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The Mixed Boundary Value Problem for Semilinear Elliptic Equation in Domains with Edges

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Abstract. In this paper, we prove the existence and uniqueness of generalized solution of the mixed boundary value problem for semilinear elliptic equation in domains with edges. In addition, some results on smoothness of generalized solutions of the problem in domain with edges are also proved.

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1. Introduction

The existence, uniqueness and smoothness of solutions of the mixed boundary value problem in domains with conical points have been studied by many authors [1, 2, 3, 4, 7, 8]. Mixed boundary problem for linear elliptic equation was studed in [9]. In this paper, we consider the mixed boundary value problem for semilinear elliptic equations in domains with edges. We will prove the existence and uniqueness of generalized solutions of the problem in the space $W^1_{2,0}(\Omega, \Gamma_1, \Gamma_2) \cap L^p(\Omega)$. Furthermore, we will prove the smoothness of generalized solutions of the problem in bounded domain Ω of 2 - dimensions with edges.

The paper is organized as follows, in Sec.2, we introduce the notations and functional spaces being used. The main results are presented in Sec.3 and Sec.4.

2. Function spaces and preliminary results

Suppose that Ω is the bounded domain in $\mathbb{R}^n, n \geq 2$. Its boundary $\partial\Omega$ consists of k - smooth manifolds $\Gamma_1, \Gamma_2, ..., \Gamma_k$ of $(n-1)$ - dimension. Furthermore, each Γ_i intersects with Γ_{i-1} or Γ_{i+1} by manifolds l_{i-1} or l_i, respectively. Without loss of generality we may assume that $\partial\Omega$ consists of two manifolds, Γ_1 and Γ_2, intersected by manifold l_0. For the any point $P \in l_0$ there are defined two half-spaces $\Gamma_1(P), \Gamma_2(P)$ of $(n-1)$ - dimension tangential to $\partial\Omega$ and the 2- dimensional plane $\pi(P)$ normal to l_0 in P. We denote by $\nu(P)$ the angle in $\pi(P)$ bounded by the rays $\Gamma_1(P) \cap \pi(P), \Gamma_2(P) \cap \pi(P)$, and by $\beta(P)$ the value of this angle.

In this paper we use following functional spaces:

- $W^k(\Omega)$ - the space consisting of all functions $u(x)$ which have generalized derivatives $\dfrac{\partial^s u}{\partial x^s}, 0 \leq s \leq k$, satisfying

$$\|u\|_{W^k(\Omega)} = \left(\int\limits_{\Omega} \sum_{s=0}^{k} \left| \frac{\partial^s u}{\partial x^s} \right|^2 \mathrm{d}x \right)^{\frac{1}{2}} < +\infty.$$

- $W^k_\alpha(\Omega)$ - the space consisting of all functions $u(x)$ which have generalized derivatives $\dfrac{\partial^s u}{\partial x^s}, 0 \leq s \leq k$, satisfying

$$\|u\|_{W^k_\alpha(\Omega)} = \left(\sum_{s=0}^{k} \int\limits_{\Omega} r^\alpha \left| \frac{\partial^s u}{\partial x^s} \right|^2 \mathrm{d}x \right)^{\frac{1}{2}} < +\infty,$$

  where $r = r(x)$ is an infinitely differentiable positive function.

- $\mathring{W}^k_\alpha(\Omega)$ - the space consisting of all functions $u(x)$ which have generalized derivatives $\dfrac{\partial^s u}{\partial x^s}, 0 \leq s \leq k$, with the norm

$$\|u\|_{\mathring{W}^k_\alpha(\Omega)} = \left(\sum_{s=0}^{k} \int\limits_{\Omega} r^{\alpha+2(s-k)} \left| \frac{\partial^s u}{\partial x^s} \right|^2 \mathrm{d}x \right)^{\frac{1}{2}} < +\infty.$$

- $W^1_{2,0}(\Omega)$ is the closure in $W^1(\Omega)$ of set consisting of all infinitely differentiable functions in Ω which vanish near $\partial\Omega$.
- $W^1_{2,0}(\Omega, \Gamma_1, \Gamma_2)$ is the closure in $W^1_{2,0}(\Omega)$ of set consisting of all infinitely differentiable functions in Ω which vanish near Γ_1.
- $E = W^1_{2,0}(\Omega, \Gamma_1, \Gamma_2) \cap L^p(\Omega)$ is the space consisting of all functions $u(x)$ satisfying

$$\|u\|_E = \|u\|_{W^1_{2,0}(\Omega,\Gamma_1,\Gamma_2)} + \|u\|_{L^p(\Omega)}.$$

Let us consider the partial differential operator

$$Lu = \sum_{i,j=1}^{n} (a_{ij} u_{x_i})_{x_j} + \sum_{i=1}^{n} a_i u_{x_i} + au,$$

where a_{ij}, a_i, a are infinitely differentiable functions in Ω and satisfy

$$(1) \qquad a_{ij} = a_{ji}, \quad i, j = 1, 2, ..., n,$$

$$(2) \qquad \nu|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \leq \nu_{-1}|\xi|^2, \quad \forall \xi \in \mathbb{R}^n, |\xi| \neq 0, \sum_{i=1}^{n} |a_i|^2 \leq M_1,$$

$$(3) \qquad -M_3 \leq a \leq -M_2, \quad M_1, M_2, M_3 > 0.$$

We consider the operator

$$\Phi : \ E \longrightarrow E^*$$

where

$$\Phi(u) = -Lu + |u|^\rho u + f,$$

and

$$(4) \qquad (L(\omega), \omega) = -\int\limits_\Omega \left[\sum_{i,j=1}^{n} a_{ij}\omega_{x_i}\omega_{x_j} - \sum_{i=1}^{n} a_i \omega_{x_i}\omega - a\omega^2 \right] \mathrm{d}x,$$

where $\omega = u - v, u, v \in E$.

Lemma 2.1. *For every u, v in E, we have following inequality*

$$(\Phi(u) - \Phi(v), u - v) \leq 0.$$

Proof. We have

$$(5) \qquad (\Phi(u) - \Phi(v), u - v) = -(L(u - v), u - v) + (|u|^\rho u - |v|^\rho v, u - v).$$

By using Cauchy's inequality and conditions (2),(3), we obtain

$$\int_{\Omega} \sum_{i=1}^{n} a_i \omega_{x_i} \omega \mathrm{d}x \le \frac{\varepsilon}{2} \|\omega_x\|^2_{L^2(\Omega)} + \frac{M_1}{2\varepsilon} \|\omega\|^2_{L^2(\Omega)}, \tag{6}$$

$$\int_{\Omega} a\omega^2 \mathrm{d}x \le -M_2 \|\omega\|^2_{L^2(\Omega)}. \tag{7}$$

From (6)-(7), we have

$$-(L(\omega),\omega) \ge \nu \|\omega_x\|^2_{L^2(\Omega)} - \frac{\varepsilon}{2}\|\omega_x\|^2_{L^2(\Omega)} - \frac{M_1}{2\varepsilon}\|\omega\|^2_{L^2(\Omega)} + M_2\|\omega\|^2_{L^2(\Omega)}. \tag{8}$$

From (8) choose $\varepsilon = \nu$, we get

$$-(L(\omega),\omega) \ge \frac{\nu}{2}\|\omega_x\|^2_{L^2(\Omega)} + \left(M_2 - \frac{M_1}{2\nu}\right)\|\omega\|^2_{L^2(\Omega)} \ge C_1 \|\omega\|_{W^1_{2,0}(\Omega,\Gamma_1,\Gamma_2)} \ge 0, \tag{9}$$

where $C_1 = \min\left(\frac{\nu}{2}, M_2 - \frac{M_1}{2\nu}\right) \ge 0, M_2 - \frac{M_1}{2\nu} \ge 0$ when $M_2 > 0$ and is large enough.

We consider

$$\begin{aligned}(|u|^\rho u - |v|^\rho v)(u-v) &= |u|^{\rho+2} + |v|^{\rho+2} - (|u|^\rho uv + |v|^\rho uv)\\ &\ge |u|^{\rho+2} + |v|^{\rho+2} - (|u|^{\rho+1}|v| + |v|^{\rho+1}|u|)\\ &= (|u|^{\rho+1} - |v|^{\rho+1})(|u| - |v|)\\ &\ge 0,\end{aligned}$$

for $\forall u, v \in E$. Hence

$$((|u|^\rho u - |v|^\rho v),(u-v)) = \int_{\Omega} (|u|^\rho u - |v|^\rho v)(u-v)\mathrm{d}x \ge 0. \tag{10}$$

From (5)-(9)-(10), we have

$$(\Phi(u) - \Phi(v), u - v) \ge 0.$$

The proof is complete. □

We recall two basic lemmas.

Lemma 2.2. [6] *Leting $u_\mu \longrightarrow u$ a.e. in $L^2(\Omega)$, u_μ is uniformly bounded in $L^p(\Omega)$ for $p = \rho + 2, \rho > 0$. Then $|u_\mu|^\rho u_\mu \to |u|^\rho u$ weakly in $L^q(\Omega)$, where $\frac{1}{p} + \frac{1}{q} = 1$.*

Puting $G(\omega) = \int\limits_{\Omega} (|u|^\rho u - |v|^\rho v)\omega \mathrm{d}x$, where $\omega = u - v$, we have following lemma:

Lemma 2.3. [5] *Leting $u, v \in E, \rho \leq \frac{2}{n-2}$ (in case $n = 2$, ρ is arbitrarily finite), then $G(\omega)$ satisfying*

$$|G(\omega)| \leq C_2 \|\omega\|^2_{L^2(\Omega)}, \quad C_2 = const > 0.$$

Lemmas 2.1, 2.2, 2.3 are basic tolls for proving the existence and uniqueness of solutions of the problem under consideration.

3. Existence and uniqueness of solution

We consider the following problem

$$Lu - |u|^\rho u = f \quad \text{in } \Omega, \tag{11}$$

$$u\Big|_{\Gamma_1} = 0, \tag{12}$$

$$\frac{\partial u}{\partial n}\Big|_{\Gamma_2} = 0, \tag{13}$$

where $\frac{\partial u}{\partial n} = \sum\limits_{i,j=1}^{n} a_{ij} \frac{\partial u}{\partial x_i} \cos(\overrightarrow{n}, x_j)$, $\overrightarrow{n}$ is the outer normal $\partial\Omega$.

Definition 3.1. *A function $u(x)$ is called a generalized solution of the problem (11)-(13) in the space E if it satisfies*

$$\int\limits_{\Omega} \left[\sum_{i,j=1}^{n} a_{ij} u_{x_i} \eta_{x_j} - \sum_{i=1}^{n} a_i u_{x_i} \eta - au\eta \right] \mathrm{d}x + \int\limits_{\Omega} |u|^\rho u \eta \mathrm{d}x = - \int\limits_{\Omega} f\eta \mathrm{d}x, \tag{14}$$

for all test function $\eta \in E$, $p = \rho + 2$.

First, we prove the existence of generalized solution of the problem (11)-(13).

Theorem 3.1. *If $f \in L^2(\Omega)$ then the problem (11)-(13) has a generalized solution $u(x)$ in the space E, where $p = \rho + 2, \rho \leq \frac{2}{n-2}$ (if $n = 2$ then ρ is arbitrary finite).*

Proof. Consider approximate solution $u^N(x)$ following the form

$$u^N(x) = \sum_{k=1}^{N} C_k^N \varphi_k(x),$$

where $\{\varphi_k(x)\}_{k=1}^{\infty}$ is basic of E which is orthonormal in $L^2(\Omega)$. We get u^N from solving the problem

$$\left(\sum_{i,j=1}^{n} a_{ij}u^N_{x_i}, \varphi_{kx_j}\right) - \left(\sum_{i=1}^{n} a_i u^N_{x_i}, \varphi_k\right) - (au^N, \varphi_k) + (|u^N|^\rho u^N, \varphi_k) = -(f, \varphi_k), \tag{15}$$

for $k = 1, 2, ..., N$.

$$u^N\Big|_{\Gamma_1} = 0, \tag{16}$$

$$\frac{\partial u^N}{\partial n}\Big|_{\Gamma_2} = 0. \tag{17}$$

The f is continuous, using the Peano theorem we get the local existence of u^N.

By multiplying both sides (15) with C_k^N, then taking the sum with respect to k from 1 to N, we obtain

$$\int_\Omega \sum_{i,j=1}^{n} a_{ij}u^N_{x_i}u^N_{x_j}\mathrm{d}x - \int_\Omega \sum_{i=1}^{n} a_i u^N_{x_i}u^N\mathrm{d}x - \int_\Omega a|u^N|^2\mathrm{d}x + \int_\Omega |u^N|^p\mathrm{d}x = -\int_\Omega fu^N\mathrm{d}x. \tag{18}$$

Using hypotheses (1)-(3) and Cauchy's inequality from (6)-(7) we get

$$\int_\Omega \sum_{i,j=1}^{n} a_{ij}u^N_{x_i}u^N_{x_j}\mathrm{d}x \geq \nu\|u^N_x\|^2_{L^2(\Omega)}, \tag{19}$$

$$\int_\Omega \sum_{i=1}^{n} a_i u^N_{x_i}u^N\mathrm{d}x \leq \frac{\varepsilon}{2}\|u^N\|^2_{L^2(\Omega)} + \frac{M_1}{2\varepsilon}\|u^N\|^2_{L^2(\Omega)}, \quad \forall\varepsilon > 0, \tag{20}$$

$$\int_\Omega a|u^N|^2\mathrm{d}x \leq -M_2\|u^N\|^2_{L^2(\Omega)}, \tag{21}$$

$$\int_\Omega fu^N\mathrm{d}x \leq \frac{1}{2}\|f\|^2_{L^2(\Omega)} + \frac{1}{2}\|u^N\|^2_{L^2(\Omega)}. \tag{22}$$

Choose $\varepsilon = \nu$. From (18)-(22) we have

$$\frac{\nu}{2}\|u^N_x\|^2_{L^2(\Omega)} + \|u^N\|^p_{L^p(\Omega)} \leq \left(\frac{M_1}{2\nu} - M_2 + \frac{1}{2}\right)\|u^N\|^2_{L^2(\Omega)} + \frac{1}{2}\|f\|^2_{L^2(\Omega)}. \tag{23}$$

Choose $M_2 > 0$ large enough such that $\delta = \frac{M_1}{2\nu} - M_2 + \dfrac{1}{2} < 0$ and put $C_3 = \min(\frac{\nu}{2}, -\delta) > 0$, we obtain

$$C_3\|u^N\|^2_{W^1_{2,0}(\Omega,\Gamma_1,\Gamma_2)} + \|u^N\|^p_{L^p(\Omega)} \leq \frac{1}{2}\|f\|^2_{L^2(\Omega)}. \tag{24}$$

Thank to (24), we have

$$\|u^N\|_{W^1_{2,0}(\Omega,\Gamma_1,\Gamma_2)} \leq C_4, \quad \|u^N\|_{L^p(\Omega)} \leq C_4, \quad C_4 = const > 0, \tag{25}$$

for $\forall N$.

By the Sobolev's imbedding theorem $W^1_{2,0}(\Omega,\Gamma_1,\Gamma_2) \hookrightarrow L^2(\Omega)$, since $\{u^N\}$ is uniformly bounded in $L^2(\Omega)$, we can choose a subsequence $\{u_\mu\} \to u(x)$ weakly a.e. in $L^2(\Omega)$). We will prove that $u(x)$ is a solution of the problem (11)-(13).

By Lemma 2.2, it is sufficient to prove that $|u_\mu|^\rho u_\mu \to |u|^p u$ weakly in $L^q(\Omega)$.

Put $\mathcal{M} = \{\eta = \sum\limits_{k=1}^{N} d_k\varphi_k(x)|\ d_k$ arbitrary$\}$. The space $\mathcal{M}$ is a subspace of the space E.

Multiplying both sides of (15) by d_k, then taking the sum with respect to k from 1 to N, we obtain

(26)
$$\int\limits_\Omega \sum_{i,j=1}^n a_{ij}u_{\mu x_i}\eta_{x_j}\mathrm{d}x - \int\limits_\Omega \sum_{i=1}^n a_i u_{\mu x_i}\eta\mathrm{d}x - \int\limits_\Omega a u_\mu \eta\mathrm{d}x + \int\limits_\Omega |u_\mu|^\rho u_\mu \eta\mathrm{d}x = -\int\limits_\Omega f\eta\mathrm{d}x.$$

For each $\eta \in \mathcal{M}$, taking limits $\mu \to \infty$, equality (26) implies

(27)
$$\int\limits_\Omega \sum_{i,j=1}^n a_{ij}u_{x_i}\eta_{x_j}\mathrm{d}x - \int\limits_\Omega \sum_{i=1}^n a_i u_{x_i}\eta\mathrm{d}x - \int\limits_\Omega a u\eta\mathrm{d}x + \int\limits_\Omega |u|^\rho u\eta\mathrm{d}x = -\int\limits_\Omega f\eta\mathrm{d}x.$$

Morever, we have

$$u_\mu\Big|_{\Gamma_1} \to u\big|_{\Gamma_1} = 0,$$
$$\frac{\partial u_\mu}{\partial n}\Big|_{\Gamma_2} \to \frac{\partial u}{\partial n}\Big|_{\Gamma_2} = 0 \text{ as } \mu \to \infty.$$

The proof is complete. □

Theorem 3.2. *If $f \in L^2(\Omega), \rho \leq \dfrac{2}{n-2}$ (if $n = 2$, ρ is arbitrary finite), then the problem (11)-(13) has at most one generalized solutions in $W^1_{2,0}(\Omega,\Gamma_1,\Gamma_2)$.*

Proof. Suppose the problem (11)-(13) has two generalized solutions u, v in $W^1_{2,0}(\Omega, \Gamma_1, \Gamma_2)$. If $\omega = u - v$, we have

(28)
$$\int_\Omega \sum_{i,j=1}^n a_{ij}\omega_{x_i}\eta_{x_j}\mathrm{d}x - \int_\Omega \sum_{i=1}^n a_i\omega_{x_i}\eta\mathrm{d}x - \int_\Omega a\omega\eta\mathrm{d}x = -\int_\Omega (|u|^\rho u - |v|^\rho v)\eta\mathrm{d}x,$$

for $\forall \eta \in W^1_{2,0}(\Omega, \Gamma_1, \Gamma_2)$.

Taking $\eta = \omega$, (28) implies that

$$\int_\Omega \sum_{i,j=1}^n a_{ij}\omega_{x_i}\omega_{x_j}\mathrm{d}x - \int_\Omega \sum_{i=1}^n a_i\omega_{x_i}\omega\mathrm{d}x - \int_\Omega a\omega^2\mathrm{d}x = -\int_\Omega (|u|^\rho u - |v|^\rho v)\omega\mathrm{d}x.$$

By using Cauchy-Bunhiacopski's inequality and Lemma 2.3 we obtain

$$\nu\|\omega_x\|^2_{L^2(\Omega)} \le \frac{\varepsilon}{2}\|\omega_x\|^2_{L^2(\Omega)} + \frac{M_1}{2\varepsilon}\|\omega\|^2_{L^2(\Omega)} - M_2\|\omega\|^2_{L^2(\Omega)} + |G(\omega)|.$$

Taking $\varepsilon = \nu$, we have

$$\frac{\nu}{2}\|\omega_x\|^2_{L^2(\Omega)} \le \big(\frac{M_1}{2\nu} - M_2 + C_2\big)\|\omega\|^2_{L^2(\Omega)}. \tag{29}$$

From (29), choose $M_2 > 0$ large enough such that $C_5 = \dfrac{M_1}{2\nu} - M_2 + C_2 < 0$ and put $C_6 = \min\left(\dfrac{\nu}{2}, -C_5\right) > 0$, we obtain

$$C_6\|\omega\|^2_{W^1_{2,0}(\Omega,\Gamma_1,\Gamma_2)} \le 0. \tag{30}$$

From (30) we have

$$u \equiv v \quad \text{in} \quad W^1_{2,0}(\Omega, \Gamma_1, \Gamma_2).$$

This implies the uniqueness of the solution. □

4. Some further results in domain of 2-dimensions

In this section, we consider smoothness of the generalized solutions in bounded domain $\Omega \in \mathbb{R}^2$ with edges. By mathematical transformation Ω equals Ω^0 and has the property

$$0 < r = \sqrt{x_1^2 + x_2^2} < +\infty, \quad 0 < \arctan\frac{x_2}{x_1} \le \beta = const.$$

Hence, boundary $\partial\Omega^0$ consist of

$$\Gamma_1^0 = \{x \mid \arctan\frac{x_2}{x_1} = 0\}, \Gamma_2^0 = \{x \mid \arctan\frac{x_2}{x_1} = \beta\}.$$

Consider the problem of Ω^0 for equation (11) and conditions:

$$u\big|_{\Gamma_1^0} = 0, \tag{31}$$

$$\frac{\partial u}{\partial n}\big|_{\Gamma_2^0} = 0. \tag{32}$$

Using the arguments as in the proof of Lemma 2.1 [8], we obtain the following lemma.

Lemma 4.1. *Let $u \in W^1_{2,0}(\Omega^0, \Gamma_1^0, \Gamma_2^0)$ and $\int\limits_{\Omega^0} r^\alpha \text{grad}^2 u \mathrm{d}x < +\infty$, then*

$$\int\limits_{\Omega^0} r^{\alpha-2}|u|^2 \mathrm{d}x \leq C_7 \int\limits_{\Omega^0} r^\alpha \text{grad}^2 u \mathrm{d}x, \tag{33}$$

where $r = \sqrt{x_1^2 + x_2^2}, \alpha \in \mathbb{R}, C_7 = const > 0$.

From Lemma 4.1, we get the following result:

If $u \in W^1_{2,0}(\Omega^0, \Gamma_1^0, \Gamma_2^0)$ and $u = 0$ when $r > d > 0$ then there exists the Friedrich's inequality

$$\int\limits_{\Omega^0} u^2 \mathrm{d}x \leq d^2 \int\limits_{\Omega^0} \text{grad}^2 u \mathrm{d}x. \tag{34}$$

Theorem 4.1. *Leting $u(x)$ for a generalized solution of problem (11)-(31)-(32), and $f \in L^2(\Omega^0)$; $u = 0$ when $r > d > 0$ (d - small enough), we have*

$$\int\limits_{\Omega^0} \text{grad}^2 u \mathrm{d}x + \int\limits_{\Omega^0} \frac{u^2}{r^2} \mathrm{d}x \leq \int\limits_{\Omega^0} |f|^2 \mathrm{d}x. \tag{35}$$

Proof. Leting $u(x)$ for a generalized solution of problem (11)-(31)-(32), we have

$$\int\limits_{\Omega^0} \left[\sum_{i,j=1}^n a_{ij} u_{x_i} \eta_{x_j} - \sum_{i=1}^n a_i u_{x_i} \eta - au\eta\right] \mathrm{d}x + \int\limits_{\Omega^0} |u|^p u\eta \mathrm{d}x = -\int\limits_{\Omega^0} f\eta \mathrm{d}x, \tag{36}$$

for $\forall \eta \in W^1_{2,0}(\Omega^0, \Gamma_1^0, \Gamma_2^0)$.

Taking $\eta = u$, we obtain

$$\int\limits_{\Omega^0} \left[\sum_{i,j=1}^n a_{ij} u_{x_i} u_{x_j} - \sum_{i=1}^n a_i u_{x_i} u - au^2\right] \mathrm{d}x + \int\limits_{\Omega^0} |u|^p \mathrm{d}x = -\int\limits_{\Omega^0} fu \mathrm{d}x.$$

By using Cauchy-Bunhiacopski's inequality and (34), we get

(37)

$$\nu \int\limits_{\Omega^0} \text{grad}^2 u \mathrm{d}x + \int\limits_{\Omega^0} |u|^p \mathrm{d}x \leq \frac{\nu}{2} \int\limits_{\Omega^0} \text{grad}^2 u \mathrm{d}x + \left[\frac{M_1}{2\nu} - M_2 + \frac{1}{2}\right] \int\limits_{\Omega^0} u^2 \mathrm{d}x + \frac{1}{2} \int\limits_{\Omega^0} |f^2| \mathrm{d}x.$$

From (34),(37) with $d > 0$ sufficiently small such that $\delta = \frac{\nu}{2} - d^2\left[\frac{M_1}{2\nu} - M_2 + \frac{1}{2}\right] > 0$, we get

$$\delta \int\limits_{\Omega^0} \mathrm{grad}^2 u \mathrm{d}x \le \frac{1}{2} \int\limits_{\Omega^0} |f|^2 \mathrm{d}x. \tag{38}$$

and then

$$\int\limits_{\Omega^0} \mathrm{grad}^2 u \mathrm{d}x \le C_8 \int\limits_{\Omega^0} |f|^2 \mathrm{d}x, \tag{39}$$

where $C_8 > 0$ depends on δ.

By Lemma 4.1 for $\alpha = 0$ we have

$$\int\limits_{\Omega^0} \frac{u^2}{r^2} \le C_9 \int\limits_{\Omega^0} \mathrm{grad}^2 u \mathrm{d}x. \tag{40}$$

From (39)-(40), the theorem is completely proved. □

Lemma 4.2. *Leting*

i) $u \in W_0^{k+m}(\Omega^0); u = 0$ *when* $r > d > 0$ *(d small enough);*

ii) $\alpha \ge 2k$.

Then

$$u \in \mathring{W}^{k+m}_{\alpha+2m}(\Omega^0).$$

Proof. Let $u \in W_0^{k+m}(\Omega^0)$, we have

$$\sum_{s=0}^{k+m} \int\limits_{\Omega^0} \left|\frac{\partial^s u}{\partial x^s}\right|^2 \mathrm{d}x < +\infty.$$

With $\alpha \ge 2k, 0 \le s \le k+m$ we get

$$\int\limits_{\Omega^0} r^{\alpha+2(s-k)} \left|\frac{\partial^s u}{\partial x^s}\right|^2 \mathrm{d}x \le \int\limits_{\Omega^0} \left|\frac{\partial^s u}{\partial x^s}\right|^2 \mathrm{d}x < +\infty. \tag{41}$$

Hence, (41) implies

$$\int\limits_{\Omega^0} r^{\alpha+2m+2(s-k-m)} \left|\frac{\partial^s u}{\partial x^s}\right|^2 \mathrm{d}x \le \int\limits_{\Omega^0} \left|\frac{\partial^s u}{\partial x^s}\right|^2 \mathrm{d}x < +\infty. \tag{42}$$

From (42) taking the sum with respect to s from 0 to $k+m$ we obtain

$$u \in \mathring{W}^{k+m}_{\alpha+2m}(\Omega^0).$$

The proof is complete. □

Lemma 4.3. *Suppose the conditions of Lemma 4.2 hold, and let* $k+m>1, \rho\in\mathbb{N}, \rho\geq 1$ *or* $\rho\in\mathbb{R}, \rho\geq k+m$*. Then*

$$|u|^{\rho}\in \mathring{W}^{k+m}_{\alpha+2m}(\Omega^0). \tag{43}$$

Proof. Considering cases

1) Case $\rho=1, 0\leq s\leq k+m$, we have

$$\int\limits_{\Omega^0}\left|\frac{\partial^s|u|}{\partial x^s}\right|^2 \mathrm{d}x=\int\limits_{\Omega^0}\left|\frac{\partial^s u}{\partial x^s}\right|^2 \mathrm{d}x<+\infty. \tag{44}$$

From (44) we have $\dfrac{\partial^s|u|}{\partial x^s}\in L^2(\Omega^0)$ and $|u|\in W_0^{k+m}(\Omega^0)$.

By Lemma 4.2, it follows that $|u|\in \mathring{W}^{k+m}_{\alpha+2m}(\Omega^0)$.

2) Case $\rho>1, 0\leq s\leq k+m$, we have

$$\frac{\partial|u|^{\rho}}{\partial x}=\rho|u|^{\rho-1}\frac{\partial|u|}{\partial x}. \tag{45}$$

From (45), we see that in terms of expansion of $\dfrac{\partial^s|u|^{\rho}}{\partial x^s}$ and its coefficients, there forms:

$$1)\ |u|^{\rho-1}\frac{\partial^s|u|}{\partial x^s},$$
$$2)\ |u|^{\gamma}\frac{\partial^{s_1}|u|}{\partial x^{s_1}}...\frac{\partial^{s_l}|u|}{\partial x^{s_l}},$$

where $0\leq\gamma<\rho-1, \sum\limits_{i=1}^{l}s_i=s; 1\leq s_i<s, 1\leq i\leq l.$

Form 1: $u\in W^{k+m}(\Omega^0), \Omega^0\subset\mathbb{R}^2$ with $k+m>1$, then the Sobolev's imbedding theorem implies that

$$W^{k+m}(\Omega^0)\hookrightarrow C(\overline{\Omega^0}).$$

with continuous injection. Hence with $u\in C(\overline{\Omega}^0)$, $|u|^{\gamma}$ continuous on $\overline{\Omega}^0$ and $|u|^{\gamma}$ bounded on $\Omega^0, \forall\gamma\geq 0$,we get $|u|^{\rho-1}$ bounded on Ω^0.

By using the result in case $\rho=1$, we have

$$\int\limits_{\Omega^0}\left|\frac{\partial^s u}{\partial x^s}\right|^2 \mathrm{d}x<+\infty.$$

Hence

$$|u|^{\rho-1}\left|\frac{\partial^s|u|}{\partial x^s}\right|\in L^2(\Omega^0).$$

Form 2: Repeating arguments which are analogous to Form 1, we have $|u|^\gamma$ bounded in $\Omega^0, 0 \leq \gamma < \rho - 1$, $u \in W_0^{k+m}(\Omega^0)$, hence $\dfrac{\partial^{s_i}|u|}{\partial x^{s_i}} \in \mathring{W}^1(\Omega)$ for $0 \leq s_i < k+m$.

By using Sobolev's imbedding theorem, we have

$$\mathring{W}^1(\Omega^0) \hookrightarrow L^p(\Omega^0), \quad \forall p \geq 1,$$

hence

$$\frac{\partial^{s_i}|u|}{\partial x^{s_i}} \in L^2(\Omega^0), \quad 1 \leq i \leq l.$$

By using Holdel's inequality we obtain

(46)

$$\int\limits_{\Omega^0} \left| |u|^\gamma \frac{\partial^{s_1}|u|}{\partial x^{s_1}} \cdots \frac{\partial^{s_l}|u|}{\partial x^{s_l}} \right|^2 \mathrm{d}x \leq C_{10} \left\| \frac{\partial^{s_1}|u|}{\partial x^{s_1}} \right\|^2_{L^{2p_1}(\Omega^0)} \cdots \left\| \frac{\partial^{s_l}|u|}{\partial x^{s_l}} \right\|^2_{L^{2p_l}(\Omega^0)} < +\infty,$$

where $\dfrac{1}{p_1} + \dfrac{1}{p_2} + \cdots + \dfrac{1}{p_l} = 1, C_{10} = const > 0$.

From (46), we see that $|u|^\gamma \dfrac{\partial^{s_1}|u|}{\partial x^{s_1}} \cdots \dfrac{\partial^{s_l}|u|}{\partial x^{s_l}} \in L^2(\Omega^0)$.

Hence $\dfrac{\partial^s |u|^\rho}{\partial x^s} \in L^2(\Omega^0)$ when $0 \leq s \leq k+m$,$|u|^\rho \in W_0^{k+m}(\Omega^0)$ and $|u|^\rho u \in \mathring{W}_0^{k+m}$.

Moreover, by Lemma 4.2, it follows that

$$|u|^\rho u \in \mathring{W}^{k+m}_{\alpha+2m}(\Omega^0).$$

This completes the proof. □

From Lemma 4.2, 4.3 and using the arguments as in the proof of Theorem 2.2 [8], we obtain the following lemma.

Lemma 4.4. *Let*

i) $f \in \mathring{W}^{k+m}_{\alpha+2m}(\Omega^0)$;
ii) $u \in W_0^{k+m}(\Omega^0) \cap \mathring{W}^0_{\alpha-2k-4}(\Omega^0)$ *is a generalized solution of problem (11)-(31)-(32) ,* $u = 0$ *when* $r > d > 0$ *(d sufficiently small);*
iii) $\alpha \geq 2k,\ k+m > 1$;
iv) $\rho \in \mathbb{N}, \rho \geq 1$ *or* $\rho \in \mathbb{R}, \rho \geq k+m-1$.

Then

$$|u|^\rho u \in \mathring{W}^{k+m+2}_{\alpha+2m}(\Omega^0).$$

Remark 1: If $u \in \mathring{W}^k_\alpha(\Omega^0), 0 \le \alpha \le 2, k \ge 1; u = 0$ for $r > d$ ($d > 0$ - sufficiently small), then $u \in W^{k-1}(\Omega^0)$.
Remark 2: If $f \in \mathring{W}^0_0(\Omega^0) = L^2(\Omega^0)$; $u = 0$ when $r > d$ ($d > 0$ - sufficiently small), $\frac{\pi}{\omega} > 1$, then $u \in \mathring{W}^2_0(\Omega^0)$.
Remark 3: If $f \in L^2(\Omega^0), u \in W^1_{2,0}(\Omega^0, \Gamma^0_1, \Gamma^0_2)$ is a generalized solution of problem (11)-(31)-(32), then $|u|^\rho u \in L^2(\Omega^0)$. Hence $F = |u|^\rho u + f \in L^2(\Omega)$ and this implies $F \in \mathring{W}^0_\alpha(\Omega^0)$. In case $k = 0$ we get $u \in \mathring{W}^2_\alpha(\Omega^0)$.

Theorem 4.2. *Let*

i) $f \in \mathring{W}^k_\alpha(\Omega^0) \cap L^2(\Omega^0), \quad 0 \le \alpha \le 2, k \ge 1$;
ii) $u \in W^1_{2,0}(\Omega^0, \Gamma^0_1, \Gamma^0_2)$ *is the generalized solution of problem (11)-(31)-(32),* $u = 0$ *for* $r > d > 0$ *(d sufficiently small);*
iii) $\frac{\pi}{\omega} > k + 2 - \frac{\alpha + 2}{2}$;
iv) $\rho \in \mathbb{N}, \rho \ge 1$ *or* $\rho \in \mathbb{R}, \rho \ge k - 1$.

Then

$$u \in \mathring{W}^{k+2}_\alpha(\Omega^0).$$

Proof. We will prove the theorem by the inductive method with respect to k.

Let $k = 1$. Since $f \in \mathring{W}^1_\alpha(\Omega^0$, hence $f \in \mathring{W}^0_\alpha(\Omega^0)$ and by Remark 2 we have $u \in \mathring{W}^2_\alpha(\Omega^0)$. We will prove $|u|^\rho u \in \mathring{W}^1_\alpha(\Omega^0)$.

a) First we prove that $|u|^\rho \in \mathring{W}^1_\alpha(\Omega^0)$, i.e.

$$\begin{aligned} &\int\limits_{\Omega^0} r^{\alpha-2}\big||u|^\rho\big|^2 \mathrm{d}x + \int\limits_{\Omega^0} r^\alpha \Big|\frac{\partial |u|^\rho}{\partial x}\Big|^2 \mathrm{d}x \\ &= \int\limits_{\Omega^0} r^{\alpha-2}|u|^{2\rho}\mathrm{d}x + \int\limits_{\Omega^0} r^\alpha (\rho|u|^{\rho-1})^2 \Big|\frac{\partial |u|}{\partial x}\Big|^2 \mathrm{d}x < +\infty, \end{aligned} \tag{47}$$

From Remark 2, $f \in L^2(\Omega^0) \Rightarrow u \in W^2(\Omega^0)$. By the Sobolev's imbedding theorem, we have

$$W^2(\Omega^0) \hookrightarrow C(\bar{\Omega}^0).$$

Hence $u \in C(\bar{\Omega}^0)$ and $|u|^S, (s \ge 0)$ continousness on $\bar{\Omega}^0$,$|u|^S$ bounded on $\bar{\Omega}^0$.

Because $|u|^{2\rho-2}$ is bounded and $u \in \mathring{W}^2_\alpha(\Omega^0)$, hence $u \in \mathring{W}^1_\alpha(\Omega^0)$ and we have

$$\int\limits_{\Omega^0} r^{\alpha-2}|u|^{2\rho}\mathrm{d}x = \int\limits_{\Omega^0} r^{\alpha-2}|u|^{2\rho-2}|u|^2\mathrm{d}x \le C_{11}\int\limits_{\Omega^0} r^{\alpha-2}|u|^2\mathrm{d}x < +\infty, \tag{48}$$

$$\int\limits_{\Omega^0} r^{\alpha}(\rho|u|^{\rho-1})^2\Big|\frac{\partial|u|}{\partial x}\Big|^2\mathrm{d}x \le C_{12}\int\limits_{\Omega^0} r^{\alpha}\Big|\frac{\partial|u|}{\partial x}\Big|^2\mathrm{d}x < +\infty, \tag{49}$$

From (48),(49), we have $|u|^{\rho} \in \mathring{W}^1_{\alpha}(\Omega^0)$

b) To prove $|u|^{\rho}u \in \mathring{W}^1_{\alpha}(\Omega^0)$. Using the argument as in section a) we have

$$\begin{aligned}&\int\limits_{\Omega^0} r^{\alpha-2}\big||u|^{\rho}u\big|^2\mathrm{d}x + \int\limits_{\Omega^0} r^{\alpha}\Big|\frac{\partial|u|^{\rho}u}{\partial x}\Big|^2\mathrm{d}x\\ &= \int\limits_{\Omega^0} r^{\alpha-2}|u|^{2(\rho+1)}\mathrm{d}x + \int\limits_{\Omega^0} r^{\alpha}\Big|\frac{\partial|u|^{\rho+1}}{\partial x}\Big|^2\mathrm{d}x < +\infty,\end{aligned} \tag{50}$$

hence $|u|^{\rho}u \in \mathring{W}^1_{\alpha}(\Omega^0)$ and $F = |u|^{\rho}u + f \in \mathring{W}^1_{\alpha}(\Omega^0)$. By Lemma 4.4, we get $u \in \mathring{W}^3_{\alpha}(\Omega^0)$.

Now, let the theorem assertion holds up to $k-1 \ge 1$, i.e. if $f \in \mathring{W}^{k-1}_{\alpha}(\Omega^0), \dfrac{\pi}{\omega} > k+1-\dfrac{\alpha+2}{2}$ then $u \in \mathring{W}^{k+1}_{\alpha}(\Omega^0)$. We need to prove this holds up to $k, (k \ge 2)$.

We have $f \in \mathring{W}^k_{\alpha}(\Omega^0), \dfrac{\pi}{\omega} > k+2-\dfrac{\alpha+2}{2}, k \ge 2$ and by inductive hypothesis we have $u \in \mathring{W}^{k+1}_{\alpha}(\Omega^0)$. By Remark 1, it follows that $u \in W^k(\Omega^0), (k \ge 2)$.

We see that in terms of the expansion of $\dfrac{\partial^s|u|^{\rho}}{\partial x^s}$ and its coefficients, there forms

$$1)\ |u|^{\rho-1}\frac{\partial^s|u|}{\partial x^s},$$

$$2)\ |u|^{\gamma}\frac{\partial^{s_1}|u|}{\partial x^{s_1}}...\frac{\partial^{s_l}|u|}{\partial x^{s_l}},$$

where $0 \le \gamma < \rho-1, \sum\limits_{i=1}^{l} s_i = s; 1 \le s_i < s, 1 \le i \le l, 0 \le s \le k$.

1) Case $s = k$. We consider form 1

$$\int\limits_{\Omega^0} r^{\alpha}(|u|^{\rho-1})^2\Big|\frac{\partial^k|u|}{\partial x^k}\Big|^2\mathrm{d}x \le C_{13}\int\limits_{\Omega^0} r^{\alpha}\Big|\frac{\partial^k|u|}{\partial x^k}\Big|^2\mathrm{d}x < +\infty, \tag{51}$$

where $C_{13} = const > 0$, hence $u \in \mathring{W}^k_{\alpha}(\Omega^0)$.

For form 2, we have

$$\begin{aligned}M &= \int\limits_{\Omega^0} r^{\alpha}\Big||u|^{\gamma}\frac{\partial^{k_1}|u|}{\partial x^{k_1}}...\frac{\partial^{k_l}|u|}{\partial x^{k_l}}\Big|^2\mathrm{d}x\\ &= \int\limits_{\Omega^0} r^{\alpha}\Big|\frac{\partial^{k_1}|u|}{\partial x^{k_1}}\Big|^2\Big||u|^{\gamma}\frac{\partial^{k_2}|u|}{\partial x^{k_2}}...\frac{\partial^{k_l}|u|}{\partial x^{k_l}}\Big|^2\mathrm{d}x < +\infty,\end{aligned} \tag{52}$$

Without loss of generality we may assume that $k_1 = \max\limits_{1\le i\le l}\{k_i\}$.

If $l = k$ then $k_1 = k_2 = ... = k_l = 1$. Using Holde's inequality and boundedness of $|u|^r$ on Ω^0, we get

$$\int\limits_{\Omega^0} |u|^{2r}|u_x|^2...|u_x|^2 \mathrm{d}x \le C_{14}\int\limits_{\Omega^0} |u_x|^2...|u_x|^2\mathrm{d}x$$

$$\le C_{14}\|u_x\|^2_{W^1(\Omega^0)}...\|u_x\|^2_{W^1(\Omega^0)} < +\infty, \tag{53}$$

where $C_{14} = const > 0$. Because $u \in W^k(\Omega^0), (k \ge 2)$, hence $u_x \in W^1(\Omega^0)$. From 52, 53, we have $M < +\infty$.

If $l < k$, then we can let $\max\limits_{1\le i\le l}\{k_i\} = k_{i1} = ... = k_{ip}$

*Leting $p = 1$, we choose $k_1 = k_{i1}$, then $k_1 > k_i$, $2 \le i \le l$. For $k = 2$, we have result as in Section a); for $k \ge 3$, we have $k > k_1$ and $k - k_i \ge 2, 2 \le i \le l$.

*Leting $1 < p \le l$ and $k \ge 2$ we get $k_i \le \frac{k}{2}, 1 \le i \le l$, hence $k - k_i \ge 2$.

For all cases, we obtain $k - k_i \ge 2, 2 \le i \le l$, hence $\dfrac{\partial^{k_i}|u|}{\partial x^{k_i}} \in W^{k-k_i}(\Omega^0)$.

Because $k - k_i \ge 2$ hence $W^{k-k_i}(\Omega^0) \hookrightarrow C(\bar{\Omega}^0), 2 \le i \le l$ with continuous injection, it follows that $\dfrac{\partial^{k_i}|u|}{\partial x^{k_i}}, 2 \le i \le l$ is bounded on Ω^0.

Moreover, $u \in \mathring{W}^{k+1}_\alpha(\Omega^0) \Rightarrow \mathring{W}^{l}_\alpha(\Omega^0)$ for all $l \le k+1$. From (52), we have

$$M \le C_{15}\int\limits_{\Omega^0} r^\alpha\Big|\frac{\partial^{k_1}|u|}{\partial x^{k_1}}\Big|^2\mathrm{d}x \le C_{15}\int\limits_{\Omega^0} r^\alpha\Big|\frac{\partial^{k_1}u}{\partial x^{k_1}}\Big|^2\mathrm{d}x < +\infty, \tag{54}$$

Hence

$$\int\limits_{\Omega^0} r^\alpha\Big|\frac{\partial^{k}|u|^p}{\partial x^{k}}\Big|^2\mathrm{d}x < +\infty. \tag{55}$$

2) Case $s < k$. Using the argument as in case $s = k$, we can prove that $|u|^\rho \in \mathring{W}^k_\alpha(\Omega^0)$. This is implied

$$|u|^\rho u \in \mathring{W}^k_\alpha(\Omega^0).$$

By Lemma 4.4, we have $u \in \mathring{W}^{k+2}_\alpha(\Omega^0)$. The proof is completed. □

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