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Exponential Decay for Nonlinear Problem in non Cylindrical Domain

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1 Introduction

Let $\widehat{Q}$ be a bounded open set of $\mathbb{R}^n_x \times (0,T)$, $T > 0$. We define

$$\Omega_s = \widehat{Q} \cap \{t = s; 0 \leq s \leq T\}$$

and suppose that the sets Ω_s are open for all s.
We represent by Γ_s the smooth boundary of Ω_s.
The lateral boundary of $\widehat{Q}$ is givem by

$$\widehat{\Sigma} = \bigcup_{0<s<T} \Gamma_s \times \{s\}$$

The boundary of $\widehat{Q}$ is define by

$$\partial\widehat{Q} = \Omega_0 \cup \widehat{\Sigma} \cup \Omega_T$$

where, Ω_0 is bounded open set of $\mathbb{R}^n_x$ with $x = (x_1, x_2, \ldots, x_n)$.
Let Ω be a bounded open set of $\mathbb{R}^n_x$ and denote by $Q = \Omega \times (0,T)$ a cylinder such that $\widehat{Q} \subset Q$.
Let Γ be the boundary of Ω also smooth and let $\Sigma = \Gamma \times (0,T)$ the lateral boundary of the cylinder Q.
Let Ω be a bounded open set in $\mathbb{R}^n$ with boundary Γ smooth and let T is a positive real number.

In the set $\widehat{Q}$ we will consider the following problem:

$$\left| \begin{array}{l} u' + \mathcal{A}u = f \\ u(0) = u_0 \end{array} \right. \tag{1}$$

where, $\mathcal{A}$ is the pseudo Laplacian operator.

The problem (1) in cylinder domain was solve in J.L.Lions [2] by Compactness Method. Also in J.L.Lions [2] was given by other solution of this problem utilizing the Monotony Method, due to M.Visik [7].
An problem in manifolds with this operator was study by authors, to appear [5].
In this work we will analyze the problem (1) in the Non Cylindric Domain $\widehat{Q}$. We will use the Penazation Method, idealized by J.L.Lions and the Monotony Method.
The proof consist in transform the problem (1) in a problem in the cylinder Q, solve and then restrict the problem to the non cylinder domain $\widehat{Q}$.

2 Notations, Hypotheses

All derivates are in the distribution sense. By $\mathcal{D}(\Omega)$ we will denote the space of the testes functions in Ω.
We will represent by $W_0^{1,p}(\Omega)$ the closed of $\mathcal{D}(\Omega)$ in $W^{1,p}(\Omega)$. The dual space of $W_0^{1,p}(\Omega)$ is denote by $W^{-1,p'}(\Omega)$, where p' denote the conjugate exponent of p, that is, $\frac{1}{p} + \frac{1}{p'} = 1$.
Let $\mathcal{A}$ the pseudo Laplacian operator, that is,

$$\begin{array}{rccc} \mathcal{A}: & W_0^{1,p}(\Omega) & \rightarrow & W^{-1,p'}(\Omega) \\ & w & \mapsto & \mathcal{A}(w) \end{array}$$

tal que

$$\mathcal{A}(w) = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\left| \frac{\partial w}{\partial x_i} \right|^{p-2} \frac{\partial w}{\partial x_i} \right), \ 2 < p < \infty.$$

We reminder that the operator $\mathcal{A}$ has the followings proprieties:
• $\mathcal{A}$ is bounded, that is, carry bounded in bounded;
• $\mathcal{A}$ monotonic, hemicontinuous, $\langle \mathcal{A}(u), u \rangle = \|u\|^p_{W_0^{1,p}}$, coercive.

We go assume the following hypotheses:
(**H1**) The family open $\{\Omega_s\}_{0<s<T}$ is increasing in the following sense.
If $t_1 \le t_2$ then $proj_{_{\mathbb{R}^n}} \Omega_{t_1} \subseteq proj_{_{\mathbb{R}^n}} \Omega_{t_2}$
(**H2**) Regularity of the boundary of $\widehat{Q}$
If $v \in W_0^{1,p}(\Omega)$ and $v = 0$ $q.s$ in $\Omega - \Omega_t$ then $v \in W_0^{1,p}(\Omega_t)$.

Finaly, we consider the function

$$M(x,t) = \left| \begin{array}{l} 1,\ in\ Q - \widehat{Q} \cup \{\Omega_0 \times \{0\}\} \\ 0,\ in\ \widehat{Q} \cup \Omega_0 \times \{0\} \end{array} \right.$$

and $\beta(u) = \frac{1}{\epsilon} M(x,t)u$, $\forall\, \epsilon > 0$.
We note that $M \in L^\infty(Q)$.

Definition 2.1 *The function $u : \widehat{Q} \to \mathbb{R}$ is a weak solution of the problem (1) if*
$u \in L^p(0,T; W_0^{1,p}(\Omega_t))$ *and*

$$\begin{array}{c} \frac{d}{dt}(u(t), v) + \langle \mathcal{A}u(t), v\rangle = (f(t), v)\ in\ D'(\Omega_t), \\ for\ all\ v \in W_0^{1,p}(\Omega_t) \\ u(0) = u_0 \end{array} \tag{2}$$

3 Main Result

In this section we will solve the follow result

Theorem 1 *Given $f \in L^{p'}(0,T; W^{-1,p'}(\Omega_t))$ and $u_0 \in W_0^{1,p}(\Omega_t)$, then there exists a unique solution of the problem (1) in the sense of the definition* 2.1*.*

The idea of proof consist in transform the problem (1) in a equivalent problem in the cylinder utilizing the penalization method.

3.1 Penalized Problem

Given $\epsilon > 0$ to each function $u_\epsilon : Q \to \mathbb{R}$ solution of the problem:

$$\left| \begin{array}{l} u_\epsilon' + \mathcal{A}u_\epsilon + \frac{1}{\epsilon} M u_\epsilon = \widetilde{f}\ in\ Q \\ u_\epsilon = 0\ on\ \Sigma \\ u_\epsilon(x,0) = \widetilde{u}_0\ in\ \Omega \end{array} \right. \tag{3}$$

where

$$\widetilde{f}(x,t) = \left| \begin{array}{l} f(x,t)\ in\ \widehat{Q} \\ 0\ in\ Q - \widehat{Q} \end{array} \right.$$

and

$$\widetilde{u}(x,0) = \left| \begin{array}{l} u_0\ in\ \Omega_0 \\ 0\ in\ \Omega - \Omega_0 \end{array} \right.$$

where $\widetilde{u}_0 \in W_0^{1,p}(\Omega)$.
From separability of $V = W_0^{1,p}(\Omega)$ there exists an hilbetian's base $(w_\nu)_\nu \subset V$. Let $V_m = [w_1, \ldots, w_m]$ be the subspace of V generate by m first vectors of $(w_\nu)_\nu$.

3.2 Approximated Problem

Consider $u_{\epsilon m}(t) \in V_m$ such that:

$$\left| \begin{array}{l} u_{\epsilon m}(t) \in V_m \\ (u'_{\epsilon m}(t), v) + (\mathcal{A}u_{\epsilon m}(t), v) + \\ \dfrac{1}{\epsilon}(Mu_{\epsilon m}(t), v) = (\widetilde{f}(t), v), \; \forall v \in V_m \\ u_{\epsilon m}(0) = \widetilde{u}_{0\epsilon m} \to \widetilde{u}_0 \end{array} \right. \tag{4}$$

Hence, the system (4) has a local solution on the interval $[0, t_m)$, with $t_m < T$. This solution can be extended to the whole interval $[0, T]$ as consequence of the priori estimates that shall be proved in the next step.

3.3 Estimates I

Considering $v = u_{\epsilon m}(t)$ in $(4)_1$ and using the proprieties of the operator $\mathcal{A}$ we have the existence of a subsequence $(u_{\epsilon\nu}) \subset (u_{\epsilon m})$ such that:

$$u_{\epsilon\nu}(T) \rightharpoonup \zeta \;\; in \;\; L^2(\Omega) \tag{5}$$

$$u_{\epsilon\nu} \overset{\star}{\rightharpoonup} u_\epsilon \;\; in \;\; L^\infty(0, T, L^2(\Omega)) \tag{6}$$

$$u_{\epsilon\nu} \rightharpoonup u_\epsilon \;\; in \;\; L^p(0, T, W_0^{1,p}(\Omega)) \tag{7}$$

$$\mathcal{A}u_{\epsilon\nu} \rightharpoonup \chi \;\; in \;\; L^{p'}(0, T, W^{-1,p'}(\Omega)) \tag{8}$$

Writing the approximated equation with ν, multiplying by $\varphi \in \mathcal{D}(0, T)$, integrating from 0 to T and integrating by parts we obtain:

$$\begin{array}{l} -\displaystyle\int_0^T (u_{\epsilon\nu}(t), v)\varphi'(t)dt + \int_0^T (\mathcal{A}u_{\epsilon\nu}(t), v)\varphi(t)dt \\ +\displaystyle\int_0^T \frac{1}{\epsilon}(Mu_{\epsilon\nu}(t), v)\varphi(t)dt = \int_0^T (\widetilde{f}(t), v)\varphi(t)dt, \\ \forall v \in V_m. \end{array} \tag{9}$$

3.4 Convergence of the term: $\frac{1}{\epsilon}(M(t)u_{\epsilon\nu}(t), v)$

As $u_{\epsilon\nu}$ is bounded in $L^\infty(0, T; L^2(\Omega)) \hookrightarrow L^2(0, T; L^2(\Omega)) = L^2(Q)$, hence $u_{\epsilon\nu}$ is bounded in $L^2(Q)$. Therefore,

$$u_{\epsilon\nu} \rightharpoonup u_\epsilon \;\; in \;\; L^2(Q) \tag{10}$$

But, $M\phi \in L^2(Q)$, because $M \in L^\infty(Q)$. Therefore $(u_{\epsilon\nu}, M\phi) \to (u_\epsilon, M\phi)$, $\forall \phi \in L^2(Q)$.

Taking to the limit in (9) when $\nu \to \infty$, using the convergence obtained and using the density of V_m in V and we have:

$$\begin{array}{l} \frac{d}{dt}(u_\epsilon(t), v) + (\chi(t), v) + \frac{1}{\epsilon}(Mu_\epsilon(t), v) = (\widetilde{f}(t), v), \\ \forall\ v\ \in V,\ in\ the\ sense\ of\ \mathcal{D}'(0,T). \end{array} \tag{11}$$

To show that, $\chi(t) = \mathcal{A}(u_\epsilon(t))$, we used the your monotony and hemicontinuity. While that the verification of $u_\epsilon(0) = \widetilde{u}_0$ and $u_{\epsilon m}(T) \rightharpoonup u_\epsilon(T)$ is done form standard.

Thus, by Teman's Lemma [6] we have

$$u'_\epsilon + \mathcal{A}u_\epsilon + \frac{1}{\epsilon}Mu_\epsilon = \widetilde{f}\ in\ \mathcal{D}'(0,T). \tag{12}$$

Multiplying (12) by $v = u_\epsilon$, we have, as in the estimates I, when $\epsilon \to 0$

$$u_\epsilon \overset{\star}{\rightharpoonup} w\ \ in\ L^\infty(0,T,L^2(\Omega)) \tag{13}$$

$$u_\epsilon \rightharpoonup w\ \ in\ L^p(0,T,W_0^{1,p}(\Omega)), \tag{14}$$

$$Mu_\epsilon \rightharpoonup Mw\ \ in\ L^2(0,T,L^2(\Omega)). \tag{15}$$

From estimates, we obtain, when $\epsilon \to 0$, $Mu_\epsilon \to 0\ in\ L^2(0,T,L^2(\Omega))$, where $Mw = 0\ \ a.s.$ in Q. Therefore

$$w = 0 \quad \text{a.e.} \quad Q - \widehat{Q} \cup \{\Omega_0 \times \{0\}\}. \tag{16}$$

De (14) e (16) and of the hypotheses (**H2**), if u to design the restriction of w the $\widehat{Q}$, we have

$$u \in L^p(0,T;W_0^{1,p}(\Omega_t))$$

3.5 Restriction the $\widehat{Q}$

The restriction of the equation (12) to $\widehat{Q}$, is

$$\begin{array}{l} (\widehat{u}'_\epsilon(t), v) + (\mathcal{A}(\widehat{u}_\epsilon(t)), v) = (f(t), v), \\ \forall\ v \in W_0^{1,p}(\Omega_t), \end{array} \tag{17}$$

where $\widehat{u}_\epsilon$ represent the restriction of u_ϵ a $\widehat{Q}$

As $\widehat{u}_\epsilon \in C_s([0,T], W_0^{1,p}(\Omega_t))$ we have that the application $t \mapsto \langle \widehat{u}_\epsilon(t), y\rangle$ is continuous for $y \in W^{-1,p'}(\Omega_t)$, hence multiplying the equation (17) by $\theta \in \mathcal{D}(0,T)$, integrating from 0 to T an integrating by parts we obtain

$$\begin{array}{l} -\displaystyle\int_0^T (\widehat{u}_\epsilon(t), v)\theta'(t)dt + \int_0^T (\mathcal{A}(\widehat{u}_\epsilon(t)), v)\theta(t)dt \\ = \displaystyle\int_0^T (f(t), v)\theta(t)dt, \qquad \forall\ \ v \in W_0^{1,p}(\Omega_t). \end{array} \tag{18}$$

As $u, \widehat{u}_\epsilon$ are the restrictions of w, u_ϵ respectively, we have of (13) and (14), when $\epsilon \longrightarrow 0$

$$\widehat{u}_\epsilon \overset{\star}{\rightharpoonup} u \ \ in \ L^\infty(0,T,L^2(\Omega_t)) \tag{19}$$

$$\widehat{u}_\epsilon \rightharpoonup u \ \ in \ L^p(0,T,W_0^{1,p}(\Omega_t)) \tag{20}$$

$$\mathcal{A}\widehat{u}_\epsilon \rightharpoonup \xi \ \ in \ L^{p'}(0,T,W^{-1,p'}(\Omega_t)) \tag{21}$$

$$\widehat{u}_\epsilon(T) \rightharpoonup \beta \ \ in \ L^2(\Omega_t) \tag{22}$$

Analogously, as in the first part of the proof, show that $\beta = u(T)$ e $\xi = \mathcal{A}u$.
Taking to the limit in (18) when $\epsilon \longrightarrow 0$ and using the convergence obtained we have

$$\frac{d}{dt}(u(t),v) + (\mathcal{A}(u(t)),v) = (f(t),v),$$

$$\forall v \in W_0^{1,p}(\Omega_t) \ em \ \mathcal{D}'(0,T).$$

As $u \in C^0([0,T], W^{-1,p'}(\Omega))$ make sense calculate $u(0)$.
Being by first part of the proof $u_\epsilon(0) = \widetilde{u}_0$ we have that $\widehat{u}_\epsilon(0) = u_0$ where we conclude $u(0) = u_0$.
For to show the uniqueness is used the monotony of the pseudo Laplacian operator $\mathcal{A}$. What that conclude the proof of the Theorem 1.

3.6 Asymptotic Behavior

The solution from Theorem 1 can be extended the interval $[0,\infty)$, hence we make sense to think in decay.
From (17) with the $v = \widehat{u}_\epsilon$, the energy of the solution associated to the restrict system (3) to $\widehat{Q}$ is given by $E_\epsilon(t) = \frac{1}{2}|\widehat{u}_\epsilon|^2$.
Taking the duality $(3)_1$ restrict to $\widehat{Q}$ com $\widehat{u}_\epsilon$ we have

$$\frac{1}{2}\frac{d}{dt}|\widehat{u}_\epsilon|^2 + \|\widehat{u}_\epsilon\|^p = 0 \tag{23}$$

where we obtain $\frac{1}{2}\frac{d}{dt}|\widehat{u}_\epsilon|^2 \leq 0$, that is, $\frac{d}{dt}E_\epsilon(t) \leq 0$, $\forall t \geq 0$.
Therefore, E_ϵ is a nonnegative increasing function.
Integrating (23) de 0 a t we have $E_\epsilon(t) + \int_0^t \|\widehat{u}_\epsilon\|_V^p = E_\epsilon(0)$. Where, we obtain

$$E_\epsilon(t) - E_\epsilon(t+1) = \int_0^t \|\widehat{u}_\epsilon(s)\|_V^p ds.$$

Using the immersion of $W_0^1(\Omega_t)$ in $L^2(\Omega_t)$ we obtain

$$\int_t^{t+1} |\widehat{u}_\epsilon|^2 ds \leq c_1 \int_t^{t+1} \|\widehat{u}_\epsilon\|_V^p.$$

Thus

$$\int_t^{t+1} |\widehat{u}_\epsilon|^2 ds \leq C[E_\epsilon(t) - E_\epsilon(t+1)] = F^2(t) \tag{24}$$

We consider now the subintervals $(t, t+\frac{1}{4})$ and $(t+\frac{3}{4}, t+1)$ of $(t, t+1)$. Using the Medium Value Theorem for integrals, we have that exists $t_1 \in (t, t+\frac{1}{4})$ such that

$$\frac{1}{4}|\widehat{u}_\epsilon| = \int_t^{t+\frac{1}{4}} |\widehat{u}_\epsilon|^2 ds \leq \int_t^{t+1} |\widehat{u}_\epsilon|^2 ds \leq F^2(t) \tag{25}$$

Where, we obtain $|\widehat{u}_\epsilon(t_1)| \leq 2F^2(t)$.
Analogously we obtain $t_2 \in (t+\frac{3}{4}, t+1)$ such that $|\widehat{u}_\epsilon(t_2)| \leq 2F^2(t)$.
Integrating the energy in $[t_1, t_2]$ and using the Medium Value Theorem for integrals, we have that exists $t^* \in (t_1, t_2)$ such that

$$(t_2 - t_1)E_\epsilon(t^*) = \int_{t_1}^{t_2} E_\epsilon(s)ds \leq F^2(t)$$

As $t_2 - t_1 > \frac{1}{2}$ we have that: $E_\epsilon(t^*) \leq 2F^2(t)$.
Let $\tau_1, \tau_2 \in [t, t+1]$ with $\tau_1 < \tau_2$ and $\tau_1 = t^*$. We have

$$E_\epsilon(\tau_2) \leq E_\epsilon(t^*) + \int_t^{t+1} \|\widehat{u}_\epsilon\|^p ds, \ \forall \tau_2 \in [t, t+1]$$

Where, we obtain

$$\sup_{t \leq s \leq t+1} E_\epsilon(s) \leq E_\epsilon(t^*) + \int_t^{t+1} \|\widehat{u}_\epsilon(s)\|^p ds$$

Thus and noting that

$$\int_t^{t+1} \|\widehat{u}_\epsilon\|^p ds \leq \frac{1}{c} F^2(t),$$

we obtain

$$\sup_{t \leq s \leq t+1} E_\epsilon(s) \leq C[E_\epsilon(t) - E_\epsilon(t+1)]$$

Therefore, by Nakao's Lemma [3], we have

$$E_\epsilon(t) \leq Ce^{-\delta t}, \ \forall \epsilon > 0.$$

We have that $\widehat{u}_\epsilon(t) \rightharpoonup u(t) \ in \ L^2(\Omega_t)$, when $\epsilon \to 0$. Using this convergence and taking to the inferior limit in the inequality above, when $\epsilon \to 0$, we obtain:

$$E(t) \leq Ce^{-\delta t}.$$

What that characterize the asymptotic behavior

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