An $M/G/1$ Retrial Queue with Non-Persistent Customers, a Second Optional Service and Different Vacation Policies

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Abstract

In this paper, we study an $M/G/1$ queue with two phases of heterogeneous service. A first essential service is provided to all arriving customers. Upon completion of this service, a customer can either opt for a second phase of service or can leave the system. A customer, who finds the server busy, either leaves the system with probability (1-α) or joins an orbit with probability α. From the orbit the customer makes repeated attempts to obtain service. We assume the inter retrial times are exponential random variables. We also assume that upon completion of a service, the server either remains in the system with probability β_0 or leaves the system for an ith type of vacation with probability $\beta_i (1 \le i \le M)$ where $\sum_{i=0}^{M} \beta_i = 1$. We obtain the probability generating functions of the system size distribution as well as the orbit size distribution in the steady state. We obtain a stochastic decomposition of the system size distribution and an expression for the additional increase in the congestion due to the presence of retrials, in the steady state. We discuss some particular cases.

Keywords: Retrial queue, non-persistent customers, second phase of optional service, different vacation policies

1 Introduction

The theory of retrial queues have been extensively applied in the study of communication and computer networks. Their special characteristic is that, a customer who finds a busy server does not leave the system or joins a queue. He joins an orbit(retrial group) from where he makes repeated attempts to

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obtain service. Several survey articles, bibliographic articles and monographs have been published on retrial queues, see [1],[2], [4] and [6].

Madan [8], Medhi[9] and Choudhury[5] have studied $M/G/1$ queues with two phases of service. The server first provides a regular service to all arriving customers, whereas only some of them receive a second phase of optional service. More recently, Artalejo and Choudhury [3] investigated a similar type of $M/G/1$ queue under classical retrial policy. Senthilkumar and Arumuganathan [10] have studied an $M/G/1$ retrial queueing system with two phase of essential service, non-persistent customers and different vacation policies.

In this paper, we examine the steady state behavior of an $M/G/1$ retrial queueing system with two phases of service, the first service being an essential service provided to all arriving customers and the second service is an optional service provided to some of the customers. A customer who finds the server busy either leaves the system with probability$(1 - \alpha)$ or joins the orbit with a probability α. Upon completion of each service, the server can remain in the system with a probability β_0 or may leave for the ith type vacation with a probability$\beta_i(1 \leq i \leq M)$ and $\sum_{i=0}^{M} \beta_i = 1$.

An example of such a model is provided by customer requests at a call center. A customer who calls up a call center is initially connected to a receptionist, who collects all the information from the customer and answers his questions(first essential service). If the customer has some technical problems to be sorted out he may ask that he be allowed to contact a technical person in the call center(second optional service). Otherwise he may be satisfied with the answers given by the receptionist. If the customer gets a busy signal upon calling the call center he may either decide to make retrial attempts to gain service or may decide to abandon this call for the time being. After attending a customer's call, the receptionist may remain in the system to attend another call or may go away to take a break or may attend to some other jobs in the call center or may decide to call up potential customers. These events can be considered as server vacations with different policies.

The rest of the paper is organized as follows. In section 2, we introduce the mathematical model of the system. In section 3, we derive the main results of the paper. In section 4, we obtain performance measures of the system. In section 5, we discuss some particular cases.

2   Mathematical Model

In this section, a single server retrial queueing system is considered. The primary customers arrive according to a Poisson process with rate λ. If a primary customer, on arrival finds the server busy, he becomes non-persistent and leaves the system with probability $(1 - \alpha)$ or with probability α, he enters into an orbit. The retrial times of the individual customers are assumed to

be i.i.d random variables with a distribution function $1 - e^{-\mu x}$. The server provides a first essential service(FES) to all arriving customers. As soon as the FES of a customer is completed, then the customer may leave the system with probability 1-r or may opt for the second optional service (SOS) with probability r. The service times S_1 of the FES and S_2 of the SOS are independent random variables having general distributions with distribution functions $B_1(.), B_2(.)$, Laplace Stieljes transforms (LST) $B_1^*(.), B_2^*(.)$ respectively. The total service time of a customer in the system is therefore given by

$$S = \begin{cases} S_1 + S_2 & \text{with probability r} \\ S_1 & \text{with probability 1-r} \end{cases}$$

The total service time S has a distribution function B(.) and has LST $B^*(s) = (1 - r)B_1^*(s) + rB_1^*(s)B_2^*(s)$. Let b_1, b_2 denote the expected values of the FES and SOS respectively. Then

$$
\begin{aligned}
E(S) &= b_1 + rb_2 \\
E(S^2) &= E(S_1^2) + rE(S_2^2) + 2rb_1b_2
\end{aligned}
$$

Let $\nu(x)$ denote the hazard function of the service time,i.e $\nu(x) = \frac{B'(x)}{1-B(x)}$.
As soon as the service is completed, the server may go for the $i^{th}(i = 1, 2 \cdots M)$ type of vacation with probability β_i, or may remain in the system to serve the next customer, if any, with probability β_0, where $\sum_{i=0}^{M} \beta_i = 1$. Let V_k be the duration of the server vacation time in the kth vacation scheme and let $V_k(x)$ and $v_k(x)$ denote the distribution function and the hazard rate function respectively of the random variable V_k $(1 \leq k \leq M)$. We assume that the interarrival times, retrial times, vacation times and service times are mutually independent of each other .

Let N(t) denote the number of customers in the orbit at time t.

The server state is denoted by, $C(t) = \begin{cases} 0, & \text{if the server is idle} \\ 1, & \text{if the server is doing either FES} \\ & \qquad \text{or SOS service} \\ 2, & \text{if the server is on vacation} \end{cases}$

We define the following supplementary variables. If C(t)=1, we define $\xi(t)$ =the elapsed service time of the customer in service at time t. If C(t) =2, $\xi(t)$= the elapsed vacation time.

The process$\{(C(t), N(t), \xi(t)), t \geq 0\}$ is a Markov process. Define

$$
\begin{aligned}
P_{0,n}(t) &= Prob\left\{C(t) = 0, N(t) = n\right\}, \ n \geq 0 \\
P_{1,n}(x,t)dx &= Prob\left\{C(t) = 1, N(t) = n, \ x \leq \xi(t) < x + dx\right\}, \ x \geq 0, \ n \geq 0 \\
P_{2,n}^k(x,t)dx &= Prob\left\{C(t) = 2, N(t) = n, \ x \leq \xi(t) < x + dx\right\}, \ x \geq 0, \ n \geq 0 \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad k = 1, 2, \cdots, M
\end{aligned}
$$

3   Steady state system size distribution

Let us assume that the system attains the steady state. From the Kolomogorov equations of the system, we have

$$\{\lambda + n\mu\} P_{0,n} = \beta_0 \int_0^\infty P_{1,n}(x)\nu(x)dx + \sum_{k=1}^M \int_0^\infty P_{2,n}^k(x)v_k(x)dx \qquad (1)$$

$$\frac{d}{dx}P_{1,n}(x) = -\{\lambda\alpha + \nu(x)\} P_{1,n}(x) + (1 - \delta_{n,0})\lambda\alpha P_{1,n-1}(x) \qquad (2)$$

$$\frac{d}{dx}P_{2,n}^k(x) = -\{\lambda\alpha + v_k(x)\} P_{2,n}^k(x) + (1 - \delta_{n,0})\lambda\alpha P_{2,n-1}^k(x) \qquad (3)$$

$$\text{for k= 1,2,..,M}$$

$$P_{1,n}(0) = \lambda P_{0,n} + (n + 1)\mu P_{0,n+1} \qquad (4)$$

$$P_{2,n}^k(0) = \beta_k \int_0^\infty P_{1,n}(x)\nu(x)dx \qquad (5)$$

$$\text{for k= 1,2,..,M}$$

Now we define the following partial generating functions. For $|z| \le 1$ and $x \ge 0$

$$P_0(z) = \sum_{n=0}^\infty P_{0,n}z^n$$

$$P_1(x, z) = \sum_{n=0}^\infty P_{1,n}(x)z^n$$

$$P_2^k(x, z) = \sum_{n=0}^\infty P_{2,n}^k(x)z^n, \quad k = 1, 2, .., M$$

The above functions along with (1) to (5) give us the following result

$$\lambda P_0(z) + z\mu P_0'(z) = \beta_0 \int_0^\infty P_1(x, z)\nu(x)dx$$

$$+ \sum_{k=1}^M \int_0^\infty P_2^k(x, z)v_k(x)dx \qquad (6)$$

$$\frac{\partial}{\partial x}P_1(x, z) = -\{\lambda\alpha + \nu(x)\} P_1(x, z) + \lambda\alpha z P_1(x, z) \qquad (7)$$

$$\frac{\partial}{\partial x}P_2^k(x, z) = -\{\lambda\alpha + v_k(x)\} P_2^k(x, z) + \lambda\alpha z P_2^k(x, z) \qquad (8)$$

$$\text{for k= 1,2,..,M}$$

$$P_1(0, z) = \lambda P_0(z) + \mu P_0'(z) \qquad (9)$$

$$P_2^k(0, z) = \beta_k \int_0^\infty P_1(x, z)\nu(x)dx \qquad (10)$$

$$\text{for k= 1,2,..,M}$$

From (7) and (8), we have

$$P_1(x, z) = P_1(0, z)e^{-\lambda\alpha(1-z)x}(1 - B(x)), \tag{11}$$

$$P_2^k(x, z) = P_2^k(0, z)e^{-\lambda\alpha(1-z)x}(1 - V_k(x)), \tag{12}$$

$$k = 1, 2, ..., M.$$

Define

$$P_1(z) = \int_0^\infty P_1(x, z)dx, \tag{13}$$

$$P_2^k(z) = \int_0^\infty P_2^k(x, z)dx, \ k = 1, 2, ..., M. \tag{14}$$

Therefore

$$P_1(z) = \frac{1 - B^*(\lambda\alpha(1-z))}{\lambda\alpha(1-z)}\{\lambda P_0(z) + \mu P_0'(z)\} \tag{15}$$

$$P_2^k(z) = \frac{1 - V_k^*(\lambda\alpha(1-z))}{\lambda\alpha(1-z)}\{\lambda P_0(z) + \mu P_0'(z)\}\beta_k B^*(\lambda\alpha(1-z)) \tag{16}$$

$$k = 1, 2, ..., M$$

From (6)

$$P_0(z) = P_0(1)\exp\left\{\frac{\lambda}{\mu}\int_1^z \frac{1 - R(u)}{R(u) - u}du\right\} \tag{17}$$

$$\text{where } R(z) = B^*(\lambda\alpha(1-z))\left\{\beta_0 + \sum_{k=1}^M \beta_k V^*(\lambda\alpha(1-z))\right\} \text{ and}$$

$$B^*(\lambda\alpha - \lambda\alpha z) = \{(1 - r) + rB_2^*(\lambda\alpha - \lambda\alpha z)\}B_1^*(\lambda\alpha - \lambda\alpha z)$$

Now the PGF of the orbit size distribution is

$$P(z) = P_0(z) + P_1(z) + \sum_{k=1}^M P_2^k(z)$$

$$= \left\{\frac{\alpha(R(z) - z) + 1 - R(z)}{\alpha(R(z) - z)}\right\}P_0(z) \tag{18}$$

From the normalizing condition, $\lim_{z\to 1} P(z) = 1$, we get

$$P_0(1) = \frac{1 - \rho\alpha}{1 + \rho(1 - \alpha)} \tag{19}$$

$$\text{where } \rho = \lambda\left\{rb_2 + b_1 + \sum_{k=1}^M \beta_k E[V_k]\right\}$$

Hence

$$P(z) = \left\{\frac{\alpha(R(z) - z) + 1 - R(z)}{\alpha(R(z) - z)}\right\}\frac{1 - \rho\alpha}{1 + \rho(1 - \alpha)}e^{\frac{\lambda}{\mu}\int_1^z \frac{1-R(u)}{R(u)-u}du} \tag{20}$$

Remark: From the expression for $P(z)$ given in (20), we see that, $P_0(1)$ must be greater than zero. Therefore a necessary condition for the existence of the steady state is $\rho\alpha < 1$.

Let K(z) be the PGF of the system size distribution. Then

$$K(z) = \left\{\frac{(\alpha-1)(R(z)-z)+(1-z)B^*(\lambda\alpha(1-z))}{\alpha(R(z)-z)}\right\}P_0(z) \quad (21)$$

3.1   Stochastic decomposition

Let K_∞ be the random variable which denotes the system size at time t in the steady state in the classical M/G/1 queue with non-persistent customers and different vacation policies. It can be verified that (See [7]) the probability generating function of K_∞ is given by

$$K_\infty(z) = \frac{(\alpha-1)\{R(z)-z\}+(1-z)B^*(\lambda\alpha(1-z))}{\alpha(R(z)-z)}\frac{1-\rho\alpha}{1+\rho(1-\alpha)} \quad (22)$$

Now introduce a random variable R_μ with the generating function

$$E(z^{R_\mu}) = \exp\left\{\frac{\lambda}{\mu}\int_1^z\frac{1-R(u)}{R(u)-u}du\right\} \quad (23)$$

The r.h.s of the above equation is $\frac{P_0(z)}{P_0(1)}$. The distribution of the random variable R_μ coincides with the conditional distribution of the number of sources of repeated calls given that the server is free. From (21),(22)and (23), we see that

$$K(z) = K_\infty(z)\ E(z^{R_\mu})$$

Thus, the random variable R_μ describes the increase in the size of the system due to the presence of retrials.

4   Performance measures

a) The mean number of customers in the orbit

$$L_q = P'(1) = \frac{\lambda}{\mu}\left(\frac{\rho\alpha}{1-\rho\alpha}\right)+\frac{\lambda^2\alpha\gamma_*}{2(1-\rho\alpha)(1+\rho(1-\alpha))}$$

$$\text{where } \gamma_* = E[S^2]+2E[S]\sum_{k=1}^M\beta_kE[V_k]+\sum_{k=1}^M\beta_kE[V_k^2]$$

b) The blocking probability that an arriving customers finds the server busy or away on vacation

$$b = 1-P_0(1) = \frac{\rho}{1+\rho(1-\alpha)}$$

c) The mean waiting time in the orbit $\quad W_q = \frac{L_q}{\lambda\alpha}$

d) The mean number of customers in the system

$$
\begin{aligned}
L & = K'(1) = \frac{\lambda}{\mu}\left(\frac{\rho\alpha}{1-\rho\alpha}\right) + \frac{\lambda E[S]}{1+\rho(1-\alpha)} + \frac{\lambda^2\alpha\gamma_*}{2(1-\rho\alpha)(1+\rho(1-\alpha))} \\
& = \frac{\lambda}{\mu}\left(\frac{\rho\alpha}{1-\rho\alpha}\right) + E[K_\infty] \qquad\qquad \text{(see [7])}
\end{aligned}
$$

The expected increase in the congestion in the system due to the presence of retrials is therefore given by $\frac{\lambda}{\mu}\left(\frac{\rho\alpha}{1-\rho\alpha}\right)$

e) The mean response time $\quad W = \frac{L}{\lambda\alpha}$

f) The steady state distribution of the server state is given by

$$
\begin{aligned}
Prob\,\{\text{server is idle}\} & = P_0(1) = \frac{1-\rho\alpha}{1+\rho(1-\alpha)} \\
Prob\,\{\text{server is busy}\} & = P_1(1) = \frac{\lambda(b_1 + rb_2)}{1+\rho(1-\alpha)} \\
Prob\,\{\text{server is on vacation}\} & = \sum_{k=1}^{M} P_2^k(1) = \frac{\lambda\sum_{k=1}^{M}\beta_k E[V_k]}{1+\rho(1-\alpha)}
\end{aligned}
$$

5 Particular cases

Case(i) If the second optional service is essential (i.e. r =1), then the PGF of the number of customers in the orbit is

$$
P(z) \;=\; \frac{\alpha(R(z)-z)+1-R(z)}{R(z)-z}P_0(z) \tag{24}
$$

$$
\text{where } R(z) \;=\; B_1^*(\lambda\alpha - \lambda\alpha z)B_2^*(\lambda\alpha - \lambda\alpha z)\left\{\beta_0 + \sum_{k=1}^{M}\beta_k V_k^*(\lambda\alpha - \lambda\alpha z)\right\}
$$

$$
P_0(z) \;=\; \frac{1-\rho\alpha}{1+\rho(1-\alpha)}\exp\left\{\frac{\lambda}{\mu}\int_1^z \frac{1-R(u)}{R(u)-u}du\right\}
$$

Equation(24) agrees with the PGF of the number of customers in the orbit in the steady state obtained by Senthil Kumar and Arumuganathan [10].

Case(ii) If $\alpha = 1$, $\beta_0 = 1$, $\beta_i = 0$, i=1,2,...,M, then the PGF of the system size becomes

$$
K(z) \;=\; \frac{(1-z)\left\{(1-r)+rB_2^*(\lambda-\lambda z)\right\}B_1^*(\lambda-\lambda z)}{\left[\left\{(1-r)+rB_2^*(\lambda-\lambda z)\right\}B_1^*(\lambda-\lambda z) - z\right]}P_0(z)
$$

$$
\text{where } P_0(z) \;=\; (1-\rho)\exp\left\{\frac{\lambda}{\mu}\int_1^z \frac{1-\left\{(1-r)+rB_2^*(\lambda-\lambda u))\right\}B_1^*(\lambda-\lambda u)}{\left\{(1-r)+rB_2^*(\lambda-\lambda u)\right\}B_1^*(\lambda-\lambda u) - u}du\right\}
$$

This is the same as equation (16) given by Artalejo and Choudhury [3] with $1 - r = q$ and $r = p$.

Case(iii) If there is no vacation, i.e, $\beta_0 = 1$, $\beta_i = 0$, i=1,2,...,M, there is no impatience, i.e, $\alpha = 1$ and the retrial rate $\mu \to \infty$, then the expression for the PGF of the system size becomes

$$K(z) = \frac{(1 - z)(1 - \rho)\left\{(1 - r) + rB_2^*(\lambda - \lambda z)\right\}B_1^*(\lambda - \lambda z)}{\left\{(1 - r) + rB_2^*(\lambda - \lambda z)\right\}B_1^*(\lambda - \lambda z) - z}$$

This is the same as equation (3.3) given by Medhi[9]

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