p<b$, then

$$\gamma_j=\frac{a+2}{2\sin^2\frac{r\pi}{a+2}}$$

If $j>(a+1)b$ such that $j-1-(a+1)b=(s-1)(k-b)+r,\ 1\le s\le a$ and $b\le q=r+b<k$ then

$$\gamma_j=\frac{a+1}{2\sin^2\frac{s\pi}{a+1}}.$$

The result follows. ∎

4. Main result

Let $T^m = \left(t_{i,j}^{(m)}\right)_{1\le i,j\le n}$ the mth power($m \in \mathbb{N}$) of the matrix T, then $T^m = PJ^mP^{-1}$.

The diagonal matrix $J = (k_{i,j})_{1\le i,j\le n}$ is such that $k_{i,j} = 0$ if $i \neq j$ and:

If $j-1 = b(r-1)+p < (a+1)b$, $1 \le r \le a+1$ and $0 \le p < b$, then $k_{j,j} = \lambda_r$.

If $j > (a+1)b$ such that $j-1-(a+1)b = (s-1)(k-b)+r$, $1 \le s \le a$ and $b \le q = r+b < k$ then $k_{j,j} = \mu_s$.

Theorem 4. *Let $T^m = \left(t_{i,j}^{(m)}\right)_{1\le i,j\le n}$ the mth power($m \in \mathbb{N}$) of the matrix T, then:*

$$t_{i,j}^{(m)} = \sum_{u=1}^{n} p_{i,u} p_{j,u} \gamma_u^m h_u, \tag{4.1}$$

where:
If $u \le (a+1)b$, $u-1 = b(r-1)+p$, $1 \le r \le a+1$, $0 \le p < b$, then:

$$\gamma_u = \lambda_r, \quad h_u = \frac{2}{a+2}\sin^2\frac{r\pi}{a+2} \quad \text{and} \quad p_{i,u} = \delta_{p,i-1 \bmod k} U_{\left[\frac{i-1}{k}\right]}\left(\cos\frac{r\pi}{a+2}\right).$$

If $(a+1)b < u$, $u-1-(a+1)b = (s-1)(k-b)+r$, $1 \le s \le a$ and $b \le q = r+b < k$, then:

$$\gamma_u = \mu_s, \; h_u = \frac{2}{a+1}\sin^2\frac{s\pi}{a+1} \quad \text{and} \quad p_{i,u} = \delta_{q,i-1 \bmod k} U_{\left[\frac{i-1}{k}\right]}\left(\cos\frac{s\pi}{a+1}\right).$$

Remark 2. *The matrix T is nonsingular if and only $\lambda_r\mu_s \neq 0$ for all r, s and the expression of the inverse $\left(t_{i,j}^{(-1)}\right)_{1\le i,j\le n}$ is obtained from the expression 4.1 by putting $m = -1$.*

5. Examples

Example 2. $k = 1$. *Tridiagonal matrix* [2].

Then $a = n$ and $b = 0$. Hence for $m \in \mathbb{N}$ (Or $m = -1$ if T is nonsingular)

$$\begin{aligned} t_{i,j}^{(m)} &= \tfrac{2}{n+1}\sum_{u=1}^{n}\left[c + 2\cos\tfrac{u\pi}{n+1}\right]^m \sin^2\tfrac{u\pi}{n+1} U_{i-1}\left(\cos\tfrac{u\pi}{n+1}\right) U_{j-1}\left(\cos\tfrac{u\pi}{n+1}\right) \\ &= \tfrac{2}{n+1}\sum_{u=1}^{n}\left[c + 2\cos\tfrac{u\pi}{n+1}\right]^m \sin\tfrac{ui\pi}{n+1}\sin\tfrac{uj\pi}{n+1}. \end{aligned}$$

Example 3. $k = 2$. *Pentadiagonal matrix* [4, 5].

Case 1 ($n = 2a,\, a \in \mathbb{N}^*$)**.** *Hence* $b = 0$.

We have for i, j : $p_{i,j} = \delta_{\overline{i-1},\overline{j-1}} U_{\left[\frac{i-1}{2}\right]}\left(\cos\frac{\left[\frac{1+j}{2}\right]\pi}{a+1}\right)$,

where $\overline{u} = 0$ *if* u *is even and* $\overline{u} = 1$ *if* u *is odd.*

Hence:
$p_{2r,2s} = U_{r-1}\left(\cos\frac{s\pi}{a+1}\right)$, $p_{2r+1,2s+1} = U_r\left(\cos\frac{(s+1)\pi}{a+1}\right)$ *and* $p_{i,j} = 0$ *if* $\overline{i} \neq \overline{j}$.

We deduce that if $\overline{i} \neq \overline{j}$ *then* $t_{i,j}^{(m)} = 0$. *Moreover:*

$$\begin{aligned} t_{2r,2s}^{(m)} &= \sum_{u=1}^{n} p_{2r,u} p_{2s,u} \gamma_u^m h_u \\ &= \sum_{v=1}^{a} h_{2v} p_{2r,2v} p_{2s,2v} \gamma_{2v}^m \\ &= \tfrac{2}{a+1} \sum_{v=1}^{a} \sin^2 \tfrac{v\pi}{a+1} U_{r-1}\left(\cos\tfrac{v\pi}{a+1}\right) U_{s-1}\left(\cos\tfrac{v\pi}{a+1}\right)\left[c + 2\cos\tfrac{v\pi}{a+1}\right]^m \\ &= \tfrac{2}{a+1} \sum_{v=1}^{a} \sin\tfrac{rv\pi}{a+1} \sin\tfrac{sv\pi}{a+1}\left[c + 2\cos\tfrac{v\pi}{a+1}\right]^m. \end{aligned}$$

and

$$\begin{aligned} t_{2r+1,2s+1}^{(m)} &= \sum_{u=1}^{n} h_u p_{2r+1,u} p_{2s+1,u} \gamma_u^m \\ &= \sum_{v=0}^{a-1} h_{2v+1} p_{2r+1,2v+1} p_{2s+1,2v+1} \gamma_{2v+1}^m \\ &= \tfrac{2}{a+1} \sum_{v=1}^{a} U_r\left(\cos\tfrac{v\pi}{a+1}\right) U_s\left(\cos\tfrac{v\pi}{a+1}\right)\left[c + 2\cos\tfrac{v\pi}{a+1}\right]^m \sin^2\tfrac{v\pi}{a+1} \\ &= \tfrac{2}{a+1} \sum_{v=1}^{a} \sin\tfrac{(r+1)v\pi}{a+1} \sin\tfrac{(s+1)v\pi}{a+1}\left[c + 2\cos\tfrac{v\pi}{a+1}\right]^m. \end{aligned}$$

Case 2 ($n = 2a + 1,\, a \in \mathbb{N}^*$)**.** *Hence* $b = 1$,
If $j \leq a + 1$ *then* $j - 1 = (r - 1)\, b + p$, *with* $p = 0$ *and* $r = j$.
We have:
$p_{i,j} = \delta_{0,\overline{i-1}} U_{\left[\frac{i-1}{2}\right]}\left(\cos\frac{j\pi}{a+2}\right)$, $h_j = \frac{2}{a+2}\sin^2\left(\frac{j\pi}{a+2}\right)$,

and $\gamma_j = c + 2\cos\frac{j\pi}{a+2}$.

If $j > a + 1$ *then* $j - 1 - (a + 1) = (k - b)(s - 1) + r$ *with* $r = 0$ *and* $s = j - (a + 1)$.
Hence
$p_{i,j} = \delta_{1,\overline{i-1}} U_{\left[\frac{i-1}{2}\right]}\left(\cos\frac{s\pi}{a+2}\right) = \delta_{1,\overline{i-1}} (-1)^{\left[\frac{i-1}{2}\right]} U_{\left[\frac{i-1}{2}\right]}\left(\cos\frac{j\pi}{a+2}\right)$,

$h_j = \frac{2}{a+2}\sin^2\left(\frac{j\pi}{a+2}\right)$, *and* $\gamma_j = c + 2\cos\frac{s\pi}{a+2} = c - 2\cos\frac{j\pi}{a+2}$.

We have for $i, j = 1, ..., n$:

$$t_{i,j}^{(m)} = \sum_{u=1}^{a+1} p_{i,u} p_{j,u} \gamma_u^m h_u + \sum_{u=a+2}^{n} p_{i,u} p_{j,u} \gamma_u^m h_u.$$

It follow that if $\overline{i} \neq \overline{j}$ *then* $t_{i,j}^{(m)} = 0$.

On the other hand:

$$\begin{aligned} t_{2r,2s}^{(m)} &= \sum_{u=1}^{n} p_{2r,u} p_{2s,u} \gamma_u^m h_u \\ &= \sum_{u=a+2}^{n} h_u p_{2r,u} p_{2s,u} \gamma_u^m \\ &= \frac{2(-1)^{r+s}}{a+1} \sum_{u=a+2}^{n} \sin^2 \frac{u\pi}{a+1} U_{r-1}\left(\cos\frac{u\pi}{a+1}\right) U_{s-1}\left(\cos\frac{u\pi}{a+1}\right) \left[c - 2\cos\frac{u\pi}{a+1}\right]^m \\ &= \frac{2}{a+1} \sum_{v=1}^{a} \sin\frac{rv\pi}{a+1} \sin\frac{sv\pi}{a+1} \left[c + 2\cos\frac{v\pi}{a+1}\right]^m . \end{aligned}$$

and

$$\begin{aligned} t_{2r+1,2s+1}^{(m)} &= \sum_{u=1}^{a+1} h_u p_{2r+1,u} p_{2s+1,u} \gamma_u^m \\ &= \frac{2}{a+2} \sum_{u=1}^{a+1} \sin^2 \frac{u\pi}{a+2} U_r\left(\cos\frac{u\pi}{a+2}\right) U_s\left(\cos\frac{u\pi}{a+2}\right) \left[c + 2\cos\frac{u\pi}{a+2}\right]^m \\ &= \frac{2}{a+2} \sum_{u=1}^{a+1} \sin\frac{(r+1)u\pi}{a+2} \sin\frac{(s+1)u\pi}{a+2} \left[c + 2\cos\frac{u\pi}{a+2}\right]^m . \end{aligned}$$

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