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Sensitivity Analysis in Fuzzy Environment

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Abstract

Data envelopment analysis (DEA) is a method to estimate the relative efficiency of decision-making units (DMUs) performing similar tasks in a production system that consumes multiple inputs to produce multiple outputs. Sensitivity analysis of specific DMU, which is under evaluation, is one of the topics of interests in data envelopment analysis. In traditional DEA, it is assumed that all inputs and outputs are exactly known . in real world, exact data may not be available. In this paper, we develop a sensitivity analysis approach for the additive model. Inputs and outputs are symmetric triangular fuzzy numbers. Variations in the data are considered for the margins of the fuzzy numbers of inputs and outputs of a specific efficient DMU and the data for the remaining DMUs are assumed fixed.

Keywords: Data envelopment Analysis; Sensitivity analysis; fuzzy data

1 Introduction

During the recent years, the issue of sensitivity and stability of data envelopment analysis results has been extensively studied. The first DEA sensitivity analysis paper by Charnes et al. [3] examined change in a single output. This is followed by a series of sensitivity analysis articles by Charnes and Neralic [5] in which sufficient conditions for simultaneous change of all outputs and all inputs of an efficient DMU which preserves efficiency were established. Seiford and Zhu [10] proposed a sensitivity analysis method that inputs or outputs could change individually, so they obtained the largest stability region.

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Traditional DEA models such as CCR, BCC [1,4] and additive models require exact data, but in more general cases, the data for evaluation are often collected from investigation to decide the natural languages such as good, medium and bad rather than a specific case. That is, the inputs and outputs are fuzzy. Based on fuzzy set theory, we can find several fuzzy approaches to the assessment of efficiency , ranking or sensitivity analysis in the DEA literature. Sengupta [11] analyzed the resulting fuzzy DEA model by using Zimmermann's method. Guo and Tanaka [7] and Leon et al. [9] employed the fuzzy ranking approach.

The main purpose of this paper is to study sensitivity analysis of the additive model in fuzzy environment. It is assumed that inputs and outputs are symmetric triangular fuzzy numbers. We restrict our attention to the case where the increase of inputs and the decrease of outputs for the efficient DMU_P, which is under evaluation are performed simultaneously. We wish to determine a region in which DMU_P remains efficient. In order to increase inputs, we decrease the margins of symmetric triangular fuzzy numbers of inputs. We decrease outputs by increasing the margins of symmetric triangular fuzzy numbers of outputs.

This paper is organized as follows: In section 2 we note symmetric triangular fuzzy numbers, and develop the additive model with symmetric triangular fuzzy numbers. Section 3 presents a sensitivity analysis method. The intent of section 4 is to provide a numerical example. The paper is concluded in section 5.

2 Preliminaries

We represent an arbitrary fuzzy number by an ordered pair of functions $\tilde{u} =: (\underline{u}(r), \overline{u}(r))$, $0 \leq r \leq 1$, which satisfy the following requirements:

- $\underline{u}$(r) is a bounded left continuous nondecreasing function over [0,1].
- $\overline{u}$(r) is a bounded left continuous nonincreasing function over [0,1].
- $\underline{u}(r)$ and $\overline{u}(r)$ are right continuous at 0
- $\underline{u}(r) \leq \overline{u}(r)$,$0 \leq r \leq 1$.

Definition 1. $M_{\tilde{u}} = Core(\tilde{u}) = \overline{u}(1) = \underline{u}(1)$; and $L_{\tilde{u}} = M_{\tilde{u}} - \underline{u}(0) \geq 0$ and $U_{\tilde{u}} = \overline{u}(0) - M_{\tilde{u}} \geq 0$ are the left and right margins of $\tilde{u} = (M_{\tilde{u}}, L_{\tilde{u}}, U_{\tilde{u}})$.

Definition 2. The fuzzy number $\tilde{t} =: (M_{\tilde{t}} - L_{\tilde{t}} + L_{\tilde{t}}r, M_{\tilde{t}} + U_{\tilde{t}} - U_{\tilde{t}}r) =: (M_{\tilde{t}}, L_{\tilde{t}}, U_{\tilde{t}})$, $0 \leq r \leq 1$ is an Asymmetric Triangular Fuzzy number. As a matter of fact $M_{\tilde{t}} - L_{\tilde{t}} + L_{\tilde{t}}r = \underline{t}(r)$ and $M_{\tilde{t}} + U_{\tilde{t}} - U_{\tilde{t}}r = \overline{t}(r)$ where $M_{\tilde{t}}, L_{\tilde{t}}, U_{\tilde{t}} \in \Re$.

A conventional fuzzy number is the symmetric triangular fuzzy number $S[c, \sigma]$ where $L_S = U_S = \sigma$ centered at c with basis 2σ.Its parametric form is $S[c, \sigma] =: (c - \sigma + r(\sigma), c + \sigma - r(\sigma)) := (c; \sigma) \quad , \; 0 \le r \le 1$ which $c, \sigma \in \Re$, c is the center and $\sigma \ge 0$ is the margin of $S[c, \sigma]$ and it is called symmetric triangular fuzzy number.

Definition 3. Let $\tilde{t} = (M_{\tilde{t}}, L_{\tilde{t}}, U_{\tilde{t}}), \tilde{u} = (M_{\tilde{u}}, L_{\tilde{u}}, U_{\tilde{u}})$ are non-symmetric triangular fuzzy numbers and $k \in \Re$, by using extension principal we can define:

1. $\tilde{t} = \tilde{u}$ if and only if $M_{\tilde{t}} = M_{\tilde{u}}$; and $L_{\tilde{t}} = L_{\tilde{u}}$ and $U_{\tilde{t}} = U_{\tilde{u}}$.

2. $\tilde{t} + \tilde{u} = (M_{\tilde{t}} + M_{\tilde{u}}, L_{\tilde{t}} + L_{\tilde{u}}, U_{\tilde{t}} + U_{\tilde{u}})$.

3.

$$k\tilde{t} = \begin{cases} (kM_{\tilde{t}}, kL_{\tilde{t}}, kU_{\tilde{t}}), \; k \ge 0 \\ (kM_{\tilde{t}}, -kL_{\tilde{t}}, -kU_{\tilde{t}}), \; k < 0 \end{cases} \tag{1}$$

Definition 4. For two fuzzy numbers in parametric forms $\tilde{t} = (\underline{t}(r), \overline{t}(r))$, $\tilde{u} = (\underline{u}(r), \overline{u}(r))$ we have $\tilde{t}\tilde{u} = \tilde{h} = (\underline{h}(r), \overline{h}(r))$ where
$\underline{h}(r) = Min\{\underline{t}(r)\underline{u}(r), \overline{t}(r)\overline{u}(r), \overline{t}(r)\underline{u}(r), \underline{t}(r)\overline{u}(r)\}$, and
$\overline{h}(r)) = Max\{\underline{t}(r)\underline{u}(r), \overline{t}(r)\overline{u}(r), \overline{t}(r)\underline{u}(r), \underline{t}(r)\overline{u}(r)\}$

Definition 5. (Ordering symmetric triangular fuzzy numbers) Let $\tilde{t} = (c_{\tilde{t}}; \sigma_{\tilde{t}})$ and $\tilde{u} = (c_{\tilde{u}}; \sigma_{\tilde{u}})$ are two symmetric triangular fuzzy numbers. We can define their ordering as follows:
We say $\tilde{t} <^* \tilde{u}$ if and only if $(c_{\tilde{t}} < c_{\tilde{u}}) \vee (c_{\tilde{t}} = c_{\tilde{u}} \wedge \sigma_{\tilde{t}} > \sigma_{\tilde{u}})$
In case of equality we have $\tilde{t} =^* \tilde{u}$ if and only if $((c_{\tilde{t}} = c_{\tilde{u}}) \wedge (\sigma_{\tilde{t}} = \sigma_{\tilde{u}}))$.
And $\tilde{t} \le^* \tilde{u}$ if and only if $(\tilde{t} <^* \tilde{u} \vee \tilde{t} =^* \tilde{u})$ it means that :

$$[(c_{\tilde{t}} < c_{\tilde{u}}) \vee (c_{\tilde{t}} = c_{\tilde{u}} \wedge \sigma_{\tilde{t}} > \sigma_{\tilde{u}})] \vee [(c_{\tilde{t}} = c_{\tilde{u}} \wedge \sigma_{\tilde{t}} = \sigma_{\tilde{u}})],$$

Now, suppose that there are n decision making units, with s outputs and m inputs. $\widetilde{Y}_j$, $\widetilde{X}_j$ are the observed vectors of outputs and inputs of DMU_j, respectively, j=1,...,n. It is assumed that, The observed values are positive symmetric triangular fuzzy numbers. Fuzzy additive model is as follows:

$$
\begin{aligned}
Max \quad & \sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+ \\
s.t. \quad & \sum_{j=1}^{n} \lambda_j \widetilde{y}_{rj} - s_r^+ =^* \widetilde{y}_{rp} \qquad r = 1, ..., s \\
& \sum_{j=1}^{n} \lambda_j \widetilde{x}_{ij} + s_i^- =^* \widetilde{x}_{ip} \qquad i = 1, ..., m \\
& \sum_{j=1}^{n} \lambda_j = 1 \\
& s_i^- \geq 0 \qquad i = 1, ..., m \\
& s_r^+ \geq 0 \qquad r = 1, ..., s \\
& \lambda_j \geq 0 \qquad j = 1, ..., n
\end{aligned}
\tag{2}
$$

We attribute $\varphi(\widetilde{a}) = c_{\widetilde{a}} - \delta\sigma_{\widetilde{a}}$ to the symmetric triangular fuzzy number $\widetilde{a} = S[c_{\widetilde{a}}, \sigma_{\widetilde{a}}]$, in which δ is a small, positive and real number by using φ the following model is obtained:

$$
\begin{aligned}
Max \quad & \sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+ \\
s.t. \quad & \sum_{j=1}^{n} \lambda_j \varphi(\widetilde{y}_{rj}) - s_r^+ = \varphi(\widetilde{y}_{rp}) \qquad r = 1, ..., s \\
& \sum_{j=1}^{n} \lambda_j \varphi(\widetilde{x}_{ij}) + s_i^- = \varphi(\widetilde{x}_{ip}) \qquad i = 1, ..., m \\
& \sum_{j=1}^{n} \lambda_j = 1 \\
& s_i^- \geq 0 \qquad i = 1, ..., m \\
& s_r^+ \geq 0 \qquad r = 1, ..., s \\
& \lambda_j \geq 0 \qquad j = 1, ..., n
\end{aligned}
\tag{3}
$$

Theorem 1. *DMU_p with symmetric triangular fuzzy inputs and outputs is Pareto-Koopmans efficient if and only if in linear programming problem (**??**)*$\sum_{i=1}^{m} s_i^{-*} + \sum_{r=1}^{s} s_r^{+*} = 0.$

Proof. Firt, suppose that in linear programming problem (**??**)$\sum_{i=1}^{m} s_i^{-*} + \sum_{r=1}^{s} s_r^{+*} \neq 0$.

Assume that $(\lambda^*, S^{-*}, S^{+*})$ is the optimal solution of (**??**). We know that $\sum_{i=1}^{m} s_i^{-*} + \sum_{r=1}^{s} s_r^{+*} > 0$, so

$$\exists \quad i : s_i^{-*} > 0 \qquad \text{or} \qquad \exists \quad r : s_r^{+*} > 0.$$

We can suppose that $s_1^{-*} > 0$. projection of a point on the boundary is as follows:

$$\begin{bmatrix} \varphi(\widetilde{x}_{ip}) - s_i^{-*} & i = 1, ..., m \\ \varphi(\widetilde{y}_{rp}) + s_r^{+*} & r = 1, ..., s \end{bmatrix}$$

so a production possibility is found whose first component of its input vector is less than the first component of DMU_p's input vector. Therefore, DMU_p is not Pareto-Koopmans efficient.

Now assume that DMU_p is not Pareto-Koopmans efficient, so

$$\exists \quad \overline{\lambda}_j \geq 0 \qquad j = 1, ..., n \qquad , \qquad \sum_{j=1}^{n} \overline{\lambda}_j = 1 \ ,$$

$$\begin{bmatrix} -\sum_{j=1}^{n} \overline{\lambda}_j \varphi(\widetilde{x}_{ip}) & i = 1, ..., m \\ \sum_{j=1}^{n} \overline{\lambda}_j \varphi(\widetilde{y}_{rp}) & r = 1, ..., s \end{bmatrix} \gneqq \begin{bmatrix} -\varphi(\widetilde{x}_{ip}) \\ \varphi(\widetilde{y}_{rp}) \end{bmatrix}$$

At least one of the inequalities is strict. For example i=1.

$$\sum_{j=1}^{n} \overline{\lambda}_j \varphi(\widetilde{x}_{1j}) + s_1^- = \varphi(\widetilde{x}_{1p}) \qquad s_1 > 0$$

$$\sum_{j=1}^{n} \overline{\lambda}_j \varphi(\widetilde{x}_{ij}) + s_i^- = \varphi(\widetilde{x}_{ip}) \qquad i = 2, ..., m$$

$$\sum_{j=1}^{n} \overline{\lambda}_j \varphi(\widetilde{y}_{rj}) - s_r^+ = \varphi(\widetilde{y}_{rp}) \qquad r = 1, ..., s$$

$$\sum_{j=1}^{n} \overline{\lambda}_j = 1$$

So $(\overline{\lambda}, S^-, S^+)$ is a feasible solution whose value of objective function is positive. Therefore, the optimal value of objective function is positive. □

We want to know that how much can we change inputs and outputs of an efficient DMU_P so that it preserves its efficiency. It is clear that by decreasing inputs and increasing outputs of DMU_p, it remains efficient, so downward variations of outputs and upward variations of inputs must be considered. If the core of a symmetric triangular fuzzy number is constant, decrease of margin of a symmetric triangular fuzzy number cause increase of the fuzzy number, so we exert the following changes on the margins of inputs and outputs of DMU_P:

$c(\widehat{\widetilde{x}}_{ip}) = c(\widetilde{x}_{ip})$, $\sigma(\widehat{\widetilde{x}}_{ip}) = \sigma(\widetilde{x}_{ip}) - \beta_i \geq 0$, $\beta_i \geq 0$ $i = 1, ..., m$
$c(\widehat{\widetilde{y}}_{rp}) = c(\widetilde{y}_{rp})$, $\sigma(\widehat{\widetilde{y}}_{rp}) = \sigma(\widetilde{y}_{rp}) + \alpha_r$, $\alpha_r \geq 0$, $\widehat{\widetilde{y}}_{rp} >^* 0$ $r = 1, ..., s$

3 Sensitivity analysis in fuzzy environment

Let a_j $j = 1, ..., m+s+n$ be the columns of the matrix and let a_P be the right hand side vector for linear programming problem(**??**). Let $(\lambda^*, S^{+*}, S^{-*})$ be the basic optimal solution of Pareto-Koopmans efficient DMU_p with the optimal basis matrix

$$B = \begin{bmatrix} \varphi(\widetilde{Y}_B) & -I_B^+ & 0 \\ \varphi(\widetilde{X}_B) & 0 & I_B^+ \\ e^T & 0 & 0 \end{bmatrix}$$

$B = [b_{ij}]_{i,j}$ $i, j = 1, ..., m+s+1$ $\Rightarrow$ $B^{-1} = [b_{ij}^{-1}]$ $i, j = 1, ..., m+s+1$
would be the inverse of matrix B. We will use the following notations:

$y_j = B^{-1}a_j$ $j = 0, 1, 2, ..., m+s+n$, $w^T = C_B^T B^{-1}$,
$z_j = C_B^T B^{-1} a_j = w^T a_j$ $j = 0, 1, 2, ..., m+s+n$

Simultaneous change of the margins of outputs and inputs means the following perturbation of the optimal basis matrix B:

$$\widehat{B} = B + D \tag{4}$$

$$D=\begin{bmatrix} 0 & \dots & 0 & -\delta\alpha_1 & 0 & \dots & 0 \\ . & & . & . & . & & . \\ . & & . & . & . & & . \\ . & & . & . & . & & . \\ 0 & \dots & 0 & -\delta\alpha_s & 0 & \dots & 0 \\ 0 & \dots & 0 & \delta\beta_1 & 0 & \dots & 0 \\ . & & . & . & . & & . \\ . & & . & . & . & & . \\ . & & . & . & . & & . \\ 0 & \dots & 0 & \delta\beta_m & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \end{bmatrix},$$

Where the distinct column corresponds to the optimal basic variable. Right hand side vector also changes as follows:

$$\widehat{a}_P = a_P + [-\delta\alpha_1 \quad \dots \quad -\delta\alpha_s \quad \delta\beta_1 \quad \dots \quad +\delta\beta_m \quad 0]^T \tag{5}$$

It can be easily showed that if $p = -\sum_{t=1}^{s} b_{kt}^{-1}(\delta\alpha_t) + \sum_{t=1}^{m} b_{k,s+t}^{-1}(\delta\beta_t)$, for matrices B^{-1} and D the following holds:

$$B^{-1}DB^{-1}D = pB^{-1}D \tag{6}$$

Because of (**??**) we can use the following theorem of Charnes and Cooper, which is proved in a more general form in [2].

Theorem 2. *Let B be a $k \times k$ matrix with inverse B^{-1}. Let D be a $k \times k$ matrix such that $B^{-1}DB^{-1}D = pB^{-1}D$ for some real scalar p. If σ is any scalar such that $p\sigma \neq -1$, then*
$(B+\sigma D)^{-1} = B^{-1}(I+\tau DB^{-1}) = (I+\tau B^{-1}D)B^{-1}$, where $\tau = -\sigma(1+p\sigma)^{-1}$.

If $\sigma = 1$, $p \neq -1$ and $\tau = \frac{-1}{(1+p)}$, then we have

$$(\widehat{B})^{-1} = (B+D)^{-1} = B^{-1}(I+\tau DB^{-1}) = (I+\tau B^{-1}D)B^{-1}, \tag{7}$$

Theorem 3. *Conditions*

$$-\tau w^T D y_j \geqslant z_j - c_j, \quad j \in N.B \tag{8}$$

are sufficient for Pareto-Koopmans efficient DMU_p with fuzzy input and output data to preserve efficiency after changes.

Proof. We know that optimality conditions of the perturbed basis are

$$\forall j \qquad \widehat{w}^T a_j - c_j \leqslant 0 \tag{9}$$

primal feasibility condition for the basic variables is

$$\widehat{y}_P = (\widehat{B})^{-1}\widehat{a}_P \geqslant 0. \tag{10}$$

We can write $\widehat{z}_j$ as

$$\widehat{z}_j = c_B^T B^{-1} a_j + \tau c_B^T B^{-1} D B^{-1} a_j = w^T a_j + \tau w^T D y_j = z_j + \tau w^T D y_j. \tag{11}$$

We know that $\widehat{z}_j - c_j = z_j - c_j = 0 \quad$ for j an index of basic variables, so we have.

$$z_j + \tau w^T D y_j - c_j \geqslant 0, \quad j \in N.B \tag{12}$$

In which $N.B$ is the set of non-basic variables. It is clear that conditions (**??**) and (**??**) are the same. Now We want to show primal feasibility of basic variables and efficiency in order to complete the proof.

$$\begin{aligned}&\widehat{y_P} = (I + \tau B^{-1} D) B^{-1}(a_P + [-\delta\alpha_1 \quad \ldots \quad -\delta\alpha_s \quad \delta\beta_1 \quad \ldots \quad \delta\beta_m \quad 0]^T) = \\ &(I + \tau B^{-1} D)(y_P + (B^{-1}D)_k) = Cg,\end{aligned} \tag{13}$$

with

$$C = (I + \tau B^{-1} D) \tag{14}$$

$$g = y_P + (B^{-1}D)_k \tag{15}$$

where $(B^{-1}D)_k$ is column k of $B^{-1}D$. Matrix C has the following structure:

$$C = \begin{bmatrix} 1 & 0 & \ldots & 0 & c_{1k} & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 & c_{2k} & 0 & \ldots & 0 \\ . & . & & . & . & . & & . \\ . & . & & . & . & . & & . \\ . & . & & . & . & . & & . \\ 0 & 0 & \ldots & 0 & 1 + c_{kk} & 0 & \ldots & 0 \\ . & . & & . & . & . & & . \\ . & . & & . & . & . & & . \\ . & . & & . & . & . & & . \\ 0 & 0 & \ldots & 0 & c_{s+m+1,k} & 0 & \ldots & 1 \end{bmatrix},$$

$$c_{jk}=\tau(B^{-1}D)_{jk}=-\tau\sum_{t=1}^{s}b_{jt}^{-1}(\delta\alpha_t)+\tau\sum_{t=1}^{m}b_{j,s+t}^{-1}(\delta\beta_t)\qquad j=1,2,...,m+s+1 \tag{16}$$

$$g_j=y_{j,P}+(B^{-1}D)_{jk}=y_{j,P}-\sum_{t=1}^{s}b_{jt}^{-1}(\delta\alpha_t)+\sum_{t=1}^{m}b_{j,s+t}^{-1}(\delta\beta_t)\qquad j=1,2,...,m+s+1 \tag{17}$$

DMU_P is efficient $\Rightarrow y_{k,p}=\lambda_p^*=1\Rightarrow g_k=y_{k,p}-\sum_{t=1}^{s}b_{kt}^{-1}(\delta\alpha_t)+\sum_{t=1}^{m}b_{k,s+t}^{-1}(\delta\beta_t)=1+p$, so

$$\begin{aligned}&\widehat{y}_{j,p}=g_j+c_{jk}g_k=y_{j,p}-\sum_{t=1}^{s}b_{jt}^{-1}(\delta\alpha_t)+\sum_{t=1}^{m}b_{j,s+t}^{-1}(\delta\beta_t)+\\&\tfrac{1}{1+p}(\sum_{t=1}^{s}b_{jt}^{-1}(\delta\alpha_t)-\sum_{t=1}^{m}b_{j,s+t}^{-1}(\delta\beta_t))(1+p)=y_{j,p}\geqslant 0\qquad j=1,2,...,m+s+1\end{aligned} \tag{18}$$

so the primal feasibility of basic variables is proved. Moreover, it is clear that the basic variables have not been changed and we have $\widehat{z}_o=z_o=0$. It is concluded that the efficiency of DMU_o has been preserved, which completes the proof □

If $1+p>0$ The system of inequalities can be written as

$$\sum_{t=1}^{s}(b_{kt}^{-1}(z_j-c_j)-w_ty_{kj})(\delta\alpha_t)-\sum_{t=1}^{m}(b_{k,s+t}^{-1}(z_j-c_j)-w_{s+t}y_{kj})(\delta\beta_t)\leqslant z_j-c_j,\qquad j\in NB \tag{19}$$

4 Numerical Example

The following example with five DMUs, one output and two outputs will be considered.

Table. 1

	DMU_1	DMU_2	DMU_3	DMU_4	DMU_5
$\widehat{y}_{1j}$	$S[2,\frac{1}{4}]$	$S[4,\frac{1}{2}]$	$S[2,\frac{1}{4}]$	$S[3,\frac{1}{4}]$	$S[2,\frac{1}{4}]$
$\widehat{x}_{1j}$	$S[4,\frac{1}{4}]$	$S[12,1]$	$S[8,1]$	$S[6,\frac{1}{2}]$	$S[2,\frac{1}{4}]$
$\widehat{x}_{2j}$	$S[6,\frac{1}{2}]$	$S[8,1]$	$S[2,\frac{1}{4}]$	$S[6,\frac{1}{2}]$	$S[8,1]$

In order to see if DMU_4 is efficient or not the following should be solved:

$$\begin{aligned} & Max s_1^- + s_2^- + s_1^+ \\ s.t. \quad & (2-\tfrac{1}{4}\delta)\lambda_1 + (4-\tfrac{1}{2}\delta)\lambda_2 + (2-\tfrac{1}{4}\delta)\lambda_3 + (3-\tfrac{1}{4}\delta)\lambda_4 + (2-\tfrac{1}{4}\delta)\lambda_5 - s_1^+ = 3-\tfrac{1}{4}\delta \\ & (4-\tfrac{1}{4}\delta)\lambda_1 + (12-\delta)\lambda_2 + (8-\delta)\lambda_3 + (6-\tfrac{1}{2}\delta)\lambda_4 + (2-\tfrac{1}{4}\delta)\lambda_5 + s_1^- = 6-\tfrac{1}{2}\delta \\ & (6-\tfrac{1}{2}\delta)\lambda_1 + (8-\delta)\lambda_2 + (2-\tfrac{1}{4}\delta)\lambda_3 + (6-\tfrac{1}{2}\delta)\lambda_4 + (8-\delta)\lambda_5 + s_2^- = 6-\tfrac{1}{2}\delta \\ & \lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5 = 1 \\ & s_1^-, s_2^-, s_1^+ \geq 0 \qquad i=1,...,m \\ & \lambda_j \geq 0 \qquad j=1,...,5 \end{aligned} \tag{20}$$

The optimal solution of the above problem is $\lambda_4^* = 1$, $\lambda_1^* = \lambda_2^* = \lambda_3^* = \lambda_5^* = 0$, $s_1^{+*} = s_1^{-*} = s_2^{+*} = 0$ and the optimal value of the objective function is zero which means that DMU_4 is Pareto-Koopmans efficient. Optimal basic variables are $\lambda_3^*, \lambda_4^*, \lambda_5^*, s_1^{-*}$. The optimal basis matrix is

$$B = \begin{bmatrix} 2.9997 & 0 & 1.9998 & 1.9998 \\ 5.9995 & 1 & 1.9998 & 7.999 \\ 5.9995 & 0 & 7.999 & 1.9998 \\ 1 & 0 & 1 & 1 \end{bmatrix},$$

with inverse

$$B^{-1} = \begin{bmatrix} 1 & 0 & 0 & -1.9998 \\ -2.0002 & 1 & 1 & -5.9988 \\ -0.6667 & 0 & 0.1667 & 0.9999 \\ -0.3333 & 0 & -0.1667 & 1.9998 \end{bmatrix},$$

We must exert the following changes on inputs and outputs of DMU_4:

$$\begin{aligned} &\widehat{\widetilde{y}}_{14} = S[3, \tfrac{1}{4}+\alpha_1] && \alpha_1 \geqslant 0 && \Rightarrow && \varphi(\widehat{\widetilde{y}}_{14}) = 3-\tfrac{1}{4}\delta - \delta\alpha_1 \\ &\widehat{\widetilde{x}}_{14} = S[6, \tfrac{1}{2}-\beta_1] && \beta_1 \geqslant 0 && \Rightarrow && \varphi(\widehat{\widetilde{x}}_{14}) = 6-\tfrac{1}{2}\delta + \delta\beta_1 \\ &\widehat{\widetilde{x}}_{24} = S[6, \tfrac{1}{2}-\beta_2] && \beta_2 \geqslant 0 && \Rightarrow && \varphi(\widehat{\widetilde{x}}_{24}) = 6-\tfrac{1}{2}\delta + \delta\beta_2' \end{aligned} \tag{21}$$

We have the following optimal basis perturbation matrix and change of the right hand side vector:

$$D = \begin{bmatrix} -\delta\alpha_1 & 0 & 0 & 0 \\ \delta\beta_1 & 0 & 0 & 0 \\ \delta\beta_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\widehat{a}_p = a_p + [-\delta\alpha_1 \quad \delta\beta_1 \quad \delta\beta_2 \quad 0]^T \tag{22}$$

$$c_B^T = [0 \qquad 1 \qquad 0 \qquad 0] \tag{23}$$

$$w^T = c_B^T B^{-1} = [-2.0002 \qquad 1 \qquad 1 \qquad -5.9988] \tag{24}$$

$p = -\delta\alpha_1 \quad , \quad \tau = \frac{-1}{1-\delta\alpha_1} \quad , \quad p \neq -1 \quad , \quad 1 + p > 0$

Sufficient conditions for preserving efficiency of DMU_4 are

$$(b_{11}^{-1}\overline{c}_j - w_1 y_{1j})(\delta\alpha_1) - (b_{12}^{-1}\overline{c}_j - w_2 y_{1j})(\delta\beta_1) - (b_{13}^{-1}\overline{c}_j - w_3 y_{1j})(\delta\beta_2) \leqslant \overline{c}_j \qquad j = 1, 2, 6, 8 \tag{25}$$

so the following system of inequalities obtains:

$0 \leqslant \delta\alpha_1 < 1 \quad , \quad \delta\beta_1 \geqslant 0 \quad , \quad \delta\beta_2 \geqslant 0$

$9.99(\delta\alpha_1) + 1.99(\delta\beta_1) + 1.99(\delta\beta_2) \leqslant 5.99$

$-\delta\alpha_1 - \delta\beta_1 + \delta\beta_2 \leqslant 1.0002$

$3 - \frac{1}{4}\delta - \delta\alpha_1 > 0 \quad , \quad \frac{1}{2} - \beta_1 > 0 \quad , \quad \frac{1}{2} - \beta_2 > 0$

The solution set of the system of inequalities shows the amount of possible changes of inputs and outputs of DMU_4.

5 Conclusion

In this paper we studied a sensitivity analysis method in fuzzy environment. Inputs and outputs were assumed symmetric triangular fuzzy numbers. We found sufficient conditions for simultaneous change of the margins of all outputs and all inputs of an efficient DMU which preserves efficiency. In other words, we found a region in which inputs and outputs of an efficient DMU, which is under consideration can change so that it remains efficient. The margins of inputs were decreased in order to increase inputs, and outputs were decreased by increasing the margins of outputs.

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