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A Variational Iterative Method for Solving the Linear and Nonlinear Klein-Gordon Equations

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Abstract

In this article, variational iterative method (VIM) is presented as an alternative method for solving the linear and nonlinear Klein Gordon equations. The method is demonstrated by several physical models of Klein Gordon equations. The present approach is highly accurate and converges rapidly.

Keywords: Variational iterative method, Linear and Nonlinear Klein-Gordon equations, Lagrange multiplier

1 Introduction

In 1999, the variational iterative method (VIM) was first time proposed by J.H.He $[1-10]$. Recently, this method is used by many researchers to study linear and nonlinear ordinary, partial and integral equations. This method is more powerful than existing techniques such as ADM $[11-20]$ and perturbation etc. The present technique require no restrictive assumptions that are used to handle nonlinear terms.

The VIM does not require specific transformation for nonlinear terms as required by other techniques. Our aim in this article is to apply VIM to find the exact solutions for Klein Gordon equations which has attracted much attention in studying solitons and condensed matter physics, investigating the interaction of solitons in collisionless plasma, quantum mechanics, relativistic physics, dispersive phenomena, the recurrence of initial state, and examining the nonlinear wave equations.

2 He's variational iterative method

To illustrate variational iterative method (VIM), we consider the general nonlinear equation

$$Lu(x,t) + Ru(x,t) + Nu(x,t) = g(x,t), \tag{1}$$

where L is linear operator R, is the remaining linear operator, $Nu(x,t)$ represents a general non-linear operator and $g(x,t)$ is source term. According to VIM $[1-10]$, a correction functional can be constructed as follow

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\xi,t)\left(Lu_n(x,\xi) + Ru_n(x,\xi) + N\widetilde{u_n}(x,\xi) - g(x,\xi)\right) d\xi,\ n \geq 0 \tag{2}$$

where $\lambda(\xi,t)$ is a general Lagrange multiplier which can be identified optimally via the variational theory [6]. The function $\widetilde{u_n}$ is a restrictive variation which means $\delta\widetilde{u_n} = 0$.Therefore, we first determine the Lagrange multiplier λ that will be identified optimally via integration by parts. The initial guess u_0 may be selected by any function that satisfies the two prescribed initial conditions [in this case]. The other components of the solution can easily be determined iteratively and consequently we may obtained exact solution by using

$$u = \lim_{n\to\infty} u_n. \tag{3}$$

We will apply VIM to four physical models. The effectiveness and the usefulness of the present method is demonstrated by finding the exact solutions to these four physical models that will be investigated.

3 The homogeneous linear Klein Gordon equations

Example 1. Consider homogeneous linear Klein Gordon equation [15]

$$\text{PDE} \qquad u_{tt} - u_{xx} + u = 0, \tag{4}$$

$$\text{I.C} \quad u(x,0) = 0, u_t(x,0) = x. \tag{5}$$

According to variational iterative method (VIM) $[1-10]$, a correct functional for Eq. (4) can be constructed as follows

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(t,\xi)\left(\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2} - \frac{\partial^2 \widetilde{u_n}(x,\xi)}{\partial x^2} + u_n(x,\xi)\right) d\xi, \tag{6}$$

where λ is Lagrange multiplier, $\widetilde{u}$ is restrictive variation, that is $\delta\widetilde{u_n} = 0$. Making the above correction functional stationary, and noting $\delta\widetilde{u_n} = 0$, we get the following stationary conditions

$$\frac{\partial^2\lambda(t,\xi)}{\partial\xi^2} = 0, \tag{7}$$

$$1 - \left.\frac{\partial\lambda(t,\xi)}{\partial\xi}\right|_{t=\xi} = 0, \tag{8}$$

$$\lambda(t,\xi)|_{t=\xi} = 0. \tag{9}$$

The Lagrange multiplier, therefore can be identified as

$$\lambda(\xi,t) = \xi - t. \tag{10}$$

Using Eq. (10) in Eq. (6) leads to the following recursive relation

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (\xi - t)\left(\frac{\partial^2 u_n(x,\xi)}{\partial\xi^2} - \frac{\partial^2\widetilde{u_n}(x,\xi)}{\partial x^2} + u_n(x,\xi)\right) d\xi. \tag{11}$$

We start with an initial approximation

$$u_0\left(x,t\right) = u\left(x,0\right) + tu_t\left(x,0\right) = xt. \tag{12}$$

and obtain the following successive approximations

$$u_1(x,t) = xt - \frac{xt^3}{3!}, \tag{13}$$

$$u_2(x,t) = xt - \frac{xt^3}{3!} + \frac{xt^5}{5!}, \tag{14}$$

$$u_3(x,t) = xt - \frac{xt^3}{3!} + \frac{xt^5}{5!} - \frac{xt^7}{7!}, \tag{15}$$

$$u_n(x,t) = x\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + ... + \frac{(-1)^n t^{2n+1}}{(2n+1)!}\right). \tag{16}$$

The approximate solution

$$u = \lim_{n\to\infty} u_n = x\sin x. \tag{17}$$

is obtained upon using the Taylor expansion of $\sin x$. The solution obtained above is same as given by Wazwaz [15].

Example 2. Consider another homogeneous linear Klein Gordon equation with initial conditions [15]

$$\text{PDE} \qquad u_{tt} - u_{xx} + u = 0, \tag{18}$$

$$\text{I.C } u(x,0) = 0, u_t(x,0) = \cosh x. \tag{19}$$

Similarly we can establish an iterative formula in the form

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (\xi - t)\left(\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2} - \frac{\partial^2 \widetilde{u_n}(x,\xi)}{\partial x^2} + u_n(x,\xi)\right) d\xi. \tag{20}$$

We will start with initial approximation given below

$$u_0\,(x,t) = u\,(x,0) + tu_t\,(x,0) = t\cosh x. \tag{21}$$

The other successive approximations are

$$u_1\,(x,t) = t\cosh x, \tag{22}$$

$$u_2\,(x,t) = t\cosh x, \tag{23}$$

$$\vdots$$

$$u_n\,(x,t) = t\cosh x. \tag{24}$$

The VIM admits that

$$u = \lim_{n->\infty} u_n, \tag{25}$$

which gives the exact solution

$$u(x,t) = t\cosh x. \tag{26}$$

4 The inhomogeneous linear Klein Gordon equations

Example 3. Consider inhomogeneous linear Klein Gordon equation with initial conditions [15]

$$\text{PDE} \qquad u_{tt} - u_{xx} + u = 2\sin x, \tag{27}$$

$$\text{I.C} \quad u(x,0) = \sin x, u_t(x,0) = 1. \tag{28}$$

We obtain the following formulation by variational iterative method

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (\xi - t)\left(\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2} - \frac{\partial^2 \widetilde{u_n}(x,\xi)}{\partial x^2} + u_n(x,\xi) - 2\sin x\right) d\xi. \tag{29}$$

Beginning with the initial approximation

$$u_0\,(x,t) = \sin x + t, \tag{30}$$

the following approximations are obtained very easily

$$u_1\,(x,t) = \sin x + t - \frac{t^3}{3!}, \tag{31}$$

$$u_2\,(x,t) = \sin x + t - \frac{t^3}{3!} + \frac{t^5}{5!}, \tag{32}$$

$$u_3\,(x,t) = \sin x + t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!}, \tag{33}$$

$$\vdots$$

$$u_n\,(x,t) = \sin x + \left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \ldots.. + \frac{(-1)^n t^{2n+1}}{(2n+1)!}\right). \tag{34}$$

We therefore have the solution

$$u = \lim_{n\to\infty} u_n, \tag{35}$$

$$u(x,t) = \sin x + \sin t. \tag{36}$$

which is the exact solution as obtained by Wazwaz [15] by using ADM. While applying ADM there appear noise terms phenomena which does not appears in VIM. So we obtain straightforward approximate solutions to Eqs. $(27-28)$.

Example 4. Consider one dimensional inhomogeneous linear Klein Gordon equation with initial conditions [15]

$$\text{PDE} \qquad u_{tt} - u_{xx} + u = -\cos x \sin t, \tag{37}$$

$$\text{I.C} \quad u(x,0) = 0, u_t(x,0) = \cos x. \tag{38}$$

By the same manipulations as illustrated in Example 1, we can obtain the following iterative formula

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (\xi - t)\left(\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2} - \frac{\partial^2 \widetilde{u_n}(x,\xi)}{\partial x^2} + u_n(x,\xi) + \cos x \sin \xi\right) d\xi. \quad (39)$$

We start with an initial approximation

$$u_0\,(x,t) = t\cos x, \quad (40)$$

which satisfies the initial conditions. With the help of recursive relation (2) the successive approximations are obtained as:

$$u_1\,(x,t) = \cos x \cos t, \quad (41)$$

$$\begin{aligned} u_2\,(x,t) &= \cos x \cos t, \quad (42) \\ &: \\ &: \end{aligned}$$

$$u_n\,(x,t) = \cos x \cos t. \quad (43)$$

The approximate solution leads to

$$u = \lim_{n\to\infty} u_n, \quad (44)$$

$$u(x,t) = \cos x \cos t. \quad (45)$$

which gives the exact solution as obtained by Wazwaz [15] using ADM. In this example, solution obtained by Wazwaz [15] using ADM contains noise terms but there is no such term in the solution obtained by VIM.

5 The inhomogeneous nonlinear Klein Gordon equations

Example 5. Consider the inhomogeneous nonlinear Klein Gordon equation [15]

$$\text{PDE} \quad u_{tt} - u_{xx} + u^2 = x^2 t^2, \quad (46)$$
$$\text{I.C} \quad u(x,0) = 0, u_t(x,0) = x. \quad (47)$$

By the same method as illustrated above, we obtain the following iterative formula

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (\xi - t)\left(\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2} - \frac{\partial^2 \widetilde{u_n}(x,\xi)}{\partial x^2} + \widetilde{u_n}^2(x,\xi) - x^2t^2\right) d\xi. \tag{48}$$

We begin with an initial approximation

$$u_0(x,t) = xt, \tag{49}$$

we obtain successive approximations

$$u_1(x,t) = xt, \tag{50}$$

$$u_2(x,t) = xt, \tag{51}$$

$$\vdots$$

$$u_n(x,t) = xt. \tag{52}$$

Its approximate solution is

$$u = \lim_{n\to\infty} u_n, \tag{53}$$

$$u(x,t) = xt.$$

which is actually the exact solution.

Example 6. Consider another inhomogeneous nonlinear Klein Gordon equation [15]

$$\text{PDE} \quad u_{tt} - u_{xx} + u^2 = 2x^2 - 2t^2 + x^4t^4, \tag{54}$$

$$\text{I.C} \quad u(x,0) = 0, u_t(x,0) = 0. \tag{55}$$

The correction functionals for (54) read

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (\xi - t)\left(\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2} - \frac{\partial^2 \widetilde{u_n}(x,\xi)}{\partial x^2} + \widetilde{u_n}^2(x,\xi) - 2x^2 + 2\xi^2 - x^4\xi^4\right) d\xi. \tag{56}$$

We can select $u_0(x,t) = x^2t^2$, by using the given initial values. Accordingly, we obtain the following successive approximations:

$$\begin{aligned} u_1(x,t) &= x^2t^2, \\ u_2(x,t) &= x^2t^2, \\ &\vdots \\ &\vdots \end{aligned} \tag{58}$$

$$u_n(x,t) = x^2t^2. \tag{59}$$

The VIM admits the use of

$$u = \lim_{n\to\infty} u_n, \tag{60}$$

$$u(x,t) = x^2t^2. \tag{61}$$

which is exact solution.

6 The Sine-Gordon equation

Example 7. Consider Sine-Gordon equation

$$\text{PDE} \qquad u_{tt} - u_{xx} = \sin u, \tag{62}$$

$$\text{I.C} \quad u(x,0) = \tfrac{\pi}{2}, u_t(x,0) = 0. \tag{63}$$

The correction functional for Eq. (62)

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (\xi - t)\left(\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2} - \frac{\partial^2 \widetilde{u_n}(x,\xi)}{\partial x^2} - \sin u\right) d\xi. \tag{64}$$

We take an initial approximation of the form

$$u_0(x,t) = \frac{\pi}{2} + t. \tag{65}$$

Using this initial approximation in (64) we obtain the following successive approximations

$$u_1(x,t) = \frac{\pi}{2} + t + 1 - \cos t, \tag{66}$$

$$u_2(x,t) = \frac{\pi}{2} + t + 1 - \cos t + \sin t - \frac{3}{4}t - \frac{\sin 2t}{8} + ..., \tag{67}$$

So we obtain series solution in case of Sine-Gordon equation upto second order approximation

$$u = \lim_{n\to\infty} u_n, \tag{68}$$

$$u(x,t) = \frac{\pi}{2} + t + 1 - \cos t + \sin t - \frac{3}{4}t - \frac{\sin 2t}{8} + ..., \tag{69}$$

It means in case of Sine-Gordon equation VIM also gives series solution.

7 Conclusion

The aim of this work is to apply this powerful Variational iterative method to investigate four physical models. The main goal is to show the usefulness of the VIM. The Variational iterative method reduces the size of calculations and there is no need of expanding nonlinearities in terms of Adomian polynomials as we do in ADM [15]. Nonlinear scientific models are arise frequently in engineering problems for expressing nonlinear phenomena. He's variational iterative method provides an efficient method for handling this nonlinear behavior. He's variational iterative method work much effectively, a few approximations can be used to achieved a high degree of accuracy.

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