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A Nonlinear Elasticity Problem Governed by Lamé System

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Abstract

In this work we study a nonlinear problem equation governed by the system of Lamé, is similar example was the partial differential equations,which operates in relativistic quantum mechanics system.We look for the existence and uniqueness of a function $u = u(x,t)$, $x \in \Omega$, $t \in (0,T)$ solution of the problem.

1 Notation

Let Ω an open bounded domain of IR^n, with regular boundary Γ.We denote by $u = \{u_1, u_2, u_3, ..., u_n\}$ a vector and T a scalar function on $Q_\tau = \Omega\times]0,\tau[$ where τ is a finite real number. λ, μ are the Lamé coefficients with $\lambda \geq 0$, $\mu > 0$, $\gamma > 0$ is a constant. Let $k > 0$ be the coefficient of the termic conductivity.Our problem is to study a similar example was the partial differential equations, which operates in relativistic quantum mechanics. It means a Ω an open bounded domain of $\mathbb{R}^n$, with regular boundary Γ.We denot by Q the cylinder $\mathbb{R}^n_x \times \mathbb{R}_t$: $Q = \Omega \times (0,T)$, with boundary Σ. L designe Lamé system define by $\mu\Delta + (\lambda + \mu)\nabla div$; λ and μ are constants Lamé with $\lambda+ \mu \geq 0$. (u_0, u_1, f) functions, $\rho > 0$.We look for the existence and uniqueness of a function $u = u(x,t)$, $x \in \Omega$, $t \in (0,T)$ solution of the problem (1; 2; 3; 4).

$$\frac{\partial^2 u}{\partial t^2} - Lu + |u|^\rho u = f,\ x \in \Omega,\ t \in (0,T) \tag{1}$$

$$u = 0 \text{ on } \Sigma \tag{2}$$

$$u(x,0)=u_0(x),\ x\in\Omega \tag{3}$$

$$\frac{\partial u}{\partial t}(x,0)=u_1(x),\ x\in\Omega \tag{4}$$

2 Existence and uniqueness of the solution

2.1 Existence of the solution

The techniques we use are those of the method of compactness

Theorem 1 *Assume that* Ω *is a bounded open, are given* f, u_0, $u_{1,}$ *with*

$$f\in L^2\left(Q\right), \tag{5}$$

$$u_0\in H_0^1\left(\Omega\right)\cap L^p\left(\Omega\right),\ p=\rho+2. \tag{6}$$

$$u_1\in L^2\left(\Omega\right). \tag{7}$$

Then there exists a function u satisfying:

$$u\in L^\infty\left(0,T;H_0^1\left(\Omega\right)\cap L^p\left(\Omega\right)\right), \tag{8}$$

$$\frac{\partial u}{\partial t}(x,0)\in L^\infty\left(0,T;L^2\left(\Omega\right)\right), \tag{9}$$

$$\frac{\partial^2 u}{\partial t^2}-Lu+\left|u\right|^\rho u=f\text{ in }Q \tag{10}$$

$$u(0)=u_0(x),\,x\in\Omega \tag{11}$$

$$\frac{\partial u}{\partial t}(0)=u_1(x),\,x\in\Omega \tag{12}$$

2.2 First step: looking for approached solutions

We introduce a sequence functions $w_1, ..., w_m, ..$ having the following properties:

$$\begin{cases} w_i \in H_0^1(\Omega) \cap L^p(\Omega) \quad \forall i; \\ \forall m,\ w_1, ...,\ w_m \text{ are linearly independent} \\ \text{combinations of linear finite } w_i \text{ are dense in} \\ H_0^1(\Omega) \cap L^p(\Omega). \end{cases} \tag{13}$$

We look for $u_m = u_m(t)$ (approximate solution) of the problem as:

$$u_m(t) = \sum_{i=1}^{i=m} g_{im}(t) w_i. \tag{14}$$

We determine the functions g_{im} with the conditions

$$\left(u_m''(t), w_j\right) + a(u_m, w_j) + \left(|u_m(t)|^\rho u_m(t), w_j\right) = \left(f(t), w_j\right),\ 1 \le j \le m, \tag{15}$$

As a is a bilinear form defined as follows:

$$a(u, v) = \lambda \sum_{i=1}^{n} \int_\Omega \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} dx + (\lambda + \mu) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \tag{16}$$

The system (15) of ordinary differential equations nonlinear be supplemented by initial conditions:

$$u_m(0) = u_{0m},\ u_{0m} = \sum_{i=1}^{m} \alpha_{im} w_i \underset{m \to \infty}{\to} u_0 \text{ in } H_0^1(\Omega) \cap L^p(\Omega), \tag{17}$$

$$u_m'(0) = u_{1m}, u_{1m} = \sum_{i=1}^{m} \beta_{im} w_i \underset{m \to \infty}{\to} u_1 \text{ in } L^2(\Omega). \tag{18}$$

Through the linear independence of $w_1, ..., w_m,$, we have $\det(w_i, w_j) \neq 0$, ie the system composed of (15), (17) and (18) admits a solution defined on $[0, t_m]$. The a priori estimates which follow show that $t_m = T$.

2.3 Second step: a priori estimates.

We multiply equation (15) index j by $g_{jm}'(t)$ was:

$$\left(u_m''(t), u_m'(t)\right) + a\left(u_m(t), u_m'(t)\right) + \left(|u_m(t)|^\rho u_m(t), u_m'(t)\right)$$

$$= \left(f\left(t\right), u_m^{'}\left(t\right)\right) \tag{19}$$

But, $\left|u_m\left(t\right)\right|^{\rho} u_m\left(t\right) \in L^{p^{'}}\left(\Omega\right)$and $p = \rho + 2$, Then according to Cauchy Schwarz were:

$$\frac{1}{2}\frac{d}{dt}\left(\left|u_m^{'}\left(t\right)\right|^2 + \left\|u_m\left(t\right)\right\|^2\right) + \frac{1}{p}\frac{d}{dt}\big(\int\limits_{\Omega}\left|u_m\left(x,t\right)\right|^p dx\big) \leq \left|f\left(t\right)\right|\left|u_m^{'}\left(t\right)\right| \tag{20}$$

So we integrate between $0, t$, we deduce:

$$\frac{1}{2}\left(\left|u_m^{'}\left(t\right)\right|^2 + \left\|u_m\left(t\right)\right\|^2\right) + \frac{1}{p}\left\|u_m\left(x,t\right)\right\|^p_{L^p(\Omega)} \leq \frac{1}{2}\left|u_{1m}\right|^2 + \frac{1}{2}\left\|u_{0m}\right\|^2$$

$$+\frac{1}{p}\left\|u_m\left(0\right)\right\|^p_{L^p(\Omega)} + \int\limits_0^t \left|f\left(\sigma\right)\right|\left|u_m^{'}\left(\sigma\right)\right| d\sigma \tag{21}$$

From $(17), (18)$ and inequality:

$$ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2.$$

We have

$$\left|f\left(\sigma\right)\right|\left|u_m^{'}\left(\sigma\right)\right| \leq \frac{1}{2}\left|f\left(\sigma\right)\right|^2 + \frac{1}{2}\left|u_m^{'}\left(\sigma\right)\right|^2.$$

Then

$$\frac{1}{2}\left(\left|u_m^{'}\left(t\right)\right|^2 + \left\|u_m\left(t\right)\right\|^2\right) + \frac{1}{p}\left\|u_m\left(x,t\right)\right\|^p_{L^p(\Omega)} \leq c + \frac{1}{2}\int\limits_0^t \left|f\left(\sigma\right)\right|^2 d\sigma$$

$$+\frac{1}{2}\int\limits_0^t \left|u_m^{'}\left(\sigma\right)\right|^2 d\sigma. \tag{22}$$

But

$$f \in L^2\left(Q\right) \Rightarrow \int\limits_0^t \left|f\left(\sigma\right)\right|^2 d\sigma \leq \text{constant}.$$

We deduce, therefore, in particular (22) that:

$$\left|u_m^{'}(t)\right|^2 \leq c^{'} + \int\limits_0^t \left|u_m^{'}(\sigma)\right|^2 d\sigma, \tag{23}$$

And after the Gronwall inequality we have:

$$\left|u_m^{'}(t)\right| \leq \text{constant.} \tag{24}$$

Then

$$\|u_m(t)\| + \|u_m(x)\|_{L^p(\Omega)} \leq \text{constant.} \tag{25}$$

By $(24), (25)$ and when $m \to \infty$ we have: u_m in a bounded set of $L^\infty(0,T;H_0^1(\Omega) \cap L^p(\Omega))$ and $u_m^{'}$ in a bounded set of $L^\infty(0,T;L^2(\Omega))$.

2.4 Third step: passage to the limit

In the second step we were u_m borned in $L^\infty(0,T;H_0^1(\Omega) \cap L^p(\Omega))$, then it is bounded in $L^2(0,T;H_0^1(\Omega))$.Since $L^\infty(0,T;H_0^1(\Omega) \cap L^p(\Omega))$.{resp. $L^\infty(0,T;L^2(\Omega))$} is the dual of $L^1(0,T;H^{-1}(\Omega)+L^{p^{'}}(\Omega))$\{resp. of $L^1(0,T;L^2(\Omega))$\}, there exists a result u_m, u_μ sush that :

$$\forall g \in L^1\left(0,T;H^{-1}(\Omega) + L^{p^{'}}(\Omega)\right) : .$$

$$\int\limits_0^T (u_\mu(t), g(t))\, dt \underset{\mu\to\infty}{\to} \int\limits_0^T (u(t), g(t))\, dt$$

Which implies:

$$u_\mu \to u \;\text{ weak in } L^\infty\left(0,T;H_0^1(\Omega) \cap L^p(\Omega)\right) \text{ and in } L^2\left(0,T;H_0^1(\Omega)\right). \tag{26}$$

So :

$$\exists u_\mu^{'} \to u^{'}\text{in } D^{'}\left(0,T;H_0^1(\Omega) \cap L^p(\Omega)\right) \Rightarrow u_\mu^{'} \to u^{'}\text{in } L^\infty\left(0,T;L^2(\Omega)\right)$$

$$\text{and in } L^2\left(0,T;L^2(\Omega)\right) \tag{27}$$

Then in particular u_m bounded in $H^1(Q)$, but we know that the next injection is compact:

$$H^1(Q) \hookrightarrow L^2(Q) \tag{28}$$

And according to the definition of compact injection, we can suppose the sequence u_μ extracted u_m satisfies (26) and (27) , then u, $u^{'}$ exists and in $L^2(Q)$ then:

$$\begin{cases} u_\mu \to u \text{ in } L^2\left(0,T;H_0^1(\Omega)\cap L^p(\Omega)\right) \text{ strong (a.e)}, \\ u_\mu^{'} \to u^{'} \text{in } L^2\left(0,T;L^2(\Omega)\right) \text{ weak (a.e)}. \end{cases} \tag{29}$$

Studying the convergence of $|u_m|^\rho u_m$:
$|u_m|^\rho u_m$ is bounded in $L^\infty\left(0,T;L^{p^{'}}(\Omega)\right)$, then
We set:

$$|u_\mu|^\rho u_\mu \to w \text{ in } L^\infty\left(0,T;L^{p^{'}}(\Omega)\right), \tag{30}$$

Showing that:

$$w = |u|^\rho u. \tag{31}$$

For this we give the following lemma:

2.4.1 Lemma:

Let O be an open bounded $\mathbb{R}_x^n \times \mathbb{R}_t$, g_μ and g des functions of $L^q(O)$, $1 < q < \infty$, such that

$$\|g_\mu\|_{L^q(O)} \leq c,\ \ g_\mu \to g \ \text{ p.p. in } O$$

Then

$$g_\mu \to g \ \text{ in } L^q \text{ weak .}$$

When we ask: $O = Q$ and $g_\mu = |u_\mu|^\rho u_\mu$, from (29) :

$$u_\mu \to u \text{ in } L^2(Q) \text{ (a.e)}$$

Therefore :

$$g_\mu = |u_\mu|^\rho u_\mu \rightharpoonup |u|^\rho u = g \text{ (a.e) in } L^{p^{'}}(\Omega)$$

Such that $p^{'} = \frac{\rho+2}{\rho+1} = q$ (for $p = \rho + 2$), and after (30) :

$$|u_\mu|^\rho u_\mu \rightharpoonup w \text{ in } L^{p^{'}}(\Omega). \tag{32}$$

Since the limit is unique, therefore:

$$g = |u|^\rho u = w.$$

We show that this solution satisfies the equation (15), so when we set $m = \mu$ and we fix j such that $\mu > j$; then:

$$\left(u^{''}_\mu(t), w_j\right) + a(u_\mu, w_j) + (|u_\mu(t)|^\rho u_\mu(t), w_j) = (f(t), w_j). \tag{33}$$

From (30) and (31)

$$\left(u^{'}_\mu, w_j\right) \rightharpoonup \left(u^{'}, w_j\right) \text{ in } L^\infty(0,T) \Rightarrow \frac{d}{dt}\left(u^{'}_\mu, w_j\right),$$

$$= \left(u^{''}_\mu, w_j\right) \rightarrow \left(u^{''}, w_j\right) \text{ in } D^{'}(0,T) \tag{34}$$

Where:

$$a(u_\mu, w_j) \rightharpoonup a(u, w_j) \text{ in } L^\infty(0,T).$$

And after (30) and (31)

$$(|u_\mu|^\rho u_\mu, w_j) \rightharpoonup (|u|^\rho u, w_j) \text{ in } L^\infty(0,T)$$

It follows therefore from (32) that:

$$\frac{d^2}{dt^2}(u, w_j) + a(u, w_j) + (|u|^\rho u, w_j) = (f, w_j).$$

According to the density of the basis $\{w_j\}$ in separable space $H^1_0(\Omega) \cap L^p(\Omega)$, we have

$$\frac{d^2}{dt^2}(u, v) + a(u, v) + (|u|^\rho u, v) = (f, v) \, \forall v \in H^1_0(\Omega) \cap L^p(\Omega) \tag{35}$$

Then the solution u satisfies (4), (5) and (6).

It remains to show that the solution u satisfies the initial conditions (7), (8) : $u(0) = u_0$, $u^{'}(0) = u_1$.

By (26) and (27) we have:

$$u_\mu \rightarrow u \text{ in } L^\infty\left(0, T; H^1_0(\Omega) \cap L^p(\Omega)\right), \tag{36}$$

$$\frac{du_\mu}{dt} = u_\mu^{'} \rightharpoonup u^{'} \text{ in } L^\infty\left(0, T; L^2(\Omega)\right). \tag{37}$$

So u_μ is continuous on $[0, T]$ then continous on 0 and then:

$$u_{0\mu} = u_\mu(0) \to u(0) = u_0 \text{ in } H_0^1(\Omega) \cap L^p(\Omega),$$

whence (7) .
And yet

$$\begin{aligned} \left(u_\mu^{''}, w_j\right) &\rightharpoonup \left(u^{''}, w_j\right) \text{ in } L^\infty(0, T), \\ \left(u_\mu^{'}, w_j\right) &\rightharpoonup \left(u^{'}, w_j\right) \text{ in } L^\infty(0, T). \end{aligned}$$

Then

$$\left(u_\mu^{'}(0), w_j\right) \rightharpoonup \left(u^{'}, w_j\right)|_{t=0} = \left(u^{'}(0), w_j\right),$$

According (14) :

$$\left(u_\mu^{'}(0), w_j\right) \rightharpoonup (u_1, w_j).$$

We have:

$$\left(u^{'}(0), w_j\right) = (u_1, w_j), \ \forall j.$$

Then:

$$u^{'}(0) = u_1,$$

Whence (8).

3 Uniqueness of solution

3.0.2 Theorem 2 :

It is located in the assumptions of Theorem 1 with:

$$\rho \leq \frac{2}{n-2} \tag{38}$$

(any finite ρ if $n = 2$).Then the solution u obtained in Theorem one is unique.

3.0.3 Preuve :

Let u, v be two solutions, in the sense of Theorem 2, then $w = u - v$ satisfies:

$$\begin{cases} w'' - Lw = |v|^{\rho} v - |u|^{\rho} u, \\ w(0) = 0, \quad w'(0) = 0, \\ w \in L^{\infty}\left(0, T; H_0^1(\Omega) \cap L^p(\Omega)\right), \\ w' \in L^{\infty}\left(0, T; L^2(\Omega)\right). \end{cases}$$

So (28) implies:

$$(w'', v) + a(w, v) = \left(|v|^{\rho} v - |u|^{\rho} u \ , v\right) \quad \forall v \in H_0^1(\Omega).$$

To replace v by w'must $w' \in H_0^1(\Omega)$ for $v \in H_0^1(\Omega)$ but $w' \in L^2(\Omega)$ then we must introduce an auxiliary function:

$$\begin{aligned} \forall s &\in \ ]0, T[\\ \Psi &: \ ]0, T[\to \mathbb{R} \end{aligned}$$

$$t \ \mapsto \ \Psi(t) = \begin{cases} -\int_t^s w(\sigma)\, d\sigma, & t \leq s; \\ 0, & t > s \end{cases}$$

$$\Psi'(t) = w(t); \qquad w_1(t) = \int_0^t w(\sigma)\, d\sigma \text{ if } \forall t \leq s.$$

Thus

$$\Psi(t) = -\int_t^s w(\sigma)\, d\sigma = w_1(t) - w_1(s) \Rightarrow \Psi(0) = -w_1(s)$$

Then (28) gives:

$$\begin{aligned} (w'', \Psi(t)) + a(w, \Psi(t)) &= \left(|v|^{\rho} v - |u|^{\rho} u \ , \Psi(t)\right) \Rightarrow \\ \frac{1}{2}|w(s)|^2 + \frac{1}{2}\|w_1(s)\|^2 &= -\int_0^s \left(|v|^{\rho} v - |u|^{\rho} u \ , \Psi(t)\right) dt \end{aligned}$$

We have

$$\begin{aligned}\left|-\left(|v|^{\rho}v-|u|^{\rho}u\ ,\Psi(t)\right)\right| &\le c\int_{\Omega}\sup\left(|u|^{\rho},|v|^{\rho}\right)|u-v|\,|\Psi(t)|\,dx= \\ &\quad c\int_{\Omega}\sup\left(|u|^{\rho},|v|^{\rho}\right)|w(t)|\,|\Psi(t)|\,dx\end{aligned}$$

According to Holder's inequality we have:

$$c\int_{\Omega}\sup\left(|u|^{\rho},|v|^{\rho}\right)|w(t)|\,|\Psi(t)|\,dx=c\int_{\Omega}\sup\left(|u|^{\rho},|v|^{\rho}\right)|w(t)|\,.$$

$$|w_1(t)-w_1(s)|\,dx\le c\left[\left\||u|^{\rho}\right\|_{L^n(\Omega)}+\left\||v|^{\rho}\right\|_{L^n(\Omega)}\right]$$

As

$$\frac{1}{n}+\frac{1}{q}+\frac{1}{2}=1,$$

Then

$$\frac{1}{q}=\frac{n-2}{2n}\Rightarrow\ q=\frac{2n}{n-2}.$$

But in Theorem 2

$$\rho\le\frac{2}{n-2}\Rightarrow\rho n\le\frac{2n}{n-2}=q\Rightarrow\rho n\le q.$$

We have

$$H_0^1(\Omega)\subset L^q(\Omega)\,,\ \frac{1}{q}=\frac{1}{2}-\frac{1}{n},n\ge3$$

Then

$$\left|\int_{\Omega}\left(|u|^{\rho}u-|v|^{\rho}v\right)\ \Psi(t)\,dx\right|\le.$$

$$c\left(\|u(t)\|^{\rho}+\|v(t)\|^{\rho}\right)\left(\|w_1(t)\|_{L^q(\Omega)}+\|w_1(s)\|_{L^q(\Omega)}\right)\|w(t)\|_{L^2(\Omega)}$$

And as $u,v\in L^{\infty}\left(0,T;H_0^1(\Omega)\right)$ for ($H_0^1(\Omega)\subset L^q(\Omega)$), we have :

$$\left|\int_{\Omega}\left(|u|^{\rho}u-|v|^{\rho}v\right)\ \Psi(t)\,dx\right|\le c\,|w(t)|\left(\|w_1(t)\|+\|w_1(s)\|\right).$$

So

$$|w\,(s)|^2 + \|w_1\,(s)\|^2 \leq c\int\limits_0^s \left(|w\,(t)|^2 + \|w_1\,(t)\|^2\right) dt.$$

According to Gronwall inequality we have:

$$\begin{array}{rcl} |w\,(s)|^2 + \|w_1\,(s)\|^2 & = & 0 \Rightarrow \\ \left\{ \begin{array}{c} w\,(s) = 0 \\ w_1\,(s) = 0 \end{array} \right. & \Rightarrow & \left\{ \begin{array}{c} u = v \\ u' = v' \end{array} \right. . \end{array}$$

Then we have the uniqueness.

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