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On a Pseudo Generalized System of Carlemann in Three Dimensions

Mohamed Said Said

Departement of Mathematics and Informatics
Faculty of Sciences and Technology
University of Ouargla 30000 Ouargla Algeria
smedsaid@yahoo.com

Abstract

In this work, we propose to study a generalized problem called pseudo Carlemann system in three dimensions. We will use the decomposition of operators method , who cut the problem into two problems which we solved summeltanely. We will prove primarily a result of uniqueness of solution. After we study the existence of the solutionsby using a sequence of approximate solutions using a semi discretization of time. Finally we make the estimates and we pass to the limit.

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1. Problem Statement

We propose to solve a system, called a pseudo Carleman system that models a phenomenon in gas kinetics. We will solve it in a bounded open domain in IR^3. This work generalizes the work done in [3]. First we will establish result of uniqueness. after we establish the existence, by giving approximations of the solution, then we will make a priori estimates and we study the convergence.
Let $\Sigma = \Omega \times \left]0, T\right[$, where $\Omega = \left]a_1, b_1\right[\times \left]a_2, b_2\right[\times \left]a_3, b_3\right[$, and T is a finite positive real number. We propose to find the functions *u, v, w* solution of the following system:

$$\begin{cases} \dfrac{\partial u}{\partial t}+\dfrac{\partial u}{\partial x}+u^2+v^2=0 \\ \dfrac{\partial v}{\partial t}+\dfrac{\partial v}{\partial y}+v^2+w^2=0 \\ \dfrac{\partial w}{\partial t}+\dfrac{\partial w}{\partial z}+w^2+u^2=0 \end{cases} \tag{1.1}$$

with the following initial and boundary values :

$$\begin{cases} u(a_1,y,z,t)=0 \\ v(x,a_2,z,t)=0 \\ w(x,y,a_3,t)=0 \\ U(x,y,z,0)=U_0(x,y,z); \quad U=(u,v,w) \end{cases} \tag{1.2}$$

Nota.- Throughout this work the norms without index are those of $L^2(\Omega)$

Theorem 1.1.- If $u,v,w\in H^1(\Omega)\cap L^\infty(\Omega)$, and $u_0,v_0,w_0\geq 0$ $in\,\Omega$. Then the problem (1.1),(1.2) admits an unic solution $u,v,w\in L^\infty(0,T,H^1(\Omega)\cap L^\infty(\Omega))$ and $u,v,w\geq 0$ $in\,\Sigma$

Proof.- The proof of this theorem will be done in two steps.

2. Uniqueness

Let (u_1,v_1,w_1) and (u_2,v_2,w_2) are two solutions of the problem (1.1) -(1.2). We pose $u=u_1-u_2,\quad v=v_1-v_2,\quad w=w_1-w_2$; substituting in (1.1), we obtain:

$$\begin{cases} \dfrac{\partial u}{\partial t}+\dfrac{\partial u}{\partial x}+u_1^2-u_2^2=v_2^2-v_1^2 \\ \dfrac{\partial v}{\partial t}+\dfrac{\partial v}{\partial y}+v_1^2-v_2^2=w_2^2-w_1^2 \\ \dfrac{\partial w}{\partial t}+\dfrac{\partial w}{\partial z}+w_1^2-w_2^2=u_2^2-u_1^2 \end{cases} \tag{2.1}$$

We multiply the first equation of (2.1) by u and we calculate the integral on Ω of the found expression, and according to $\left\langle \dfrac{\partial u}{\partial x},u\right\rangle\geq 0$ and $\left(u_1^2-u_2^2\right)\left(u_1-u_2\right)=\left(u_1+u_2\right)\left(u_1-u_2\right)^2\geq 0$ because $u\geq 0$, we obtain :

$$\left\langle \frac{\partial u}{\partial t}, u \right\rangle \leq \int_\Omega (u_1 - u_2)(v_2^2 - v_1^2)\, dxdydz\,. \tag{2.2}$$

By hypothesis we have $u_1, u_2, v_1, v_2, w_1, w_2 \in L^\infty(\Sigma)$ then:

$$\left| \int_\Omega (v_2^2 - v_1^2)(u_1 - u_2)\, dxdydz \right| \leq \int_\Omega |(v_1 + v_2)(v_1 - v_2)(u_1 - u_2)|\, dxdydz$$
$$\leq 2M \int_\Omega |u(x,y,z,t)\; v(x,y,z,t)|\; dxdydz \leq 2M \|u\| \|v\|$$

where $M = \max(\|u_i\|_\infty, \|v_i\|_\infty; \|w_i\|_\infty)$. According to (2.2), we deduce that

$$\frac{d}{dt}\|u(t)\|^2 \leq 4M \|u(t)\|\, \|v(t)\| \leq 2M \left(\|u(t)\|^2 + \|v(t)\|^2 \right) \tag{2.3}$$

By applying the same method to the second and the third equations of the system(2.1), we get the inequalities similar to (2.3).By Adding these inequalities and applying the lemma of Gronwal [1]; [9], we deduce that:
$\|u(t)\|^2 + \|v(t)\|^2 + \|w(t)\|^2 = 0$, then $u_1 = u_2$; $v_1 = v_2$; $w_1 = w_2$

3 Approximate solutions

To establish the existence of the solution of the problem (1.1) -(1.2), we will use the decomposition method exposed in [3],which decomposes the problem (1.1)-(1.2) into two problems as follows:

Let $k = \Delta t$ is the time step and assume that we know $U = (u, v, w)$ at the time nk denoted U^n. We will determine the function U^{n+1} which is the approximation of U at the time *(n +1) k* , we make this in two-steps.
In the first step, we decompose the system (1.1) into two problems which the first is the following:

$$\begin{cases} \dfrac{\partial u}{\partial t} + u^2 + v^2 = 0 \\ \dfrac{\partial v}{\partial t} + v^2 + w^2 = 0 \\ \dfrac{\partial w}{\partial t} + w^2 + u^2 = 0 \end{cases} \tag{3.1}$$

and the second problem is:

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial x} = 0 \\ \dfrac{\partial v}{\partial t} + \dfrac{\partial v}{\partial y} = 0 \\ \dfrac{\partial w}{\partial t} + \dfrac{\partial w}{\partial z} = 0 \end{cases} \tag{3.2}$$

After a half discretization of (3.1) and (3.2) compared to *t*, and assuming that $\{u^n, v^n, w^n\}$ is known, we will determine : $\{u^{n+1/2}, v^{n+1/2}, w^{n+1/2}\}$ the approximate solution of (3.1). From this solution, which has become known now, the system (3.2) gives us $u^{n+1}; v^{n+1}$ and w^{n+1}. By incrementing the time, we get a sequence of approximate solutions which will study the convergence.The half discretization of (3.1) is:

$$\begin{cases} u^{n+1/2} - u^n + k\left[\left(u^{n+1/2}\right)^2 + \left(v^{n+1/2}\right)^2\right] = 0 \\ v^{n+1/2} - v^n + k\left[\left(v^{n+1/2}\right)^2 + \left(w^{n+1/2}\right)^2\right] = 0 \\ w^{n+1/2} - w^n + k\left[\left(w^{n+1/2}\right)^2 + \left(u^{n+1/2}\right)^2\right] = 0 \end{cases} \tag{3.3}$$

Also, the half discretization of (3.2) is:

$$\begin{cases} u^{n+1} - u^{n+1/2} + k\dfrac{\partial u^{n+1}}{\partial x} = 0 \\ v^{n+1} - v^{n+1/2} + k\dfrac{\partial v^{n+1}}{\partial y} = 0 \\ w^{n+1} - w^{n+1/2} + k\dfrac{\partial w^{n+1}}{\partial z} = 0 \end{cases} \tag{3.4}$$

The explicit solution of (3.4) is :

$$\begin{cases} u^{n+1}(x,y,z) = \dfrac{1}{k}\displaystyle\int_{a_1}^{x} u^{n+1/2}(\xi,y,z)\exp\left(\dfrac{\xi - y - z}{k}\right)d\xi \\ v^{n+1}(x,y,z) = \dfrac{1}{k}\displaystyle\int_{a_2}^{x} v^{n+1/2}(x,\xi,z)\exp\left(\dfrac{\xi - x - z}{k}\right)d\xi \\ w^{n+1}(x,y,z) = \dfrac{1}{k}\displaystyle\int_{a_1}^{x} w^{n+1/2}(\xi,y,z)\exp\left(\dfrac{\xi - y - x}{k}\right)d\xi \end{cases} \tag{3.5}$$

We pose $u^0 = u_0, v^0 = v_0$, $w^0 = w_0$. This completely definies the sequence $\{u^n, v^n, w^n\}$.

4 Priori estimates

Let $k = \dfrac{T}{N}$, where $N \in IN$. We introduce the following functions: $u_{ik}(t) = u^{n+i/2}, v_{ik}(t) = v^{n+i/2}, w_{ik}(t) = w^{n+i/2}, i = 1,2;\ t \in [nk,(n+1)k[$
$n = 0,1,2,...,N-1$. And $u_k(t), v_k(t), w_k(t)$ are linear in the interval $[nk,(n+1)k[$ such that $u_k(nk) = u^n, v_k(nk) = v^n, w_k(nk) = w^n$. We need following lemma:

Lemma 4.1.- The functions $u_k, v_k, w_k, u_{ik}, v_{ik}, w_{ik}$ remain, when k tend to zero, in a bounded domain in $L^\infty(0,T;H^1(\Omega) \cap L^\infty(\Omega))$.

To prove this lemma, we will use the following result:

Lemma 4.2.- If we pose $C_0 = \|u_0\|_{L^\infty(\Omega)} + \|v_0\|_{L^\infty(\Omega)} + \|w_0\|_{L^\infty(\Omega)}$. Then $\|u^{n+i/2}\|_{L^\infty(\Omega)} + \|v^{n+i/2}\|_{L^\infty(\Omega)} + \|w^{n+i/2}\|_{L^\infty(\Omega)} \le C_0$; $i = 1,2$, moreover these functions are positive.

Proof.- (Of the lemma 4.2). The proof of this lemma is easy, it uses the formula (3.5) . For $i = 1,2$, we pose :

$$A^{n+i/2} = \|u^{n+i/2}\|_{L^\infty(\Omega)}; B^{n+i/2} = \|v^{n+i/2}\|_{L^\infty(\Omega)}; C^{n+i/2} = \|w^{n+i/2}\|_{L^\infty(\Omega)}$$

According to (3.5), we obtain that : $A^{n+1} \le A^{n+1/2}; B^{n+1} \le B^{n+1/2}; C^{n+1} \le C^{n+1/2}$

To complete the proof of the lemma 4.2 , we must we show that: $A^{n+1/2} + B^{n+1/2} + C^{n+1/2} \le A^n + B^n + C^n$. For this it suffices to use the expression (3.3): The first equation of the system (3.3) is written as:

$$u^{n+1/2} - u^n + k\left[\left(u^{n+1/2}\right)^2 + \left(v^{n+1/2}\right)^2\right]$$

Since $k\left[\left(w^{n+1/2}\right)^2 + \left(u^{n+1/2}\right)\right]^2 \ge 0$, then $u^{n+1/2} - u^n \le 0$

We use the same method for the second and the third equations of the system (3.3) And by adding these inequalities, we obtain the desired result. According to the lemma 4.2, we deduce that the functions $u_k, v_k, w_k, u_{ik}, v_{ik}, w_{ik}; i = 1,2$, remain when k tend to zero, in a bounded domain in $L^\infty(0,T;L^\infty(\Omega))$.

To show that these functions remain bounded in $L^\infty(0,T;H^1(\Omega))$, we will use the following lemma:

Lemma 4.3.- It exists a constante $C_1 \ge 0$ such that :

$$\left\{\left\|\frac{\partial u^{n+i/2}}{\partial x_j}\right\|;\left\|\frac{\partial v^{n+i/2}}{\partial x_j}\right\|;\left\|\frac{\partial w^{n+i/2}}{\partial x_j}\right\|\right\}\leq C_1;\ \ i=1,2;\ \ j=1,2,3 \qquad (4.1)$$

Where $x_1 = x;\ x_2 = y;\ x_3 = z$

Proof.- It suffices to etablish the following inequalities :

$$\left\|\frac{\partial u^{n+1}}{\partial x_j}\right\|\leq\left\|\frac{\partial u^{n+1/2}}{\partial x_j}\right\|;\ \left\|\frac{\partial v^{n+1}}{\partial x_j}\right\|\leq\left\|\frac{\partial v^{n+1/2}}{\partial x_j}\right\|\quad\left\|\frac{\partial w^{n+1}}{\partial x_j}\right\|\leq\left\|\frac{\partial w^{n+1/2}}{\partial x_j}\right\|;\ \ j=1,2,3 \qquad (4.2)$$

We calculate the derivative of the first equation of the system (3.4) compared to the variable x_j, $j=1,2,3$, we get the following expression:

$$\begin{cases}\dfrac{\partial u^{n+1}}{\partial x_j}-\dfrac{\partial u^{n+1/2}}{\partial x_j}+k\dfrac{\partial}{\partial x_1}\left(\dfrac{\partial u^{n+1}}{\partial x_j}\right)=0\\ \left.\dfrac{\partial u^{n+1}}{\partial x_j}\right|_{x_j=a_j}=0\quad j=1,2,3\end{cases}$$

After we calculate the scalar product in $L^2(\Omega)$, of the found formula with $\frac{\partial u^{n+1}}{\partial x_j}$,

we obtain the following expression: $\left\|\frac{\partial u^{n+1}}{\partial x_j}\right\|^2\leq\left\|\frac{\partial u^{n+1/2}}{\partial x_j}\right\|\left\|\frac{\partial u^{n+1}}{\partial x_j}\right\|$.

After, we follow the same method for the other equations of (3.4) we obtain the similar inequalities relating to the functions *v* and *w*.

To complete the proof of the lemma 4.3, we need the following lemma:

Lemma 4.4.- There is a constante $C_2>0$ such that for *i=1,2,3*, we have:

$$\left\|D_i u^{n+1/2}\right\|^2+\left\|D_i v^{n+1/2}\right\|^2+\left\|D_i w^{n+1/2}\right\|^2\leq C_2\left(\left\|D_i u^n\right\|^2+\left\|D_i v^n\right\|^2+\left\|D_i w^n\right\|^2\right) \qquad (4.3)$$

where $D_i=\frac{\partial}{\partial x_i}$

Proof .- We calculate the derivative of the first equation of (3.3); we obtain:

$$D_i u^{n+1/2}-D_i u^n+2ku^{n+1/2}D_i u^{n+1/2}+2kv^{n+1/2}D_i v^{n+1/2}=0\ \text{ then}$$

$$\left\|D_i u^{n+1/2}\right\|\leq\left\|D_i u^n\right\|+2kC_0\left(\left\|D_i u^{n+1/2}\right\|+\left\|D_i v^{n+1/2}\right\|\right) \qquad (4.4)$$

by using the same method for the second and the third equations of (3.3), we obtain the similar inequalities relating to the functions v and w
To simplifiy the calculations, we pose

$$X = \left\|D_i u^{n+1/2}\right\|; Y = \left\|D_i v^{n+1/2}\right\|; Z = \left\|D_i w^{n+1/2}\right\|; \ A = \left\|D_i u^n\right\|; B = \left\|D_i v^n\right\|; C = \left\|D_i w^n\right\|$$

And $\gamma = 2kC_0$. The equations (4.4); (4.5) becomes:

$$X \leq A + \gamma(X + Y), \ Y = B + \gamma(Y + Z), \ Z \leq A + \gamma(Z + X) \quad .$$

After calculations we will obtain: $X^2 + Y^2 + Z^2 \leq \frac{1}{(1-3\gamma)^2}\left(A^2 + B^2 + C^2\right)$

this completes the proof of the lemma 4.4

Now we pursue the proof of the lemma 4.3. For this we pose:

$$\beta_{n+i/2} = \sum_{j=1}^{3}\left(\left\|D_j u^{n+i/2}\right\|^2 + \left\|D_j v^{n+i/2}\right\|^2 + \left\|D_j w^{n+i/2}\right\|^2\right) \ i = 1,2.$$

Using the formula (4.2), we deduce that: $\beta_{n+1} \leq \beta_{n+1/2}$.

According to the lemma 4.4, we deduce that $\exists\, C_3 > 0, \ \beta_n, \beta_{n+1/2}, \beta_{n+1} \leq C_3$. This completes the proof of the lemma 4.3. Thus the proof of lemma 4.2 is completed.

5 Passage to the limit

According to lemma 4.1, we can extract the sequences
$u_k, v_k, w_k, u_{ik}, v_{ik}, w_{ik} \quad i = 1,2$, such that:
$u_k \to u, v_k \to v, w_k \to w, u_{ik} \to u_i, v_{ik} \to v_i, w_{ik} \to w_i$ weak star in the space $L^\infty(0,T;H^1(\Omega) \cap L^\infty(\Omega))$. But this convergence is not sufficient to take the limit in the nonlinear term, then we will need another result, it is given by the following lemma :

Lemma 5.1.- When k tend to zero, the functions $\frac{\partial u_k}{\partial t}, \frac{\partial v_k}{\partial t}, \frac{\partial w_k}{\partial t}$ remain in a bounded domain in $L^\infty(0,T;L^2(\Omega))$

Nota.- In this paragraph all the norms are taken in $L^\infty(0,T;L^2(\Omega))$

Proof.- Adding the first equations contained in (3.3) and (3.4), we obtain:

$$u^{n+1} - u^n + k\frac{\partial u^{n+1}}{\partial x} + k\left[\left(u^{n+1/2}\right)^2 + \left(v^{n+1/2}\right)^2\right] = 0 \qquad (5.1)$$

this is equivalent to : $$\frac{\partial u_k}{\partial t} + \frac{\partial u_{2k}}{\partial x} + (u_{1k})^2 + (v_{1k})^2 = 0 \qquad (5.2)$$

using the same method for the other equations, we obtain for the expressions (3.3) and (3.4), we obyain the equations similar of the equation (5.2) and we apply the lemma 4.1. This completes the proof of the lemma 5.1.
According to the injection of $H^1(\Sigma)$ in $L^2(\Sigma)$ is compacte [2] , and according to the estimates on u_k, v_k, w_k, we can suppose that $u_k \to u, v_k \to v, w_k \to w$ strongly in $L^2(\Sigma)$, and we have: $\|u_k - u_{2k}\| \le \sup_{0\le n\le N-1} \|u^{n+1} - u^n\|$

Using the lemma 4.1 and the formula (5.1), we deduce that:

$$\|u_k - u_{2k}\| \le kC_5 \;;\; \|v_k - v_{2k}\| \le kC_5 \;;\; \|w_k - w_{2k}\| \le kC_5$$

According to the strong convergence of the sequences in $L^2(\Sigma)$, we deduce that: $u_{2k} \to u, v_{2k} \to v, w_{2k} \to w$ strongly in $L^2(\Sigma)$
Then we can to assume that: $u_{1k} \to u, v_{1k} \to v, w_{1k} \to w$ strongly in $L^2(\Sigma)$
We have now : $(u_{1k})^2 \to (u)^2, (v_{1k})^2 \to (v)^2, (w_{1k})^2 \to (w)^2$ weakly in $L^2(\Sigma)$
Then we can now to take the limit in the equation (5.3). Thus we see that the functions *u,v,w* satisfies the system (1.1). According to the lemma 5.1, we obtain:
$\frac{\partial u_k}{\partial t} \to \frac{\partial u}{\partial t}, \ \frac{\partial v_k}{\partial t} \to \frac{\partial v}{\partial t}, \frac{\partial w_k}{\partial t} \to \frac{\partial w}{\partial t}$ weakly in $L^\infty(0,T;L^2(\Omega))$
Consequently $u_k(0) \to u(0), v_k(0) \to v(0), w_k(0) \to w(0)$ weakly in $L^2(\Omega)$ and $u(0) = u_0; v(0) = v_0; w(0) = w_0$

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