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Further Study on the Levitin-Polyak Well-Posedness for Vector Equilibrium Problems

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Abstract

In this paper, we study Levitin-Polyak well-posedness for vector equilibrium problems with functional constraints. Two sufficient conditions of (generalized) Levitin-Polyak well-posedness are derived for vector equilibrium problems.

Keywords: Vector equilibrium problems; Levitin-Polyak well-posedness; Approximating solution sequence

1 Introduction and preliminaries

Well-posedness of unconstrained and constrained scalar optimization problems was first introduced and studied in Tykhonov [8] and Levitin and Polyak [4], respectively. Since then, the concept of well-posedness has been generalized to variational inequality problems, equilibrium problems [3, 6]. Very recently, Li [5] and Peng[7] studied Levitin-Polyak-type well-posedness for vector equilibrium problems with abstract constraints and with explicit constraint, respectively. In this paper, we give two easier sufficient conditions to derive Levitin-Polyak well-posedness for vector equilibrium problems. The results in this paper generalize and extend some known results in literature [5, 7].

Let (X, d) be a locally convex Hausdorff topological vector space and (Y, d_1) be a metric space. Let $X_1 \subseteq X$, $K \subseteq Y$ be nonempty and closed sets. Let Z be a convex Hausdorff topological vector space ordered by a pointed, closed and convex cone C with nonempty interior int$C$. Let $h(x, y) : X_1 \times X_1 \to Z$ be a continuous mapping and $g : X_1 \to Y$ be a continuous mapping.

Let

$$X_0 = \{x \in X_1 : g(x) \in K\}.$$

Consider the following vector equilibrium problems with functional constraints:

$$\text{(VEP)} \qquad \begin{array}{c} \text{Find } \bar{x} \in X_0 \text{ such that} \\ h(\bar{x}, y) \notin -intC, \quad \forall y \in X_0. \end{array}$$

Let (P, d_1) be a metric space, $P_1 \subseteq P$ and $p \in P$. In the sequel, we denote by $d_{P_1}(p) = \inf\{d_1(p, p') : p' \in P_1\}$ the distance function from point p to set P_1.

For a convex Hausdorff topological vector space V, we denote by V^* its dual space. For any set $C \subset V$, we will denote the (positive) polar cone of C by

$$C^* = \{\lambda : \lambda(c) \geq 0, \forall c \in C\}$$

Let $e \in intC$ be fixed. Denote

$$C^{*0} = \{\lambda \in C^* : \lambda(e) = 1\}.$$

It is known that C^{*0} is a weak*-compact set[1].

Throughout this paper, we always assume that the feasible set $X_0 \neq \emptyset$ and $h(x, x) = 0, \forall x \in X_1$.

Definition 1.1.

(i) A sequence $\{x_n\} \subseteq X_1$ is called a Levitin-Polyak approximating solution sequence if there exist $\{\epsilon_n\} \subseteq R^1_+$ with $\epsilon_n \to 0$ such that

$$d_{X_0}(x_n) \leq \epsilon_n \tag{1}$$

$$h(x_n, y) + \epsilon_n e \notin -intC, \quad \forall y \in X_0 \tag{2}$$

(ii) $\{x_n\} \subseteq X_1$ is called a generalized Levitin-Polyak approximating solution sequence if there exists $\{\epsilon_n\} \subseteq R^1_+$ with $\epsilon_n \to 0$ such that (2) *holds and*

$$d_K(g(x_n)) \leq \epsilon_n \tag{3}$$

Definition 1.2. (VEP) *is said to be (generalized) Levitin-Polyak well-posed if the solution set $\bar{X}$ of* (VEP) *is nonempty, and for (generalized) Levitin-Polyak approximating solution sequence $\{x_n\}$, there exist a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and $\bar{x} \in \bar{X}$ such that $x_{n_j} \to \bar{x}$.*

Definition 1.3.

(i) A mapping $h(x,y) : X_1 \times X_1 \to Y$ *is said to be C-monotone if for all* $x, y \in X_1$*, the following relation holds:*

$$h(x,y) + h(y,x) \leq_C 0.$$

(ii) $h(x,.)$ *is said to be C-convex on* X_1 *if for all* $y_1, y_2 \in X_1, \forall t \in [0,1]$*, the following relation holds:*

$$h(x, ty_1 + (1-t)y_2) \leq_C th(x,y_1) + (1-t)h(x,y_2).$$

(iii) h *is said to be C-coercive on* $X_3 \subset X_1$*, if* X_3 *is bounded or there exists* $x_0 \in X_3$ *such that*

$$\lim_{x \in X_3, ||x|| \to +\infty} \frac{-h(x,x_0)}{||x||} = +\infty,$$

where $+\infty$ *is an imaginary element satisfying* $+\infty - c \in C, \forall c \in C$*.*

2 Main results

Under C-monotonity and C-coercivity of the mapping h, we give easier verified sufficient conditions to obtain Levitin-Polyak well-posedness for vector equilibrium problems.

Theorem 2.1. Let X be finite dimensional and the solution set $\bar{X}$ be nonempty. Let X_1, X_0 be nonempty closed and convex subsets of X. Suppose that there exist $\delta_1 > 0$ such that h is C-monotone and C-coercive on $X_1(\delta_1) = \{x \in X_1 : d_{X_0}(x) \leq \delta_1\}$. Suppose that $h(x,.)$ is C-convex on $X_1(\delta_1)$ and $h(x,.)$ is Gateaux differentiable on $X_1(\delta_1)$. Further assume that the set $Q = \{x \in X_0 :$ for any $x_1 \in X_1(\delta_1)$, there exists $t_0 \in (0,1)$such that $x_1 + t_0(x_0 - x_1) \in X_0\}$ is nonempty. Then, (VEP) is Levitin-Polyak well-posed.

Proof. Let $\{x_n\}$ be a Levitin-Polyak approximating solution sequence. Then, there exist $\{\epsilon_n\} \subseteq R^1_+$ with $\epsilon_n \to 0$ satisfying (1) and (2). From (1), we can assume without loss of generality that $\{x_n\} \subseteq X_1(\delta_1)$. Let us show by contradiction that $\{x_n\}$ is bounded. Let $x_0' \in Q$. Then, there exists $t_0 \in (0,1)$ such that $x_0 + t_0(x_0' - x_0) \in X_0$, where $x_0 \in X_1(\delta_1)$ is the element in the definition of C-coercivity of h on $X_1(\delta_1)$. Thus, from (2), we have

$$h(x_n, x_0 + t_0(x_0' - x_0)) + \epsilon_n e \notin -intC.$$

Since $h(x,.)$ is C-convex, one has

$$h(x_n, x_0 + t_0(x_0' - x_0)) - (1-t_0)h(x_n, x_0) - t_0 h(x_n, x_0') \in -C.$$

That is

$$(1-t_0)h(x_n,x_0)+t_0h(x_n,x_0^{'})+\epsilon_n e\notin -intC. \quad (4)$$

By the C-monotonicity of h on $X_1(\delta_1)$, we have

$$h(x_n,x_0^{'})+h(x_0^{'},x_n)\in -C \quad (5)$$

The combination of (4) and (5) yields

$$(1-t_0)h(x_n,x_0)-t_0h(x_0^{'},x_n)+\epsilon_n e\notin -intC. \quad (6)$$

By the C-convexity of h on $X_1(\delta_1)$, we obtain from [1] Proposition 1.63 that

$$h(x_0^{'},x_n)\geq_C h(x_0^{'},x_0^{'})+Dh(x_0^{'},x_0^{'})(x_n-x_0^{'}), \quad (7)$$

where $Dh(x_0^{'},x_0^{'})$ is Gateaux derivative at $(x_0^{'},x_0^{'})$. The combination of (6) and (7)and $h(x,x)=0,\forall x\in X_1$ yields

$$(1-t_0)h(x_n,x_0)-t_0Dh(x_0^{'},x_0^{'})(x_n-x_0^{'})+\epsilon_n e\notin -intC. \quad (8)$$

If $\{x_n\}$ is unbounded, we assume without loss of generality that $||x_n||\to+\infty$. Then, from the C-coercivity of h on $X_1(\delta_1)$, we have

$$\lim_{x_n\in X_3,||x_n||\to+\infty}\frac{h(x_n,x_0)}{||x_n||}=-\infty,$$

Furthermore, it is obvious that $\{\frac{<Dh(h(x_0^{'},x_0^{'})),x_n-x_0^{'}>}{||x_n||}\}$ is bounded. Dividing (8) by $||x_n||$ and passing to the limit, a contradiction arises. Hence, $\{x_n\}$ is bounded. Thus, we can find a subsequence $\{x_{n_j}\}$ and $\bar{x}\in X_1(\delta_1)$ such that $x_{n_j}\to\bar{x}$. Taking the limit in (1) (with x_n replaced by x_{n_j}), we have $\bar{x}\in X_0$. Further arguing as in the proof of Theorem 2.1 shows that $\bar{x}\in\bar{X}$. □

Corollary 2.1. Let X be finite dimensional and the solution set $\bar{X}$ be nonempty. Let X_1,X_0 be nonempty closed and convex subsets of X. Suppose that there exist $\delta_1>0$ such that h is C-monotone and C-coercive on $X_2(\delta_1)=\{x\in X_1 : d_K(g(x))\leq\delta_1\}$. Suppose that $h(x,.)$ is C-convex on $X_2(\delta_1)$ and $h(x,.)$ is Gateaux differentiable on $X_2(\delta_1)$. Further assume that the set $Q=\{x\in X_0 : \text{for any } x_1\in X_2(\delta_1), \text{there exists } t_0\in(0,1) \text{such that } x_1+t_0(x_0-x_1)\in X_0\}$ is nonempty. Then, (VEP) is generalized Levitin-Polyak well-posed.

Assumption 2.1. Suppose that X is finite dimensional, X_1 is a nonempty closed and convex subset of X. Suppose that Y is a normed space and $K\subset Y$

is a nonempty, closed and convex cone with nonempty interior int$K$. Suppose that $g(x)$ is K-concave on X_1. Suppose that $h(x,y)$ is C-monotone and $h(x,.)$ is C-convex.

Let $\lambda \in C^{*0}$. Consider the following (scalar) equilibrium problem of finding $\bar{x} \in X_0$ such that:

$$(EP_\lambda) \qquad \lambda(h(\bar{x},y)) \geq 0, \forall y \in X_0.$$

Denote by $\bar{X}_\lambda$ the solution set EP_λ.

For $\lambda \in C^{*0}$, define

$$R_\lambda = \bigcap_{y \in X_0} \{d \in {X_0}^\infty : \lambda(h(y, y+td)) \leq 0, \forall t > 0\}.$$

The next lemma follows immediately from [2].

Lemma 2.1. *Let Assumption 2.1 hold and $\lambda \in C^{*0}$. Then, $\bar{X}_\lambda$ is nonempty and compact if and only if $R_\lambda = \{0\}$.*

Theorem 2.2. Let Assumption 2.1 hold. If for any $\lambda \in C^{*0}$, the solution set $\bar{X}_\lambda$ of EP_λ is nonempty and compact, then (VEP) is Levitin-Polyak well-posed.

Proof. Let $\{x_k\} \subset X_1$ satisfy (1) and (2). From (1), we can assume that without loss of generality that $\{x_k\} \subset X_1(\delta_1)$. Now we show that $\{x_k\}$ is bounded. Suppose to the contrary that $\{x_k\}$ is unbounded. We assume without loss of generality that $||x_k|| \to +\infty$. Note that

$$h(x_k, y) + \epsilon_k e \notin -intC, \forall y \in X_0.$$

By the C-convexity of $h(x_k, \cdot)$ on X_1, we have $\lambda_k \in C^{*0}$ such that

$$\lambda_k(h(x_k, y)) \geq -\epsilon_k, \forall y \in X_0.$$

Since C^{*0} is $\omega^* -$ compact, we can assume without loss of generality that

$$\omega^* - \lim_{k \to +\infty} \lambda_k = \bar{\lambda} \in C^{*0}$$

By the C-monotonicity of $h(x,y)$ on X_0, we have that

$$\lambda(h(x_k, y) + h(y, x_k)) \leq 0, \forall y \in X_0, \lambda \in C^{*0}.$$

For $\lambda_k \in C^{*0}$, one has that

$$\lambda_k(h(y, x_k)) \leq \epsilon_k, \forall y \in X_0.$$

We can assume without loss of generality that

$$\lim_{k\to+\infty}\frac{x_k-y}{||x_k-y||}=d$$

Thus, $d\in X_1(\delta_1)^\infty$ (In fact, it can be easily shown that $d\in X_0^\infty$) and $||d||=1$. For any $y\in X_0$ and $x_k\in X_1(\delta_1)$ and $0<t<||x_k-y||$, we have

$$\begin{aligned}\bar{\lambda}(h(y,y+td)) &= \lim_{k\to+\infty}\lambda_k(h(y,y+t\tfrac{x_k-y}{||x_k-y||}))\\ &\leq \limsup_{k\to+\infty}\left[\tfrac{||x_k-y||-t}{||x_k-y||}\lambda_k(h(y,y))+\tfrac{t}{||x_k-y||}\lambda_k(h(y,x_k))\right]\leq 0\end{aligned}$$

since $h(y,y)=0$ and $\lambda_k(h(y,x_k))\leq\epsilon_k$. That is, $0\neq d\in R_{\bar{\lambda}}$, contradicting the fact that the solution set of the problem $EP_{\bar{\lambda}}$ is compact. Consequently, $\{x_k\}$ is bounded. There exist a subsequence $\{x_{k_j}\}$ of $\{x_k\}$ and $\bar{x}\in X_1(\delta_1)$ such that $x_{k_j}\to\bar{x}$. Taking the limit in (1)(with x_k replaced by x_{k_j}), we have $\bar{x}\in X_0$. Moreover, Taking the limit in (2) (with x_k replaced by x_{k_j}), we obtain $h(\bar{x},y)\notin -intC,\forall y\in X_0$. So $\bar{x}\in\bar{X}$. □

Corollary 2.2. Let Assumption 2.1 hold. If for any $\lambda\in C^{*0}$, the solution set $\bar{X}_\lambda$ of EP_λ is nonempty and compact, then (VEP) is generalized Levitin-Polyak well-posed.

Remark 2.1. Necessary and/or sufficient conditions for the nonemptiness and compactness of a scalar monotone equilibrium problem were given in [2]. By Theorem 2.2, the sufficient conditions can be applied to verify (generalized) Levitin-Polyak well-posedness of (VEP).

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