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On the Kernel of the Black-Scholes Equation in the Form of White Noise

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Abstract

In this paper, we study the well known equation which is the Black-Scholes equation in the form of white noise. We found the kernel of such equation and obtained some interesting properties of such kernel.

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1 Introduction

In financial mathematics, the famous equation named the Black-Scholes equation plays an important role in solving the option price of stocks. The Black-Scholes equation is given by

$$\frac{\partial u(s,t)}{\partial t} + rs\frac{\partial u(s,t)}{\partial s} + \frac{\sigma^2}{2}s^2\frac{\partial^2 u(s,t)}{\partial s^2} - ru(s,t) = 0 \tag{1}$$

with the terminal condition

$$u(s,t) = (s-p)^+ \tag{2}$$

for $0 \leq t \leq T$ where $u(s,t)$ is the option price at time t, r is the interest rate, s is the price of stock at time t, σ is the volatility of stock and p is the strike price.

In this work, we transformed the option price $u(s,t)$ to $V(\xi,t)$ where ξ is the white noise and ξ can be obtained from the Geometric Brownian motion of the stock model

$$ds = \mu s dt + \sigma s dB$$

where μ is the drift, σ is the volatility and B is the Brownian motion. Thus (1) can be transformed to the equation

$$\frac{\partial V(\xi,t)}{\partial t}+\frac{1}{2t^2}\frac{\partial^2 V(\xi,t)}{\partial \xi^2}+\left(\frac{r}{t\sigma}-\frac{\sigma}{2t}\right)\frac{\partial V(\xi,t)}{\partial \xi}-rV(\xi,t)=0. \qquad (3)$$

We study (3) for the case $0\leq t\leq 1$ with the condition

$$V(\xi,1)=f(\xi) \qquad (4)$$

where $f(\xi)$ is the given generalized function. We can show that $\xi=\frac{dB}{dt}$ is a tempered distribution and $V(\xi,t)$ is also a tempered distribution see [1]. Thus we can apply the Fourier transform to (3) and (4), we obtain

$$V(\xi,t)=K(\xi,t)*f(\xi) \qquad (5)$$

in the convolution form as a solution of (3) with satisfies the condition (4) and $K(\xi,t)$ is the kernel of the form

$$K(\xi,t)=\sqrt{\frac{t}{2\pi(1-t)}}e^{-(1-t)r}exp\left[\frac{-t\left(\xi-\left(\frac{r}{\sigma}-\frac{\sigma}{2}\right)\ln t\right)^2}{2(1-t)}\right]. \qquad (6)$$

It can be shown that $\lim_{t\to 1}K(\xi,t)=\delta(\xi)$. Thus from (5)

$$V(\xi,1)=\lim_{t\to 1}V(\xi,t)=\delta(\xi)*f(\xi)=f(\xi)$$

thus (4) holds.

2 Preliminary Notes

Recall the stock model

$$ds=\mu sdt+\sigma sdB \qquad (7)$$

or

$$ds=\mu sdt+\sigma s\dot{B}(t)dt$$

where B is the Wiener process or the Brownian motion, $\dot{B}(t)=\frac{dB}{dt}$ is the white noise. From (7), by using the Itô's formula, we obtain

$$\int_0^t d\left(\ln s(\tau)\right)=\left(\mu-\frac{\sigma^2}{2}\right)\int_0^t d\tau+\sigma\int_0^t \dot{B}(\tau)d\tau \quad \text{where} \quad 0\leq\tau\leq t.$$

Thus

$$\ln s(t)-\ln s_0=\left(\mu-\frac{\sigma^2}{2}\right)t+\sigma\int_0^t \dot{B}(\tau)d\tau \qquad (8)$$

where $s(0) = s_0$.

Now consider $\int_0^t \dot{B}(\tau)d\tau$, we regard $\dot{B}(\tau)$ as a stochastic process with smooth sample path. By the mean value theorem, there exists some point τ^* for $0 < \tau^* < t$ such that

$$\int_0^t \dot{B}(\tau^*)d\tau = \dot{B}(\tau^*)\int_0^t d\tau = \dot{B}(\tau^*)t$$

where $\dot{B}(\tau^*)$ is the mean value of such integral and let $\xi(\tau^*)$ denote the mean value $\dot{B}(\tau^*)$. Actually $\xi(\tau^*)$ also the white noise. Thus from (7), $\ln\left(\frac{s}{s_0}\right) = \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma\xi t$ or

$$\xi = \frac{1}{t\sigma}\ln\left(\frac{s}{s_0}\right) - \frac{\mu}{\sigma} + \frac{\sigma}{2} \tag{9}$$

Let us discuss briefly some properties of ξ. It has been shown that ξ is a tempered distribution, that is $\forall\xi \in \mathcal{S}'(\mathbb{R})-$the space of tempered distribution and for any testing function $\varphi \in \mathcal{S}(\mathbb{R})-$the Schwartz space, define $<\xi, \varphi>$ on $\mathcal{S}'(\mathbb{R})$, we can apply the Minlos Theorem to obtain a probability measure μ on its dual space $\mathcal{S}'(\mathbb{R})$ such that

$$\int\limits_{\mathcal{S}'(\mathbb{R})} e^{ix\varphi}d\mu(x) = e^{-\frac{1}{2}|\varphi|_0^2}, \quad \forall\varphi \in \mathcal{S}(\mathbb{R})$$

where $|\varphi|_0^2 = \int\limits_{\mathbb{R}} |\varphi|^2 dx$ see [1]. Moreover $<\xi, \varphi>$ has the Gaussain distribution with mean 0 and variance $|\varphi|_0^2$.

Now we show how to construct equation (3). From (9)

$$\frac{\partial\xi}{\partial s} = \frac{1}{t\sigma}\cdot\frac{s_0}{s}\cdot\frac{1}{s_0} = \frac{1}{t\sigma s},$$

thus

$$\frac{\partial u(s,t)}{\partial s} = \frac{\partial V(\xi,t)}{\partial s} = \frac{\partial V(\xi,t)}{\partial \xi}\cdot\frac{\partial\xi}{\partial s} = \frac{1}{t\sigma s}\frac{\partial V(\xi,t)}{\partial\xi},$$

and

$$\begin{aligned}
\frac{\partial^2 u(s,t)}{\partial s^2} &= \frac{\partial^2 V(\xi,t)}{\partial s^2} = \frac{\partial}{\partial\xi}\left(\frac{1}{t\sigma s}\frac{\partial V(\xi,t)}{\partial\xi}\right)\cdot\frac{\partial\xi}{\partial s} \\
&= \left(\frac{1}{t\sigma s}\frac{\partial^2 V(\xi,t)}{\partial\xi^2} + \frac{1}{t\sigma}\frac{\partial V(\xi,t)}{\partial\xi}\left(-\frac{1}{s^2}\right)\frac{\partial s}{\partial\xi}\right)\frac{\partial\xi}{\partial s} \\
&= \frac{1}{t^2\sigma^2 s^2}\frac{\partial^2 V(\xi,t)}{\partial\xi^2} - \frac{1}{t\sigma s^2}\frac{\partial V(\xi,t)}{\partial\xi}
\end{aligned}$$

substitute into (1) we obtain (3).

Definition 2.1. Let $f(x)$ is a locally integrable function. The Fourier transform $\widehat{f}(\omega)$ of $f(x)$ is defined by

$$\widehat{f}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega x} f(x)dx \tag{10}$$

and the inverse Fourier transform of $\widehat{f}(\omega)$ also defined by

$$f(x) = \mathcal{F}^{-1}\widehat{f}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x}\widehat{f}(\omega)d\omega \tag{11}$$

3 Main Results

Theorem 3.1. *Given the white noise form of the Black-Scholes equation*

$$\frac{\partial V(\xi,t)}{\partial t} + \frac{1}{2t^2}\frac{\partial^2 V(\xi,t)}{\partial \xi^2} + \left(\frac{r}{t\sigma} - \frac{\sigma}{2t}\right)\frac{\partial V(\xi,t)}{\partial \xi} - rV(\xi,t) = 0 \tag{12}$$

for $0 \leq t \leq 1$ *and* ξ *is the white noise given by* (9) *with the condition*

$$V(\xi,1) = f(\xi) \tag{13}$$

where $f(\xi)$ *is the given generalized function. Then we obtain*

$$V(\xi,t) = K(\xi,t) * f(\xi) \tag{14}$$

as the solution of (12) *where*

$$K(\xi,t) = \sqrt{\frac{t}{2\pi(1-t)}}e^{-(1-t)r}\exp\left[\frac{-t\left(\xi - \left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)\ln t\right)^2}{2(1-t)}\right]. \tag{15}$$

is the kernel of (12).

Proof. Take the Fourier transform defined by (10) to both sides of (12) and obtain

$$\frac{\partial \widehat{V}(\omega,t)}{\partial t} - \frac{\omega^2}{2t^2}\widehat{V}(\omega,t) + \frac{1}{t}\left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)i\omega\widehat{V}(\omega,t) - r\widehat{V}(\omega,t) = 0.$$

Thus

$$\widehat{V}(\omega,t) = C(\omega)e^{rt}e^{-\frac{\omega^2}{2t} - i\omega\left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)\ln t}.$$

Now, from (13),

$$\widehat{V}(\omega,1) = \widehat{f}(\omega).$$

Thus

$$C(\omega) = e^{-r+\frac{\omega^2}{2}}\widehat{f}(\omega),$$

and we obtain

$$\widehat{V}(\omega,t)=e^{-(1-t)r}\exp\left(\frac{-(1-t)}{2t}\omega^2-i\omega\left(\frac{r}{\sigma}-\frac{\sigma}{2}\right)\ln t\right)\widehat{f}(\omega),$$

for $0<t\leq 1$. Now

$$|\widehat{V}(\omega,t)|\leq\mid e^{-(1-t)r}\mid\mid e^{\frac{-(1-t)}{2t}\omega^2}\mid\mid\widehat{f}(\omega)\mid.$$

Let $M=\max\mid\widehat{f}(\omega)\mid$, thus

$$|\widehat{V}(\omega,t)|\leq\mid e^{-(1-t)r}\mid\mid e^{\frac{-(1-t)}{2t}\omega^2}\mid M\leq K.$$

where K is a constant. It follows that $\widehat{V}(\omega,t)$ is bounded for any fixed t with $0<t\leq 1$. Moreover $\widehat{V}(\omega,t)$ is a tempered distribution. Now, from (11),

$$\begin{aligned}
&V(\xi,t)\\
&=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\omega\xi}\widehat{V}(\omega,t)d\omega\\
&=\frac{1}{2\pi}e^{-(1-t)r}\int_{-\infty}^{\infty}\exp\left[\frac{-(1-t)}{2t}\omega^2-i\omega\left(\frac{r}{\sigma}-\frac{\sigma}{2}\right)\ln t+i\omega\xi\right]\widehat{f}(\omega)d\omega\\
&=\frac{1}{2\pi}e^{-(1-t)r}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\\
&\qquad\exp\left[\frac{-(1-t)}{2t}\left(\omega^2-\frac{2\omega ti\left(\xi-y-\left(\frac{r}{\sigma}-\frac{\sigma}{2}\right)\ln t\right)}{1-t}\right)\right]f(y)dyd\omega\\
&=\frac{1}{2\pi}e^{-(1-t)r}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left[\frac{-t\left(\xi-y-\left(\frac{r}{\sigma}-\frac{\sigma}{2}\right)\ln t\right)^2}{2(1-t)}\right]\times\\
&\qquad\exp\left[\frac{-(1-t)}{2t}\left(\omega-\frac{it\left(\xi-y-\left(\frac{r}{\sigma}-\frac{\sigma}{2}\right)\ln t\right)^2}{1-t}\right)\right]f(y)dyd\omega\\
&=\frac{1}{2\pi}e^{-(1-t)r}\int_{-\infty}^{\infty}\exp\left[\frac{-t\left(\xi-y-\left(\frac{r}{\sigma}-\frac{\sigma}{2}\right)\ln t\right)^2}{2(1-t)}\right]\times\\
&\qquad\int_{-\infty}^{\infty}\exp\left[\frac{-(1-t)}{2t}\left(\omega-\frac{it\left(\xi-y-\left(\frac{r}{\sigma}-\frac{\sigma}{2}\right)\ln t\right)^2}{1-t}\right)\right]d\omega f(y)dy.
\end{aligned}$$

Let $u=\sqrt{\frac{(1-t)}{2t}}\left(\omega-\frac{it\left(\xi-y-\left(\frac{r}{\sigma}-\frac{\sigma}{2}\right)\ln t\right)}{1-t}\right)$, then $du=\sqrt{\frac{1-t}{2t}}d\omega$ and we have

$d\omega = \sqrt{\frac{2t}{1-t}}du$. Thus

$$\begin{aligned} V(\xi,t) &= \frac{1}{2\pi}e^{-(1-t)r}\sqrt{\frac{2t}{1-t}}\int_{-\infty}^{\infty}\exp\left[\frac{-t\left(\xi-y-\left(\frac{r}{\sigma}-\frac{\sigma}{2}\right)\ln t\right)^2}{2(1-t)}\right] \\ &\qquad \left(\int_{-\infty}^{\infty}e^{-u^2}du\right)f(y)dy \\ &= \frac{\sqrt{\pi}}{2\pi}\sqrt{\frac{2t}{1-t}}e^{-(1-t)r}\int_{-\infty}^{\infty}\exp\left[\frac{-t\left(\xi-y-\left(\frac{r}{\sigma}-\frac{\sigma}{2}\right)\ln t\right)^2}{2(1-t)}\right]f(y)dy \\ &= \sqrt{\frac{t}{2\pi(1-t)}}e^{-(1-t)r}\int_{-\infty}^{\infty}\exp\left[\frac{-t\left(\xi-y-\left(\frac{r}{\sigma}-\frac{\sigma}{2}\right)\ln t\right)^2}{2(1-t)}\right]f(y)dy. \end{aligned}$$

Let

$$K(\xi,t) = \sqrt{\frac{t}{2\pi(1-t)}}e^{-(1-t)r}\exp\left[\frac{-t\left(\xi-\left(\frac{r}{\sigma}-\frac{\sigma}{2}\right)\ln t\right)^2}{2(1-t)}\right] \quad \text{for } 0<t\leq 1.$$

Then $V(\xi,t) = K(\xi,t) * f(\xi)$ for $0<t\leq 1$.
Thus we obtain (15). $K(\xi,t)$ is called the kernel of (12). □

Theorem 3.2. (The properties of $K(\xi,t)$).
The kernel $K(\xi,t)$ define by (15) has the following properties

(i) $K(\xi,t)$ *satisfies equation* (12).

(ii) $K(\xi,t)$ *is a tempered distribution, that is* $K(\xi,t)\in\mathcal{S}'(\mathbb{R})$.

(iii) $K(\xi,t)>0$ *for* $0<t\leq 1$.

(iv) $e^{(1-t)r}\int_{-\infty}^{\infty}K(\xi,t)d\xi = 1$.

(v) $\lim_{t\to 1}K(\xi,t) = \delta(\xi)$ *where* $\delta(\xi)$ *is the Dirac-delta distribution.*

(vi) $K(\xi,t)$ *is a Gaussian(or normal) distribution with mean* $e^{-(1-t)r}\left(\frac{r}{\sigma}-\frac{\sigma}{2}\right)\ln t$, *variance* $e^{-2(1-t)r}\frac{(1-t)}{t}$.

Proof. (i) By computing directly, $K(\xi,t)$ satisfies (12).
(ii) Since $K(\xi,t)$ is a Gaussian function and $K(\xi,t)\in\mathcal{L}(\mathbb{R})$ where $\mathcal{L}(\mathbb{R})$ is the space of integrable function on the real $\mathbb{R}$. It follows that $K(\xi,t)$ is a tempered distribution.

(iii) $K(\xi, t) > 0$ for $0 < t \leq 1$ is obvious.
(iv)

$$\begin{aligned}
& e^{(1-t)r} \int_{-\infty}^{\infty} K(\xi,t) d\xi \\
& \quad = e^{(1-t)r} \int_{-\infty}^{\infty} \sqrt{\frac{t}{2\pi(1-t)}} e^{-(1-t)r} \exp\left[\frac{-t\left(\xi - \left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)\ln t\right)^2}{2(1-t)}\right] \\
& \quad = \sqrt{\frac{t}{2\pi(1-t)}} \int_{-\infty}^{\infty} \exp\left[\frac{-t\left(\xi - \left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)\ln t\right)^2}{2(1-t)}\right] d\xi.
\end{aligned}$$

Let $u = \frac{\sqrt{t}\left(\xi - \left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)\ln t\right)}{\sqrt{2(1-t)}}$, then $d\xi = \sqrt{\frac{2(1-t)}{t}} du$. Thus

$$\begin{aligned}
e^{(1-t)r} \int_{-\infty}^{\infty} K(\xi,t) d\xi &= \sqrt{\frac{t}{2\pi(1-t)}} \int_{-\infty}^{\infty} e^{-u^2} \sqrt{\frac{2(1-t)}{t}} du \\
&= \sqrt{\frac{t}{2\pi(1-t)}} \sqrt{\frac{2(1-t)}{t}} \sqrt{\pi} \\
&= 1.
\end{aligned}$$

Thus $\quad e^{(1-t)r} \int_{-\infty}^{\infty} K(\xi,t) d\xi = 1.$
(v)

$$\begin{aligned}
\lim_{t\to 1} K(\xi,t) &= \lim_{t\to 1} e^{-(1-t)r} \lim_{t\to 1} \sqrt{\frac{t}{2\pi(1-t)}} \exp\left[\frac{-t\left(\xi - \left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)\ln t\right)^2}{2(1-t)}\right] \\
&= \lim_{t\to 1} \sqrt{\frac{t}{2\pi(1-t)}} \exp\left[\frac{-t\left(\xi - \left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)\ln t\right)^2}{2(1-t)}\right] \\
&= \delta(\xi) \qquad \text{see [2,pp 36-37].}
\end{aligned}$$

(vi) Now, the mean

$$\begin{aligned}
E\left(K(\xi,t)\right) &= E\left(e^{-(1-t)r} \sqrt{\frac{t}{2\pi(1-t)}} \exp\left[\frac{-t\left(\xi - \left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)\ln t\right)^2}{2(1-t)}\right]\right) \\
&= e^{-(1-t)r} E\left(\sqrt{\frac{t}{2\pi(1-t)}} \exp\left[\frac{-\left(\xi - \left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)\ln t\right)^2}{\frac{2(1-t)}{t}}\right]\right) \\
&= e^{-(1-t)r} \left(\frac{r}{\sigma} - \frac{\sigma}{2}\right) \ln t.
\end{aligned}$$

Since the function in the is a Gaussian or normal distribution. Thus $E\left(K(\xi,t)\right)$ is the mean $e^{-(1-t)r}\left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)\ln t$.

The variance

$$\begin{aligned} V\left(K(\xi,t)\right) &= V\left(e^{-(1-t)r}\sqrt{\frac{t}{2\pi(1-t)}}\exp\left[\frac{-t\left(\xi-\left(\frac{r}{\sigma}-\frac{\sigma}{2}\right)\ln t\right)^2}{2(1-t)}\right]\right) \\ &= e^{-2(1-t)r}\left(\frac{1-t}{t}\right) \qquad \text{for } 0<t\leq 1. \end{aligned}$$

□

Note: The solution $V(\xi,t)$ of (12) of the Theorem (3.1) is called the option price in the white noise form where the white noise ξ can be computed from (8) when the price s of the stock is Known. The solution $V(\xi,t)$ can be written by

$$V(\xi,t)=K(\xi,t)*f(\xi)$$

or for $0<t\leq 1$,

$$V(\xi,t)=e^{-(1-t)r}\sqrt{\frac{t}{2\pi(1-t)}}\int_{-\infty}^{\infty}\exp\left[\frac{-t\left(\xi-y-\left(\frac{r}{\sigma}-\frac{\sigma}{2}\right)\ln t\right)^2}{2(1-t)}\right]f(y)dy,$$

$$e^{(1-t)r}V(\xi,t)=\sqrt{\frac{t}{2\pi(1-t)}}\int_{-\infty}^{\infty}\exp\left[\frac{-t\left(\xi-y-\left(\frac{r}{\sigma}-\frac{\sigma}{2}\right)\ln t\right)^2}{2(1-t)}\right]f(y)dy.$$

The left hand side of the above equation is the value of the option price of the riskless interest rate r (constant) at the time $1-t$ $(0\leq t\leq 1)$ which equals the convolution form on the right hand side.

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