Applied Mathematical Sciences, Vol. 4, 2010, no. 40, 1955 - 1966

Hyper $\mathcal{N}$-Ideals of Hyper BCK-Algebras

Young Bae Jun

Department of Mathematics Education (and RINS)
Gyeongsang National University, Chinju 660-701, Korea
skywine@gmail.com

Min Su Kang*

Department of Mathematics
Hanyang University, Seoul 133-791, Korea
sinchangmyun@hanmail.net

Abstract

The notions of (weak, s-weak, strong) hyper $\mathcal{N}$-ideals are introduced, and several related properties are investigated. Relations among weak hyper $\mathcal{N}$-ideals, s-weak hyper $\mathcal{N}$-ideals and strong hyper $\mathcal{N}$-ideals are discussed. A characterization of a weak hyper $\mathcal{N}$-ideal is provided.

Mathematics Subject Classification: 06F35, 03G25.

Keywords: Hyper BCK-algebra, (strong, weak) hyper BCK-ideal, (weak, s-weak, strong) hyper $\mathcal{N}$-ideal.

1 Introduction

A (crisp) set A in a universe X can be defined in the form of its characteristic function $\mu_A : X \to \{0, 1\}$ yielding the value 1 for elements belonging to the set A and the value 0 for elements excluded from the set A. So far most of the generalization of the crisp set have been conducted on the unit interval $[0, 1]$ and they are consistent with the asymmetry observation. In other words, the generalization of the crisp set to fuzzy sets relied on spreading positive information that fit the crisp point $\{1\}$ into the interval $[0, 1]$. Because no negative meaning of information is suggested, we now feel a need to deal with negative information. To do so, we also feel a need to supply mathematical tool. To attain such object, Jun et al. [2] introduced a new function which is called

*Corresponding author.

negative-valued function, and constructed $\mathcal{N}$-structures. They applied $\mathcal{N}$-structures to BCK/BCI-algebras, and discussed $\mathcal{N}$-subalgebras and $\mathcal{N}$-ideals in BCK/BCI-algebras. The hyper structure theory (called also multialgebras) was introduced in 1934 by Marty [7] at the 8th congress of Scandinavian Mathematiciens. In [6], Jun et al. applied the hyper structures to BCK-algebras, and introduced the concept of a hyper BCK-algebra which is a generalization of a BCK-algebra. They also introduced the notion of a (weak, s-weak, strong) hyper BCK-ideal, and gave relations among them. Harizavi [1] studied prime weak hyper BCK-ideals of lower hyper BCK-semilattices. Jun et al. discussed the notion of hyperatoms and scalar elements of hyper BCK-algebras (see [3]).

In this paper, we introduce the notions of (weak, s-weak, strong) hyper $\mathcal{N}$-ideals, and investigate several related properties. We provide relations among weak hyper $\mathcal{N}$-ideals, s-weak hyper $\mathcal{N}$-ideals and strong hyper $\mathcal{N}$-ideals. We also discuss a characterization of a weak hyper $\mathcal{N}$-ideal.

2 Preliminaries

We include some elementary aspects of hyper BCK-algebras that are necessary for this paper, and for more details we refer to [4], [5], and [6].

Let H be a nonempty set endowed with a hyperoperation "$\circ$". For two subsets A and B of H, denote by $A \circ B$ the set $\bigcup_{a\in A, b\in B} a \circ b$. We shall use $x \circ y$ instead of $x \circ \{y\}$, $\{x\} \circ y$, or $\{x\} \circ \{y\}$.

By a *hyper BCK-algebra* we mean a nonempty set H endowed with a hyperoperation "$\circ$" and a constant 0 satisfying the following axioms:

(HK1) $(x \circ z) \circ (y \circ z) \ll x \circ y$,

(HK2) $(x \circ y) \circ z = (x \circ z) \circ y$,

(HK3) $x \circ H \ll \{x\}$,

(HK4) $x \ll y$ and $y \ll x$ imply $x = y$,

for all $x, y, z \in H$, where $x \ll y$ is defined by $0 \in x \circ y$ and for every $A, B \subseteq H$, $A \ll B$ is defined by $\forall a \in A$, $\exists b \in B$ such that $a \ll b$. In such case, we call "$\ll$" the *hyperorder* in H.

Note that the condition (HK3) is equivalent to the condition:

$$(\forall x, y \in H)\ (x \circ y \ll \{x\}). \tag{2.1}$$

In any hyper BCK-algebra H, the following hold:

(a1) $x \circ 0 \ll \{x\},\ 0 \circ x \ll \{0\}\ 0 \circ 0 \ll \{0\}$, (a2) $(A \circ B) \circ C = (A \circ C) \circ B,\ A \circ B \ll A,\ 0 \circ A \ll \{0\}$,

(a3) $0 \circ 0 = \{0\}$,

(a4) $0 \ll x$ and $x \ll x$,

(a5) $A \ll A$,

(a6) $A \subseteq B \ \Rightarrow\ A \ll B$,

(a7) $0 \circ x = \{0\}$ and $0 \circ A = \{0\}$,

(a8) $A \ll \{0\} \ \Rightarrow\ A = \{0\}$,

(a9) $A \circ B \ll A$,

(a10) $x \in x \circ 0$,

(a11) $x \circ 0 \ll \{y\} \ \Rightarrow\ x \ll y$,

(a12) $y \ll z \ \Rightarrow\ x \circ z \ll x \circ y$,

(a13) $x \circ y = \{0\} \ \Rightarrow\ (x \circ z) \circ (y \circ z) = \{0\},\ x \circ z \ll y \circ z$,

(a14) $A \circ \{0\} = \{0\} \ \Rightarrow\ A = \{0\}$

for all $x, y, z \in H$ and for all nonempty subsets A, B and C of H.

A nonempty subset I of a hyper BCK-algebra H is said to be a *hyper BCK-ideal* of H if it satisfies

(I1) $0 \in I$,

(I2) $(\forall x \in H)\ (\forall y \in I)\ (x \circ y \ll I \ \Rightarrow\ x \in I)$.

A nonempty subset I of a hyper BCK-algebra H is called a *strong hyper BCK-ideal* of H if it satisfies (I1) and

(I3) $(\forall x \in H)\ (\forall y \in I)\ ((x \circ y) \cap I \neq \emptyset \ \Rightarrow\ x \in I)$.

Note that every strong hyper BCK-ideal of a hyper BCK-algebra is a hyper BCK-ideal.

A nonempty subset I of a hyper BCK-algebra H is called a *weak hyper BCK-ideal* of H if it satisfies (I1) and

(I4) $x \circ y \subseteq I$ and $y \in I$ imply $x \in I$ for all $x, y \in H$.

Table 1: Cayley table

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0,a\}$	$\{0,a\}$
b	$\{b\}$	$\{a,b\}$	$\{0,a,b\}$

3 Hyper $\mathcal{N}$-ideals

Denote by $\mathcal{F}(H,[-1,0])$ the collection of functions from a set H to $[-1,0]$. We say that an element of $\mathcal{F}(H,[-1,0])$ is a *negative-valued function* from H to $[-1,0]$ (briefly, $\mathcal{N}$*-function* on H). By an $\mathcal{N}$*-structure* we mean an ordered pair (H,f) of H and an $\mathcal{N}$-function f on H. In what follows, let H denote a hyper BCK-algebra and f an $\mathcal{N}$-function on H unless otherwise specified. For any subset S of H, we denote by $\bigvee\limits_{s\in S} f(s)$ and $\bigwedge\limits_{s\in S} f(s)$ the $\sup\limits_{s\in S} f(s)$ and $\inf\limits_{s\in S} f(s)$, respectively.

Definition 3.1. A *hyper* $\mathcal{N}$*-ideal* of H is an $\mathcal{N}$-structure (H,f) in which f satisfies the following two conditions:

$$(\forall x,y\in H)\ \big(x \ll y\ \Rightarrow\ f(x)\le f(y)\big), \tag{3.1}$$

$$(\forall x,y\in H)\ \Big(f(x)\le \max\Big\{\bigvee_{b\in x\circ y} f(b),\ f(y)\Big\}\Big). \tag{3.2}$$

Example 3.2. Let $H=\{0,a,b\}$ be a hyper BCK-algebra with the Cayley table which is given in Table 1. Let (H,f) be an $\mathcal{N}$-structure in which f is given by

$$f=\begin{pmatrix} 0 & a & b\\ -0.7 & -0.4 & -0.2\end{pmatrix}.$$

It is easily verified that (H,f) is a hyper $\mathcal{N}$-ideal of H.

Definition 3.3. An $\mathcal{N}$-structure (H,f) in H is called a *strong hyper* $\mathcal{N}$*-ideal* of H if the following inequalities are valid:

$$(\forall x,y\in H)\ \Big(\bigvee_{c\in x\circ x} f(c)\le f(x)\le \max\Big\{\bigwedge_{d\in x\circ y} f(d),\ f(y)\Big\}\Big). \tag{3.3}$$

Example 3.4. Let $H=\{0,a,b\}$ be a hyper BCK-algebra with the Cayley table which is given in Table 2. Let (H,f) be an $\mathcal{N}$-structure in which f is

Table 2: Cayley table

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{a\}$
b	$\{b\}$	$\{b\}$	$\{0, b\}$

given by

$$f = \begin{pmatrix} 0 & a & b \\ -0.8 & -0.6 & -0.3 \end{pmatrix}.$$

It is easily verified that (H, f) is a strong hyper $\mathcal{N}$-ideal of H.

Definition 3.5. An $\mathcal{N}$-structure (H, f) in H is called an *s-weak hyper $\mathcal{N}$-ideal* of H if it satisfies the following two conditions:

$$(\forall x \in H)\ \big(f(0) \leq f(x)\big), \tag{3.4}$$

$$(\forall x, y \in H)\ \big(\exists\, b \in x \circ y)\ (f(x) \leq \max\{f(b), f(y)\}\big). \tag{3.5}$$

Definition 3.6. An $\mathcal{N}$-structure (H, f) in H is called a *weak hyper $\mathcal{N}$-ideal* of H if it satisfies

$$(\forall x, y \in H)\ \Big(f(0) \leq f(x) \leq \max\Big\{\bigvee_{b \in x \circ y} f(b),\ f(y)\Big\}\Big). \tag{3.6}$$

Theorem 3.7. *Every s-weak hyper $\mathcal{N}$-ideal is a weak hyper $\mathcal{N}$-ideal.*

Proof. Let an $\mathcal{N}$-structure (H, f) in H be an s-weak hyper $\mathcal{N}$-ideal of H and let $x, y \in H$. Then there exist $b \in x \circ y$ such that

$$f(x) \leq \max\{f(b), f(y)\}.$$

Since $f(b) \leq \bigvee_{d \in x \circ y} f(d)$, it follows that

$$f(x) \leq \max\Big\{\bigvee_{d \in x \circ y} f(d),\ f(y)\Big\}.$$

Hence (H, f) is a weak hyper $\mathcal{N}$-ideal of H. □

The converse of Theorem 3.7 is not true as seen in the following example.

Example 3.8. Let $H = \mathbb{N} \cup \{0, \alpha\}$, where $\alpha(\neq 0) \notin \mathbb{N}$. Define a hyperoperation "$\circ$" on H as follows:

$$x \circ y := \begin{cases} \{0\} & \text{if } \ x = 0, \\ \{0, x\} & \text{if } \ (x \leq y,\ x \in \mathbb{N}) \text{ or } (x \in \mathbb{N},\ y = \alpha), \\ \{x\} & \text{if } \ x > y,\ x \in \mathbb{N}, \\ \{0\} \cup \mathbb{N} & \text{if } \ x = y = \alpha, \\ \mathbb{N} & \text{if } \ x = \alpha,\ y \in \mathbb{N}, \\ \{\alpha\} & \text{if } \ x = \alpha,\ \ y = 0. \end{cases}$$

Then $(H, \circ, 0)$ is a hyper BCK-algebra. Let (H, f) be an $\mathcal{N}$-structure in which f is given by

$$f = \begin{pmatrix} 0 & 1 & 2 & 3 & \cdots & \alpha \\ -4+3 & -4+3.1 & -4+3.14 & -4+3.141 & \cdots & -4+\pi \end{pmatrix}.$$

Then (H, f) is a weak hyper $\mathcal{N}$-ideal of H, but it is not an s-weak hyper $\mathcal{N}$-ideal of H.

An $\mathcal{N}$-structure (H, f) in H is said to satisfy the **sup** *property* if for any nonempty subset T of H there exists $x_0 \in T$ such that $f(x_0) = \bigvee_{x \in T} f(x)$.

Note that, in a finite hyper BCK-algebra, every $\mathcal{N}$-structure satisfies the **sup** property. The following example shows that there exists an $\mathcal{N}$-structure which does not satisfy the **sup** *property.*

Example 3.9. Let $H = \mathbb{N} \cup \{0\} \cup \{\alpha, \beta\}$, where $\alpha(\neq 0) \notin \mathbb{N}$ and $\beta(\neq 0) \notin \mathbb{N}$ with $\alpha \neq \beta$. Define a hyperoperation "$\circ$" on H as follows:

$$x \circ y := \begin{cases} \{0, x\} & \text{if } \ (x \leq y,\ x, y \in \mathbb{N} \cup \{0\}) \text{ or } (x \in \mathbb{N} \cup \{0\}, y \in \{\alpha, \beta\}) \\ \{x\} & \text{if } \ x > y,\ x, y \in \mathbb{N} \cup \{0\}, \\ \{\alpha\} & \text{if } \ x = \alpha,\ y \neq \alpha, \\ \{\beta\} & \text{if } \ x = \beta,\ y \neq \beta, \\ \{0\} & \text{if } \ x = y = \alpha \text{ or } x = y = \beta. \end{cases}$$

Then $(H, \circ, 0)$ is a hyper BCK-algebra (see [1]). Consider an $\mathcal{N}$-structure (H, f) in which f is defined by

$$f = \begin{pmatrix} 0 & 1 & 2 & 3 & \cdots & \alpha & \beta \\ -2 & -2+1.4 & -2+1.41 & -2+1.414 & \cdots & 0 & 0 \end{pmatrix}.$$

Let $T = \mathbb{N} \cup \{0\} \subseteq H$. Then $\bigvee_{y \in T} f(y) = -2 + \sqrt{2}$. But there does not exist $x_0 \in T$ such that $f(x_0) = -2 + \sqrt{2}$. Hence (H, f) does not satisfy the **sup** *property.*

The following example shows that there exists an $\mathcal{N}$-structure which satisfies the **sup** *property.*

Example 3.10. Let $H = \mathbb{N} \cup \{0\}$ and define a hyperoperation "$\circ$" on H as follows:

$$x \circ y := \begin{cases} \{0, x\} & \text{if } x \leq y, \\ \{x\} & \text{if } x > y. \end{cases}$$

Then $(H, \circ, 0)$ is a hyper BCK-algerba.

(1) Let (H, f) be an $\mathcal{N}$-structure in which f is defined by

$$f(n) := \begin{cases} 0 & \text{if } n \in \{0, 2, 4, \cdots\}, \\ \alpha & \text{if } n \in \{1, 3, 5, \cdots\}. \end{cases}$$

with $\alpha \in [-1, 0)$. Then (H, f) satisfies the **sup** *property.*

(2) Let $(H; g)$ be an $\mathcal{N}$-structure where

$$g = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & \cdots \\ -2+1 & -2+1.7 & -2+1.73 & -2+1.732 & -2+1.7320 & \cdots \end{pmatrix}.$$

Then $(H; g)$ is a hyper $\mathcal{N}$-ideal of H. Let $T = \mathbb{N} \subseteq H$. Then $\bigvee_{y \in T} g(y) = -2 + \sqrt{3}$. But there does not exist $x_0 \in T$ such that $g(x_0) = -2 + \sqrt{3}$. Hence $(H; g)$ does not satisfy the **sup** *property.*

A weak hyper $\mathcal{N}$-ideal may not be an s-weak hyper $\mathcal{N}$-ideal . But we have the following proposition.

Proposition 3.11. *If (H, f) is a weak hyper $\mathcal{N}$-ideal of H satisfying the* **sup** *property, then (H, f) is an s-weak hyper $\mathcal{N}$-ideal of H.*

Proof. Since (H, f) satisfies the **sup** property, there exists $a_0 \in x \circ y$ such that $f(a_0) = \bigvee_{a \in x \circ y} f(a)$. It follows from (3.6) that

$$f(x) \leq \max\Big\{ \bigvee_{a \in x \circ y} f(a),\ f(y) \Big\} = \max\{f(a_0), f(y)\}.$$

This completes the proof. □

Since every $\mathcal{N}$-structure (H, f) in H satisfies the **sup** property in a finite hyper BCK-algebra H, the concept of weak hyper $\mathcal{N}$-ideals and s-weak hyper $\mathcal{N}$-ideals coincide in a finite hyper BCK-algebra.

Proposition 3.12. *Let (H, f) be a strong hyper $\mathcal{N}$-ideal of H and let $x, y \in H$. Then*

(1) $f(0) \leq f(x)$.

(2) $x \ll y \implies f(x) \leq f(y)$.

(3) $(\forall b \in x \circ y)\ (f(x) \leq \max\{f(b), f(y)\})$.

Proof. (1) Since $0 \in x \circ x$ for all $x \in H$, we have $f(0) \leq \bigvee_{b \in x \circ x} f(b) \leq f(x)$, which proves (1).

(2) Let $x, y \in H$ be such that $x \ll y$. Then $0 \in x \circ y$ and so $\bigwedge_{d \in x \circ y} f(d) \leq f(0)$. It follows from (3.3) and (1) that

$$f(x) \leq \max\Big\{\bigwedge_{d \in x \circ y} f(d),\ f(y)\Big\} \leq \max\{f(0),\ f(y)\} = f(y).$$

(3) Let $x, y \in H$. Since

$$f(x) \leq \max\Big\{\bigwedge_{d \in x \circ y} f(d),\ f(y)\Big\} \leq \max\{f(b),\ f(y)\}$$

for all $b \in x \circ y$, we have the desired result. □

The following corollaries are straightforward.

Corollary 3.13. *If (H, f) is a strong hyper $\mathcal{N}$-ideal of H, then*

$$(\forall x, y \in H)\ \Big(f(x) \leq \max\Big\{\bigvee_{b \in x \circ y} f(b),\ f(y)\Big\}\Big).$$

Corollary 3.14. *Every strong hyper $\mathcal{N}$-ideal is both an s-weak hyper $\mathcal{N}$-ideal (and hence a weak hyper $\mathcal{N}$-ideal) and a hyper $\mathcal{N}$-ideal .*

Proposition 3.15. *Let (H, f) be a hyper $\mathcal{N}$-ideal of H and let $x, y \in H$. Then*

(1) $f(0) \leq f(x)$.

(2) *If (H, f) satisfies the* **sup** *property, then*

$$(\exists\, a \in x \circ y)\ (f(x) \leq \max\{f(a),\ f(y)\}). \tag{3.7}$$

Proof. (1) Since $0 \ll x$ for all $x \in H$, it follows from (3.1) that $f(0) \leq f(x)$.

(2) If (H, f) satisfies the **sup** property, then there exists $a_0 \in x \circ y$ such that $f(a_0) = \bigvee_{a \in x \circ y} f(a)$. Hence

$$f(x) \leq \max\Big\{\bigvee_{a \in x \circ y} f(a),\ f(y)\Big\} = \max\{f(a_0),\ f(y)\}.$$

This completes the proof. □

Corollary 3.16. (1) *Every hyper $\mathcal{N}$-ideal is a weak hyper $\mathcal{N}$-ideal .*

(2) *If (H, f) is a hyper $\mathcal{N}$-ideal of H satisfying the* **sup** *property, then (H, f) is an s-weak hyper $\mathcal{N}$-ideal of H.*

Proof. Straightforward. □

In Proposition 3.15, if a hyper $\mathcal{N}$-ideal (H, f) does not satisfy the **sup** property, then (3.7) is not valid. In fact, in Example 3.8, (H, f) is a hyper $\mathcal{N}$-ideal of H and (H, f) does not satisfy the **sup** property. Also (H, f) does not satisfy (3.7).

The following example shows that the converse of Corollary 3.14 and Corollary 3.16(1) may not be true.

Example 3.17. (1) Consider the hyper BCK-algebra H in Example 3.2. Let (H, f) be an $\mathcal{N}$-structure in which f is given by

$$f = \begin{pmatrix} 0 & a & b \\ -0.9 & -0.7 & -0.4 \end{pmatrix}.$$

Then we can see that (H, f) is a hyper $\mathcal{N}$-ideal of H and hence it is also a weak hyper $\mathcal{N}$-ideal of H. But it is not a strong hyper $\mathcal{N}$-ideal of H since

$$\max\Big\{\bigwedge_{w\in b\circ a} f(w),\ f(a)\Big\} = \max\{f(a),\ f(a)\} = -0.7 \ngeq -0.4 = f(b).$$

(2) Consider the hyper BCK-algebra H in Example 3.2. Let (H, f) be an $\mathcal{N}$-structure in which f is given by

$$f = \begin{pmatrix} 0 & a & b \\ -0.8 & -0.4 & -0.6 \end{pmatrix}.$$

Then (H, f) is both a weak hyper $\mathcal{N}$-ideal of H and an s-weak hyper $\mathcal{N}$-ideal of H. But it is not a hyper $\mathcal{N}$-ideal of H since $a \ll b$ but $f(a) \nleq f(b)$.

For an $\mathcal{N}$-structure (H, f) in a set H, the *closed β-cut* of (H, f) is denoted by $C(f; \beta)$, and is defined as follows:

$$C(f;\beta) := \{x \in H \mid f(x) \leq \beta\},\ \beta \in [-1, 0].$$

Theorem 3.18. *Let (H, f) be an $\mathcal{N}$-structure in H. Then (H, f) is a weak hyper $\mathcal{N}$-ideal of H if and only if it satisfies:*

$$(\forall \beta \in [-1, 0])\ (C(f;\beta) \neq \emptyset\ \Rightarrow\ C(f;\beta) \text{ is a weak hyper BCK-ideal of } H).$$

Proof. Assume that (H, f) is a weak hyper $\mathcal{N}$-ideal of H and let $\beta \in [-1, 0]$ be such that $C(f;\beta) \neq \emptyset$. It is clear from (3.6) that $0 \in C(f;\beta)$. Now let $x, y \in H$ be such that $x \circ y \subseteq C(f;\beta)$ and $y \in C(f;\beta)$. Then $x \circ y \subseteq C(f;\beta)$ implies that for every $b \in x \circ y$, $b \in C(f;\beta)$. It follows that $f(b) \leq \beta$ for all $b \in x \circ y$ so that $\bigvee_{b \in x \circ y} f(b) \leq \beta$. Using (3.6) we have

$$f(x) \leq \max\{ \bigvee_{b \in x \circ y} f(b),\, f(y)\} \leq \beta,$$

which implies that $x \in C(f;\beta)$. Consequently, $C(f;\beta)$ is a weak hyper BCK-ideal of H.

Conversely, suppose that every nonempty closed β-cut $C(f;\beta)$ is a weak hyper BCK-ideal of H for all $\beta \in [-1, 0]$. Let $f(x) = \beta$ for $x \in H$. By (I1), $0 \in C(f;\beta)$. Hence $f(0) \leq \beta = f(x)$. Now for any $x, y \in H$ let $\beta = \max\Big\{ \bigvee_{d \in x \circ y} f(d),\, f(y)\Big\}$. Then $y \in C(f;\beta)$, and for each $b \in x \circ y$ we have

$$f(b) \leq \bigvee_{d \in x \circ y} f(d) \leq \max\Big\{ \bigvee_{d \in x \circ y} f(d),\, f(y)\Big\} = \beta.$$

Hence $b \in C(f;\beta)$, which imply that $x \circ y \subseteq C(f;\beta)$. Combining $y \in C(f;\beta)$ and $C(f;\beta)$ being weak hyper BCK-ideal of H, we conclude that $x \in C(f;\beta)$, and so

$$f(x) \leq \beta = \max\Big\{ \bigvee_{d \in x \circ y} f(d),\, f(y)\Big\}.$$

This completes the proof. □

Lemma 3.19. [3] *Let A be a subset of H. If I is a hyper BCK-ideal of H such that $A \ll I$, then A is contained in I.*

Theorem 3.20. *Let (H, f) be an $\mathcal{N}$-structure in H. Then (H, f) is a hyper $\mathcal{N}$-ideal of H if and only if for every $\beta \in [-1, 0]$, the the nonempty closed β-cut $C(f;\beta)$ is a hyper BCK-ideal of H.*

Proof. Assume that (H, f) is a hyper $\mathcal{N}$-ideal of H and $C(f;\beta) \neq \emptyset$ for any $\beta \in [-1, 0]$. It is clear that $0 \in C(f;\beta)$ by Proposition 3.15(1). Let $x, y \in H$ be such that $x \circ y \ll C(f;\beta)$ and $y \in C(f;\beta)$. Then $x \circ y \ll C(f;\beta)$ implies that for every $b \in x \circ y$ there is $b_0 \in C(f;\beta)$ such that $b \ll b_0$, so $f(b) \leq f(b_0)$ by (3.1). It follows that $f(b) \leq f(b_0) \leq \beta$ for all $b \in x \circ y$ so that $\bigvee_{b \in x \circ y} f(b) \leq \beta$. Then

$$f(x) \leq \max\Big\{ \bigvee_{b \in x \circ y} f(b),\, f(y)\Big\} \leq \beta,$$

which implies that $x \in C(f;\beta)$. Consequently, $C(f;\beta)$ is a hyper BCK-ideal of H.

Conversely, suppose that for each $\beta \in [-1,0]$ the closed β-cut $C(f;\beta)$ is a hyper BCK-ideal of H. Let $x, y \in H$ be such that $x \ll y$ and $f(y) = \beta$. Then $y \in C(f;\beta)$, and so $x \ll C(f;\beta)$. It follows from Lemma 3.19 that $x \in C(f;\beta)$ so that $f(x) \leq \beta = f(y)$. Now for any $x, y \in H$ let $\beta = \max\{\bigvee_{d \in x \circ y} f(d),\, f(y)\}$. Then $y \in C(f;\beta)$, and for each $b \in x \circ y$ we have

$$f(b) \leq \bigvee_{d \in x \circ y} f(d) \leq \max\Big\{\bigvee_{d \in x \circ y} f(d),\, f(y)\Big\} = \beta.$$

Hence $b \in C(f;\beta)$, which imply that $x \circ y \subseteq C(f;\beta)$. Using (a6), we get $x \circ y \ll C(f;\beta)$. Combining $y \in C(f;\beta)$ and $C(f;\beta)$ being a hyper BCK-ideal of H, we conclude that $x \in C(f;\beta)$, and so

$$f(x) \leq \beta = \max\Big\{\bigvee_{d \in x \circ y} f(d),\, f(y)\Big\}.$$

This completes the proof. □

Theorem 3.21. *If (H, f) is a strong hyper $\mathcal{N}$-ideal of H, then for every $\beta \in [-1,0]$, the nonempty closed β-cut $C(f;\beta)$ is a strong hyper BCK-ideal of H.*

Proof. Let $\beta \in [-1,0]$ be such that $C(f;\beta) \neq \emptyset$. Then there exist $b \in C(f;\beta)$, and so $f(b) \leq \beta$. Using Proposition 3.15(1), we get $f(0) \leq f(b) \leq \beta$. Thus $0 \in C(f;\beta)$. Let $u, v \in H$ be such that $(u \circ v) \cap C(f;\beta) \neq \emptyset$ and $v \in C(f;\beta)$. Then we can take $b_0 \in (u \circ v) \cap C(f;\beta)$ and so $f(b_0) \leq \beta$. Hence

$$f(u) \leq \max\Big\{\bigwedge_{b \in u \circ v} f(b),\, f(v)\Big\} \leq \max\{f(b_0),\, f(v)\} \leq \beta,$$

which implies $u \in C(f;\beta)$. Consequently, $C(f;\beta)$ is a strong hyper BCK-ideal of H. □

We now consider the converse of Theorem 3.21.

Theorem 3.22. *Let H satisfy $|x \circ y| < \infty$ for all $x, y \in H$. Let (H, f) be an $\mathcal{N}$-structure in H in which the nonempty closed β-cut $C(f;\beta)$ is a strong hyper BCK-ideal of H for every $\beta \in [-1,0]$. Then (H, f) is a strong hyper $\mathcal{N}$-ideal of H.*

Proof. Since every strong hyper BCK-ideal is a hyper BCK-ideal, it follows from Theorem 3.20 that (H, f) is a hyper $\mathcal{N}$-ideal of H. Note that $x \circ x \subseteq x \circ H \ll \{x\}$ for all $x \in H$. It follows that $c \ll x$ for every $c \in x \circ x$ so that $f(c) \leq$

$f(x)$ for all $c \in x \circ x$. Hence $\bigvee_{c \in x \circ x} f(c) \leq f(x)$. Let $\max\left\{\bigwedge_{d \in x \circ y} f(d),\, f(y)\right\} = \beta$. Then $\bigwedge_{d \in x \circ y} f(d) \leq \beta$ and $f(y) \leq \beta$. Since $|x \circ y| < \infty$ for all $x, y \in H$, there exist $d_0 \in x \circ y$ such that $f(d_0) \leq \beta$ and $f(y) \leq \beta$. Then $(x \circ y) \cap C(f;\beta) \neq \emptyset$, and $y \in C(f;\beta)$. Since $C(f;\beta)$ is a strong hyper BCK-ideal, it follows that $x \in C(f;\beta)$ so that $f(x) \leq \beta = \max\left\{\bigwedge_{d \in x \circ y} f(d),\, f(y)\right\}$. Therefore (H, f) is a strong hyper $\mathcal{N}$-ideal of H. □

Acknowledgements

The first author was supported by the fund of sabbatical year program (2009), Gyeongsang National University.

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Received: January, 2010