Applied Mathematical Sciences, Vol. 4, 2010, no. 40, 1975 - 1999

A Combined Interval Net DEA and BSC for Evaluating Organizational Efficiency

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Abstract

Measuring the performance of a production system has been an important task in management for purposes of control, planning, etc. Traditional studies in data envelopment analysis (DEA) view systems as a whole when measuring the efficiency, ignoring the operation of individual processes within a system.but in Network DEA we are allowed to consider the evaluation of changes occurred within the process. In this research, we propose a methodology named CINDB(Combined Interval Net DEA and BSC) to evaluate the performance of organization considering financial and non-financial perspectives, and we know The input and output measures for the integrated DEA-BSC model are grouped in "cards" which are associated with " BSC ". BSC clear representation of the relationship and logic between the key performance indicators(KPI) of 4 perspectives - financial, customer, internal process, and learning and growth.

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Keywords: balanced scorecard ,Data envelopment analysis, Network flows

1 Introduction

Measuring the performance of a production system has been an important task in management for purposes of control, planning, etc. One technique widely applied to measure the relative efficiency of a set of production systems, or decision making units (DMUs), which utilize the same inputs to produce the same outputs, is data envelopment analysis (DEA). Conventionally, the system is treated as a black box, in which only inputs and outputs of the black box are considered in measuring the efficiency. The performance of the component processes interacting with each other in the system are not considered. DEA is a mathematical programming technique that claborates the relative efficiency of multiple decision-making units (DMUs) on the basis of observed inputs and outputs, which may be expressed with different types of metrics. The BSC is a management tool composed of a collection of measures, arranged in groups, and denoted as cards. The measures are related to four major managerial perspectives, and are aimed at providing top managers with a comprehensive view of their business.The cards offer balanced evaluation of the organizational performance along financial, marketing, operational and strategic dimensions. BSC combines financial and operational measures, and focuses both on the short- and long-term objectives of the organization. It was motivated by the realization that traditional financial measures by themselves are inadequate in providing a complete and useful overview of organizational performance.

There has been limited study in the literature to tie between BSC and DEA. Rouse, Putterill and Ryan (2002) described the performance analytical tools and frameworks used to support change-management in the aircraft servicing and maintenance division by BCS-DEA methodology. Also Eilat, Golany and Shtub (2005) proposed and demonstrated a methodology for the construction and analysis of efficient, effective and balanced portfolios of R and D projects by using BSC-DEA. The method that we propose in this paper uses an extended DEA model, which quantifies some of the qualitative concepts embedded in the BSC approach.

I think , application of DEA to evaluate the BSC result may be a good solution to the implementation of the BSC. Richard (2003) argues that DEA is suitable for measuring the best practice of the BSC indicators. The efficiency frontier as measured by DEA can be used to specifically investigate the efficiency of decision-making units (DMUs). The slack could be used as the evaluation of a firm's efficiency on those BSC indicators. (Rickards 2003). [18]

The integrated interval net DEA-BSC model addresses four common goals that firms are trying to accomplish: (1) achieving strategic objectives (effectiveness goal); (2) optimizing the usage of resources in generating desired outputs (efficiency goal); (3) obtaining balance (balance goal); and (4) obtaining Cause and Effect in Perspectives . The model is applicable for every organizations for-profit. The contribution of the model that is presented in this paper is both conceptual, and excutive for any given DMUthat are devoted to specific output/input measures.[18] In this present paper, there will be, in the first section,we provides an introduction to DEA and BSC . next section we have view on combined Network DEA and BSC model. Then in the section.3, we will develop the combined Network DEA Models and BSc on interval inputs and outputs. Finally, in the last section, we will have conclusions.

2 LITERATURE REVIEW

2.1 DEA models

Data envelopment analysis (DEA) is a non-parametric technique that ranks units based on their relative ability to convert inputs into outputs. (Charnes et al., 1978; Schinnar et al., 1990). DEA uses linear programming methodology to define a production frontier for decision-making units. One main advantage of DEA is that it allows several inputs and several outputs to be considered at the same time. Assume a set of observed DMUs, DMUj; j = 1,. . .,n, associated with m inputs, xij; i = 1,. . .,m, and s outputs, yrj; r = 1,. . ., s. In the method originally proposed by Charnes et al. (1978), (often referred to as the DEA-CCR model) the efficiency of the DMUp is defined as follows. Model (1) intput oriented - CRS model

$$
\begin{aligned}
\text{Min} \quad & \theta \\
\text{s.t.} \quad & \sum_{j=1}^{n} \lambda_j x_{ij} \leq \theta x_{ip}, \quad \text{i=1,...,m} \\
& \sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{rp}, \quad \text{r=1,...,s} \\
& \lambda_j \geq 0, \quad \text{j=1,...,n.}
\end{aligned}
\tag{1}
$$

If is the optimum value, then DMUp is said to be efficient .The DEA Model (1) is known as an input- oriented model. An input oriented model defines a unit to be "relatively inefficient" with respect to a sample when it is clear that other units in the sample could be producing the same level of outputs as the unit in question, while consuming less inputs. Conversely, an output oriented model classifies a unit to be "relatively inefficient" if other units in the sample are able to produce a higher level of outputs with the same level of inputs as the unit in question. Additionally, models can stipulate either constant or variable return to scale. A constant return to scale model assumes that for a given unit, output levels are always proportional to the level of inputs; on the other hand, a variable return to scale model allows for the level of outputs grow proportionally higher or lower than a corresponding increase in inputs.

2.2 Balanced Scorecard

In today,s complex competitive environment, firms need to be agile and flexible. As a result, availability of the right information at the right time for both decision making and performance evaluation has become critical. A popular performance measurement scheme suggested by Kaplan and Norton (1992) is the balanced scorecard that employs performance metrics from financial, customer, business process, and technology perspectives. By combining these different perspectives, the balanced scorecard helps managers understand the interrelationships and tradeoffs between alternative performance dimensions and leads to improved decision making and problem solving. As can be seen from Fig. 1, the intention of the BSC approach was to translate the vision and strategy of a business unit into objectives and measures in four different areas: the financial, customer, internal business-process and learning and growth perspectives.

1-Customer perspective: The aim is to identify the customer and market segments in which the business unit will compete and the measures of the business unit's performance in these targeted segments.

2- Internal perspective: The measures focus on the internal processes that will have the greatest impact on customer satisfaction and achieving an organization's financial objectives.

3- Innovation and learning perspective: It identifies the infrastructure that the organization must build to create long-term growth and improvement.

4- Financial perspective: It is valuable in summarizing the readily measurable economic consequences of actions already taken. It indicates whether a company's strategy, implementation, and execution are contributing to bottom-line improvement .

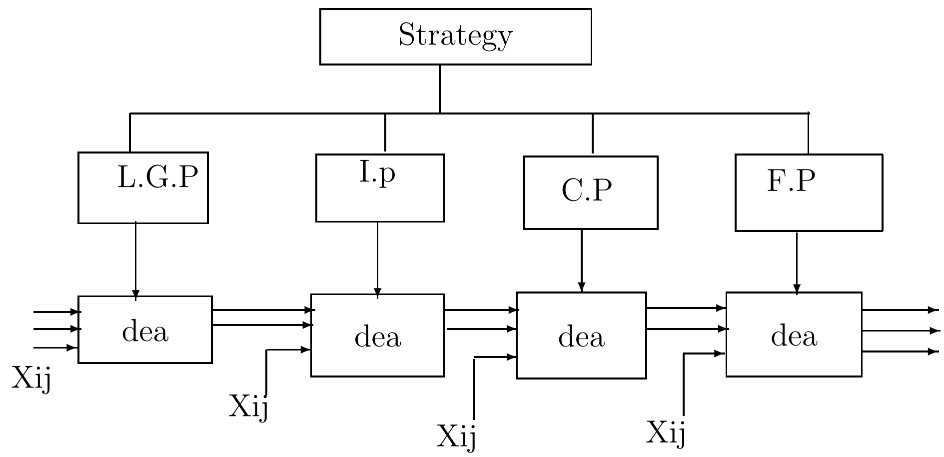
Through the years, the Balanced Scorecard has evolved, from the performance measurement tool originally introduced by Kaplan and Norton (1992), to a tool for implementing strategies (Kaplan and Norton, 1996) and a framework for determining the alignment of an organisation's human, information and organisation capital with its strategy (Kaplan and Norton, 2004a)

3 the methodology combined Network DEA and BSC

Most researches showed that BSC is a useful technique to calculate performance measures. Ziegenfuss (2000), demonstrated the use of the "Balanced Scorecard" methodology in selecting performance measures for internal auditing departments. And we know ,DEA presents a model for evaluating the performance of a set of comparable decision making units (DMUs). Each DMU is evaluated in terms of a set of outputs that represent its successes, and a set of inputs that represent the resources .but we want combined this two methodology for evaluated organization . DEA can be a useful tool in setting benchmarks and evaluating BSC results. The organizations were evaluated via the BSC-DEA model, where each organization is expressed by inputs as resources and outputs as objectives to be attained. In this study is to find out the relationships among four output perspectives. For such an objective, a structure equation model is employed to test the interrelationships of all the variables in the entire model. The proposed structural equation model is shown in Figure 1.

The DEA-BSC model generalizes this model, To get the strategic orientation, it relies on the BSC methodology, which provides a systemization concept where strategic objectives are formulated looking from several different perspectives [5]. This systemization checks whether all the important aspects are measured and it elucidates compensation effects between measures. For each objective, one or more measures are formulated. While some objectives are formulated into quantitative measures, other can be measured using only qualitative measures. Using the BSC methodology, a prespective (card) is defined to be a group of measures. The BSC structure is comprised of four perspectives , which represent four mutually exclusive groups of measures. The ,,highest hierarchy level includes a single card, denoted by Co which includes all existing output measures and we called strategy .

Fig1-combined BSC and DEA model



The next level includes the four perspectives $C1...C4$. In order to reflect the desired balance, a decision maker can set limits on what may be regarded as suitable lower and upper bounds for the relative importance on each card. If we show lower and upper bounds with Lck and Uck we will have:

$$L_{ck} \leq \frac{\sum\limits_{r \in ck} u_r Y_{rp}}{\sum\limits_{r \in co} u_i y_{rp}} \leq U_{ck} k = 1...4$$

$$L_{ck} \leq \frac{\sum\limits_{i \in ck} v_r x_{ip}}{\sum\limits_{i \in co} v_i x_{ip}} \leq U_{ck} k = 1...4$$

Now we want introduce the mathematical formulations of the proposed network-DEA and bsc model and efficiency measures in this section. Following the formulation of LP (1) shown earlier and limitation of (2) , we limit our discussion to the output-oriented measure only, and the technology is assumed to exhibit constant returns-to-scale (CRS). (we know a DEA Model is output oriented if it seeks to increase outputs without increasing inputs. Our approach to the Network DEA Model is an extension of that used in the four-stage DEA Model.

The network DEA model of fig(3) can be formalized and thats Model is a linear program and has a dual. When the redundant constraints model are eliminated, it can be formulated as:(we Suppose, lower and upper bounds in four stages balanced scored card are presented in Table 1).

	L and G	L and G	I.P	I.P	C	C	F	F
lowerandupper	*lower* 0.2	*upper* 0.4	*lower* 0.3	*upper* 0.6	*lower* 0.2	*upper* 0.4	*lower* 0.2	*upper* 0.5

Table 1. Balance bounds used for prespectives.

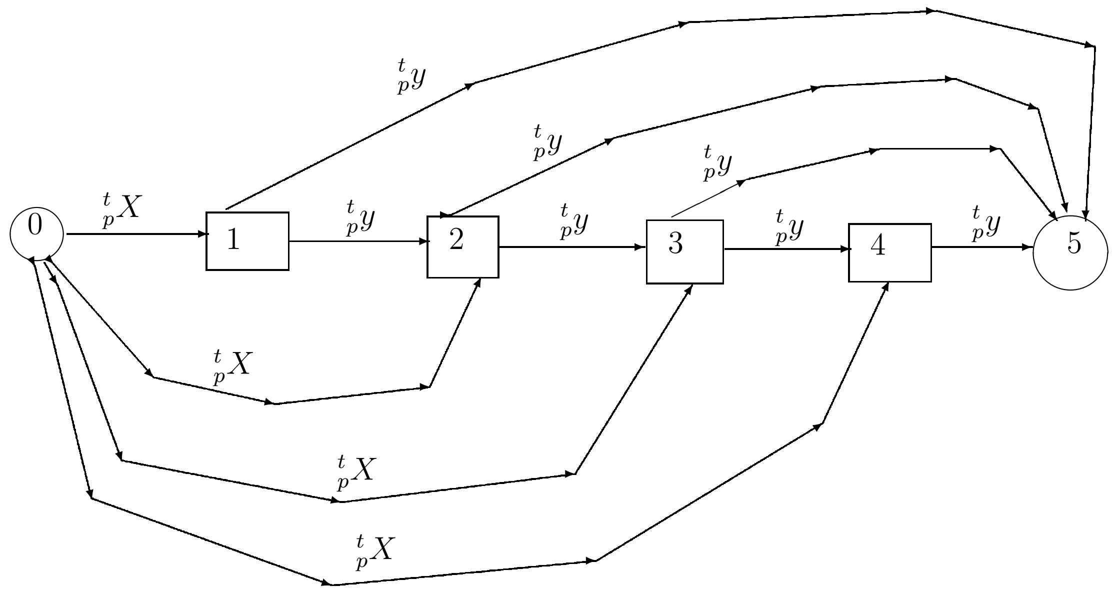


Fig2-combined BSC and DEA model

$\sum_{j=1}^{n} \gamma_j \, {}^{5}_{2}y_{rj} \leq (0.2\mu_9 + 0.7\mu_{10} + 0.2\mu_{11} + 0.2\mu_{12} - 0.4\mu_{13} + 0.4\mu_{14} - 0.4\mu_{15} - 0.5\mu_{16} + 1){}^{5}_{2}y_{rp}$

$\sum_{j=1}^{n} \beta_j \, {}^{5}_{3}y_{rj} \leq (0.2\mu_9 + 0.3\mu_{10} + 0.8\mu_{11} + 0.2\mu_{12} - 0.4\mu_{13} - 0.6\mu_{14} + 0.6\mu_{15} - 0.5\mu_{16} + 1){}^{5}_{3}y_{rp}$

$\sum_{j=1}^{n} \alpha_j \, {}^{5}_{4}y_{rj} \leq (0.2\mu_9 + 0.3\mu_{10} + 0.2\mu_{11} + 0.8\mu_{12} - 0.4\mu_{13} - 0.6\mu_{14} - 0.4\mu_{15} + 0.5\mu_{16} + 1){}^{5}_{4}y_{rp}$

. $\sum_{j=1}^{n} \lambda_j \, {}^{1}_{0}x_{ij} \leq (0.8\mu_1 - 0.3\mu_2 - 0.2\mu_3 - 0.2\mu_4 - 0.6\mu_5 + 0.6\mu_6 + 0.4\mu_7 + 0.5\mu_8 + \theta){}^{1}_{0}x_{ip}$

$\sum_{j=1}^{n} \gamma_j \, {}^{2}_{0}x_{ij} \leq (-0.2\mu_1 + 0.7\mu_2 - 0.2\mu_3 - 0.2\mu_4 + 0.4\mu_5 - 0.4\mu_6 + 0.4\mu_7 + 0.5\mu_8 + \theta){}^{2}_{0}x_{ip}$

$$\sum_{j=1}^{n}\beta_j\ {}_0^3x_{ij} \le (-0.2\mu_1 - 0.3\mu_2 + 0.8\mu_3 - 0.2\mu_4 + 0.4\mu_5 + 0.6\mu_6 - 0.6\mu_7 + 0.5\mu_8 + \theta)_0^3x_{ip}$$

$$\sum_{j=1}^{n}\alpha_j\ {}_0^4x_{ij} \le (-0.2\mu_1 - 0.3\mu_2 - 0.2\mu_3 + 0.8\mu_4 + 0.4\mu_5 + 0.6\mu_6 + 0.4\mu_7 - 0.5\mu_8 + \theta)_0^4x_{ip}$$

$$\sum_{j=1}^{n}\lambda_j\ {}_1^2y_{rj} \le (-0.9\mu_9 + 0.3\mu_{10} + 0.2\mu_{11} + 0.2\mu_{12} + 0.6\mu_{13} - 0.6\mu_{14} - 0.4\mu_{15} - 0.5\mu_{16})_1^2y_{rp}$$

$$\sum_{j=1}^{n}\gamma_j\ {}_1^2y_{rj} \le (-0.2\mu_1 + 0.7\mu_2 - 0.2\mu_3 - 0.2\mu_4 + 0.4\mu_5 - 0.4\mu_6 + 0.4\mu_7 + 0.5\mu_8)_1^2y_{rp}$$

$$\sum_{j=1}^{n}\gamma_j\ {}_2^3y_{rj} \le (0.2\mu_9 - 0.7\mu_{10} + 0.2\mu_{11} + 0.2\mu_{12} - 0.4\mu_{13} + 0.4\mu_{14} - 0.4\mu_{15} - 0.5\mu_{16})_2^3y_{rp}$$

$$\sum_{j=1}^{n}\beta_j\ {}_2^3y_{rj} \le (-0.2\mu_1 - 0.3\mu_2 + 0.8\mu_3 - 0.2\mu_4 + 0.4\mu_5 + 0.6\mu_6 - 0.6\mu_7 + 0.5\mu_8)_2^3y_{rp}$$

$$\sum_{j=1}^{n}\beta_j\ {}_3^4y_{rj} \le (+0.2\mu_9 + 0.3\mu_{10} - 0.8\mu_{11} + 0.2\mu_{12} - 0.4\mu_{13} - 0.6\mu_{14} + 0.6\mu_{15} - 0.5\mu_{16})_3^4y_{rp}$$

$$\sum_{j=1}^{n}\alpha_j\ {}_3^4y_{rj} \le (-0.2\mu_1 - 0.3\mu_2 - 0.2\mu_3 + 0.8\mu_4 + 0.4\mu_5 + 0.6\mu_6 + 0.4\mu_7 - 0.5\mu_8)_3^4y_{rp}$$

$$\lambda_j \ge 0, \alpha_j \ge 0, \beta_j \ge 0, \gamma_j \ge 0, \mu_j \ge 0$$

There are fourteen constraints in this model, If $\theta = 1$ and $S_i^+ = S_r^- = 0$,then the system is efficient. Multipliers $\lambda_j, \alpha_j, \beta_j, \gamma_j, \mu_j$ are associated with processes 1, 2, 3and 4.

4 Interval Network DEA and BSC

In this section, our goal is to develop the previous section models on interval dataSupposethat:$x_{ij} = [x^l_{ij}, x^u_{ij}]$, $y_{rj} = [y^l_{rj}, y^u_{rj}]$,$j = 1, ..., n, r = 1, ..., s, i = 1, ..., m$, Assume that the feasible region of the problem (4) is called S1.

Now suppose thatDMUp has the highest possible input and the lowest possible output, and the other of DMUs have the lowest possible inputs; and the highest possible outputs i.e:(x^l_{ip}, x^u_{ip}), $(y^l_{rp}, y^u_{rp}) j = 1, ..., n$ and $j \neq p$ The problem (4) can be formolized as:

$$\sum_{j=1}^{n} \gamma_j\ {}_2^5y^u_{rj} \le (0.2\mu_9 + 0.7\mu_{10} + 0.2\mu_{11} + 0.2\mu_{12} - 0.4\mu_{13} + 0.4\mu_{14} - 0.4\mu_{15} - 0.5\mu_{16} + 1){}_2^5y^l_{rp}$$

$$\sum_{j=1}^{n} \beta_j\ {}_3^5y^u_{rj} \le (0.2\mu_9 + 0.3\mu_{10} + 0.8\mu_{11} + 0.2\mu_{12} - 0.4\mu_{13} - 0.6\mu_{14} + 0.6\mu_{15} - 0.5\mu_{16} + 1){}_3^5y^l_{rp}$$

$$\sum_{j=1}^{n} \alpha_j\ {}_4^5y^u_{rj} \le (0.2\mu_9 + 0.3\mu_{10} + 0.2\mu_{11} + 0.8\mu_{12} - 0.4\mu_{13} - 0.6\mu_{14} - 0.4\mu_{15} + 0.5\mu_{16} + 1){}_4^5y^l_{rp}$$

$$\sum_{j=1}^{n} \lambda_j\ {}_0^1x^l_{ij} \le (0.8\mu_1 - 0.3\mu_2 - 0.2\mu_3 - 0.2\mu_4 - 0.6\mu_5 + 0.6\mu_6 + 0.4\mu_7 + 0.5\mu_8 + \theta){}_0^1x^u_{ip}$$

$$\sum_{j=1}^{n} \gamma_j\ {}_0^2x^l_{ij} \le (-0.2\mu_1 + 0.7\mu_2 - 0.2\mu_3 - 0.2\mu_4 + 0.4\mu_5 - 0.4\mu_6 + 0.4\mu_7 + 0.5\mu_8 + \theta){}_0^2x^u_{ip}$$

$$\sum_{j=1}^{n} \beta_j\ {}_0^3x^l_{ij} \le (-0.2\mu_1 - 0.3\mu_2 + 0.8\mu_3 - 0.2\mu_4 + 0.4\mu_5 + 0.6\mu_6 - 0.6\mu_7 + 0.5\mu_8 + \theta){}_0^3x^u_{ip}$$

$$\sum_{j=1}^{n} \alpha_j\ {}_0^4x^l_{ij} \le (-0.2\mu_1 - 0.3\mu_2 - 0.2\mu_3 + 0.8\mu_4 + 0.4\mu_5 + 0.6\mu_6 + 0.4\mu_7 - 0.5\mu_8 + \theta){}_0^4x^u_{ip}$$

$$\sum_{j=1}^{n} \lambda_j \, {}_1^2y_{rj}^u \leq (-0.9\mu_9+0.3\mu_{10}+0.2\mu_{11}+0.2\mu_{12}+0.6\mu_{13}-0.6\mu_{14}-0.4\mu_{15}-0.5\mu_{16}){}_1^2y_{rp}L$$

$$\sum_{j=1}^{n} \gamma_j \, {}_1^2y_{rj}^l \leq (-0.2\mu_1+0.7\mu_2-0.2\mu_3-0.2\mu_4+0.4\mu_5-0.4\mu_6+0.4\mu_7+0.5\mu_8){}_1^2y_{rp}^u$$

$$\sum_{j=1}^{n} \gamma_j \, {}_2^3y_{rj}^u \leq (0.2\mu_9-0.7\mu_{10}+0.2\mu_{11}+0.2\mu_{12}-0.4\mu_{13}+0.4\mu_{14}-0.4\mu_{15}-0.5\mu_{16}){}_2^3y_{rp}^l$$

$$\sum_{j=1}^{n} \beta_j \, {}_2^3y_{rj}^l \leq (-0.2\mu_1-0.3\mu_2+0.8\mu_3-0.2\mu_4+0.4\mu_5+0.6\mu_6-0.6\mu_7+0.5\mu_8){}_2^3y_{rp}^u$$

$$\sum_{j=1}^{n} \beta_j \, {}_3^4y_{rj}^u \leq (+0.2\mu_9+0.3\mu_{10}-0.8\mu_{11}+0.2\mu_{12}-0.4\mu_{13}-0.6\mu_{14}+0.6\mu_{15}-0.5\mu_{16}){}_3^4y_{rp}^l$$

$$\sum_{j=1}^{n} \alpha_j \, {}_3^4y_{rj}^l \leq (-0.2\mu_1-0.3\mu_2-0.2\mu_3+0.8\mu_4+0.4\mu_5+0.6\mu_6+0.4\mu_7-0.5\mu_8){}_3^4y_{rp}^u$$

$$\lambda_j \geq 0, \alpha_j \geq 0, \beta_j \geq 0, \gamma_j \geq 0, \mu_j \geq 0$$

Assume that the feasible region of the problem (5) is called S2.

Now suppose that DMUp has the lowest possible input and the highest possible output, and the other of DMUs have the highest possible inputs; and the lowest possible outputs, i.e: (x_{ip}^l, x_{ip}^u) , $(y_{rp}^l, y_{rp}^u) j = 1, ..., n$ and $j \neq p$ The problem (4)can be formolized as:

$$\sum_{j=1}^{n} \gamma_j \, {}_2^5y_{rj}^l \leq (0.2\mu_9+0.7\mu_{10}+0.2\mu_{11}+0.2\mu_{12}-0.4\mu_{13}+0.4\mu_{14}-0.4\mu_{15}-0.5\mu_{16}+1){}_2^5y_{rp}^u$$

$$\sum_{j=1}^{n} \beta_j \ {}_3^5y_{rj}^l \leq (0.2\mu_9 + 0.3\mu_{10} + 0.8\mu_{11} + 0.2\mu_{12} - 0.4\mu_{13} - 0.6\mu_{14} + 0.6\mu_{15} - 0.5\mu_{16} + 1){}_3^5y_{rp}^u$$

$$\sum_{j=1}^{n} \alpha_j \ {}_4^5y_{rj}^l \leq (0.2\mu_9 + 0.3\mu_{10} + 0.2\mu_{11} + 0.8\mu_{12} - 0.4\mu_{13} - 0.6\mu_{14} - 0.4\mu_{15} + 0.5\mu_{16} + 1){}_4^5y_{rp}^u$$

$$\sum_{j=1}^{n} \lambda_j \ {}_0^1x_{ij}^u \leq (0.8\mu_1 - 0.3\mu_2 - 0.2\mu_3 - 0.2\mu_4 - 0.6\mu_5 + 0.6\mu_6 + 0.4\mu_7 + 0.5\mu_8 + \theta){}_0^1x_{ip}^l$$

$$\sum_{j=1}^{n} \gamma_j \ {}_0^2x_{ij}^u \leq (-0.2\mu_1 + 0.7\mu_2 - 0.2\mu_3 - 0.2\mu_4 + 0.4\mu_5 - 0.4\mu_6 + 0.4\mu_7 + 0.5\mu_8 + \theta){}_0^2x_{ip}^l$$

$$\sum_{j=1}^{n} \beta_j \ {}_0^3x_{ij}^u \leq (-0.2\mu_1 - 0.3\mu_2 + 0.8\mu_3 - 0.2\mu_4 + 0.4\mu_5 + 0.6\mu_6 - 0.6\mu_7 + 0.5\mu_8 + \theta){}_0^3x_{ip}^l$$

$$\sum_{j=1}^{n} \alpha_j \ {}_0^4x_{ij}^u \leq (-0.2\mu_1 - 0.3\mu_2 - 0.2\mu_3 + 0.8\mu_4 + 0.4\mu_5 + 0.6\mu_6 + 0.4\mu_7 - 0.5\mu_8 + \theta){}_0^4x_{ip}^l$$

$$\sum_{j=1}^{n} \lambda_j \ {}_1^2y_{rj}^l \leq (-0.9\mu_9 + 0.3\mu_{10} + 0.2\mu_{11} + 0.2\mu_{12} + 0.6\mu_{13} - 0.6\mu_{14} - 0.4\mu_{15} - 0.5\mu_{16}){}_1^2y_{rp}^u$$

$$\sum_{j=1}^{n} \gamma_j \ {}_1^2y_{rj}^u \leq (-0.2\mu_1 + 0.7\mu_2 - 0.2\mu_3 - 0.2\mu_4 + 0.4\mu_5 - 0.4\mu_6 + 0.4\mu_7 + 0.5\mu_8){}_1^2y_{rp}^l$$

$$\sum_{j=1}^{n} \gamma_j \ {}_2^3y_{rj}^l \leq (0.2\mu_9 - 0.7\mu_{10} + 0.2\mu_{11} + 0.2\mu_{12} - 0.4\mu_{13} + 0.4\mu_{14} - 0.4\mu_{15} - 0.5\mu_{16}){}_2^3y_{rp}^u$$

$$\sum_{j=1}^{n} \beta_j \ {}_2^3y_{rj}^u \leq (-0.2\mu_1 - 0.3\mu_2 + 0.8\mu_3 - 0.2\mu_4 + 0.4\mu_5 + 0.6\mu_6 - 0.6\mu_7 +$$

$0.5\mu_8)_2^3 y_{rp}^l$

$$\sum_{j=1}^{n} \beta_j \ {}_3^4 y_{rj}^l \leq (+0.2\mu_9 + 0.3\mu_{10} - 0.8\mu_{11} + 0.2\mu_{12} - 0.4\mu_{13} - 0.6\mu_{14} + 0.6\mu_{15} - 0.5\mu_{16})_3^4 y_{rp}^u$$

$$\sum_{j=1}^{n} \alpha_j \ {}_3^4 y_{rj}^u \leq (-0.2\mu_1 - 0.3\mu_2 - 0.2\mu_3 + 0.8\mu_4 + 0.4\mu_5 + 0.6\mu_6 + 0.4\mu_7 - 0.5\mu_8)_3^4 y_{rp}^l$$

$$\lambda_j \geq 0, \alpha_j \geq 0, \beta_j \geq 0, \gamma_j \geq 0, \mu_j \geq 0$$

Assume that the feasible region of the problem (6) is called S3.

Theorem:If$\theta^*, \theta^{l*} and \theta^{u*}$ are the optimal values of the problems (4) , (5) and(6)respectively and for each i , j and r, :$x_{ij} = [x_{ij}^l, x_{ij}^u]$, $y_{rj} = [y_{rj}^l, y_{rj}^u]$, then, $\theta^{l*\leq}\theta^* \leq \theta^{u*}$

Proof: since the problem; (4)is minimized, and this have objective function , thus in order to proof $\theta^{l*\leq}\theta^* \leq \theta^{u*}$, we should just enough proof that:$s_3 \subset s_1 \subset s_2$. First we proof that $s_3 \subset s_1$, Assume $(\lambda, \theta)\varepsilon s_3$ so:

therfor $\sum_{j=1, j\neq p}^{n} \lambda_j \ {}_1^5 y_{rj}^l + \lambda_j \ {}_1^5 y_{rp}^u \geq \phi_1 \sum_{j=1}^{n} \lambda_j \ {}_1^5 y_{rp}^u \Longrightarrow (\phi_1 - \lambda_j)\lambda_j \ {}_1^5 y_{rp}^u \leq \sum_{j=1, j\neq p}^{n} \lambda_j \ {}_1^5 y_{rj}^l$

$$\Longrightarrow (\phi_1 - \lambda_j)\lambda_j \ {}_1^5 y_{rp}^u \leq \sum_{j=1, j\neq p}^{n} \lambda_j \ {}_1^5 y_{rj} \Longrightarrow$$

$$\sum_{j=1, j\neq p}^{n} \lambda_j \ {}_1^5 y_{rj} \geq (\phi_1 - \lambda_j)\lambda_j \ {}_1^5 y_{rp}$$

$$\Longrightarrow \sum_{j=1, j\neq p}^{n} \lambda_j \ {}_1^5 y_{rj} \geq (\phi_1 - \lambda_j)\lambda_j \ {}_1^5 y_{rp} \Longrightarrow$$

$$\sum_{j=1}^{n} \lambda_j \ {}_1^5 y_{rj} \geq \phi_1 \ {}_1^5 y_{rp}$$

therfor $\sum_{j=1,j\neq p}^{n} \gamma_j\ {}^5_2y^l_{rj} + \gamma_j\ {}^5_2y^u_{rp}\ \geq\ \phi_1 \sum_{j=1}^{n} \gamma_j\ {}^5_2y^u_{rp} \Longrightarrow (\phi_1 - \gamma_j)\gamma_j\ {}^5_2y^u_{rp}\ \leq$
$\sum_{j=1,j\neq p}^{n} \gamma_j\ {}^5_2y^l_{rj}$

$\Longrightarrow (\phi_1 - \gamma_j)\gamma_j\ {}^5_2y^u_{rp} \leq \sum_{j=1,j\neq p}^{n} \gamma_j\ {}^5_2y_{rj} \Longrightarrow$

$\sum_{j=1,j\neq p}^{n} \gamma_j\ {}^5_2y_{rj} \geq (\phi_1 - \gamma_j)\gamma_j\ {}^5_2y_{rp}$

$\Longrightarrow \sum_{j=1,j\neq p}^{n} \gamma_j\ {}^5_2y_{rj} \geq (\phi_1 - \gamma_j)\lambda_j\ {}^5_1y_{rp} \Longrightarrow$

$\sum_{j=1}^{n} \gamma_j\ {}^5_2y_{rj} \geq \phi_1\ {}^5_2y_{rp}$

therfor $\sum_{j=1,j\neq p}^{n} \beta_j\ {}^5_3y^l_{rj} + \beta_j\ {}^5_3y^u_{rp}\ \geq\ \phi_1 \sum_{j=1}^{n} \beta_j\ {}^5_3y^u_{rp} \Longrightarrow (\phi_1 - \beta_j)\beta_j\ {}^5_3y^u_{rp}\ \leq$
$\sum_{j=1,j\neq p}^{n} \beta_j\ {}^5_2y^l_{rj}$

$\Longrightarrow (\phi_1 - \beta_j)\ {}^5_3y^u_{rp} \leq \sum_{j=1,j\neq p}^{n} \beta_j\ {}^5_3y_{rj} \Longrightarrow$

$\sum_{j=1,j\neq p}^{n} \beta_j\ {}^5_3y_{rj} \geq (\phi_1 - \beta_j)\ {}^5_3y_{rp}$

$\Longrightarrow \sum_{j=1,j\neq p}^{n} \beta_j\ {}^5_3y_{rj} \geq (\phi_1 - \beta_j)\ {}^5_3y_{rp} \Longrightarrow$

$\sum_{j=1}^{n} \beta_j\ {}^5_3y_{rj} \geq \phi_1\ {}^5_3y_{rp}$

therfor $\sum_{j=1,j\neq p}^{n} \alpha_j \ {}_4^5y_{rj}^l + \alpha_j \ {}_4^5y_{rp}^u \ \geq \ \phi_4 \sum_{j=1}^{n} \beta_j \ {}_4^5y_{rp}^u \Longrightarrow (\phi_4 - \alpha_j) \ {}_4^5y_{rp}^u \ \leq \sum_{j=1,j\neq p}^{n} \alpha_j \ {}_4^5y_{rj}^l$

$$\Longrightarrow (\phi_4 - \alpha_j) \ {}_4^5y_{rp}^u \leq \sum_{j=1,j\neq p}^{n} \alpha_j \ {}_4^5y_{rj} \Longrightarrow$$

$$\sum_{j=1,j\neq p}^{n} \alpha_j \ {}_4^5y_{rj} \geq (\phi_1 - \alpha_j) \ {}_4^5y_{rp}$$

$$\Longrightarrow \sum_{j=1,j\neq p}^{n} \beta_j \ {}_4^5y_{rj} \geq (\phi_1 - \alpha_j) \ {}_4^5y_{rp} \Longrightarrow$$

$$\sum_{j=1}^{n} \alpha_j \ {}_4^5y_{rj} \geq \phi_4 \ {}_4^5y_{rp}$$

therfor $\sum_{j=1,j\neq p}^{n} \lambda_j \ {}_0^1x_{ij}^u + \lambda_j\_0^1x_{ip}^l \ \leq \ (\phi_5 + \theta) \ {}_0^1x_{ip}^l \Longrightarrow \sum_{j=1,j\neq p}^{n} \lambda_j \ {}_0^1x_{ij}^u \ \leq (\phi_5 + \theta - \lambda_j) \ {}_0^1x_{ip}^l \Longrightarrow$

$$\sum_{j=1,j\neq p}^{n} \lambda_j \ {}_0^1x_{ij}^u \leq (\phi_5 + \theta - \lambda_j) \ {}_0^1x_{ip} \Longrightarrow$$

$$(\phi_5 + \theta - \lambda_j) \ {}_0^1x_{ip} \geq \sum_{j=1,j\neq p}^{n} \lambda_j \ {}_0^1x_{ij}^u \Longrightarrow$$

$$(\phi_5 + \theta - \lambda_j) \ {}_0^1x_{ip} \geq \sum_{j=1,j\neq p}^{n} \lambda_j \ {}_0^1x_{ij} \Longrightarrow$$

$$\sum_{j=1}^{n} \lambda_j \ {}_0^1x_{ij}^u \leq (\phi_5 + \theta) \ {}_0^1x_{ip}$$

therfor $\sum_{j=1,j\neq p}^{n} \gamma_j \ {}_0^2x_{ij}^u + \gamma_j\_0^2x_{ip}^l \leq (\phi_6 + \theta) \ {}_0^2x_{ip}^l \Longrightarrow \sum_{j=1,j\neq p}^{n} \gamma_j \ {}_0^2x_{ij}^u \leq (\phi_5 + \theta - \gamma_j) \ {}_0^2x_{ip}^l \Longrightarrow$

$$\sum_{j=1,j\neq p}^{n} \lambda_j \ {}_0^2x_{ij}^u \leq (\phi_6 + \theta - \gamma_j) \ {}_0^2x_{ip} \Longrightarrow$$

$$(\phi_6 + \theta - \gamma_j) \ {}_0^2x_{ip} \geq \sum_{j=1,j\neq p}^{n} \gamma_j \ {}_0^2x_{ij}^u \Longrightarrow$$

$$(\phi_6 + \theta - \gamma_j) \ {}_0^2x_{ip} \geq \sum_{j=1,j\neq p}^{n} \lambda_j \ {}_0^2x_{ij} \Longrightarrow$$

$$\sum_{j=1}^{n} \gamma_j \ {}_0^2x_{ij}^u \leq (\phi_6 + \theta) \ {}_0^2x_{ip}$$

therfor $\sum_{j=1,j\neq p}^{n} \beta_j \ {}_0^3x_{ij}^u + \beta_j\_0^3x_{ip}^l \leq (\phi_7 + \theta) \ {}_0^3x_{ip}^l \Longrightarrow \sum_{j=1,j\neq p}^{n} \beta_j \ {}_0^3x_{ij}^u \leq (\phi_7 + \theta - \beta_j) \ {}_0^3x_{ip}^l \Longrightarrow$

$$\sum_{j=1,j\neq p}^{n} \lambda_j \ {}_0^3x_{ij}^u \leq (\phi_7 + \theta - \beta_j) \ {}_0^3x_{ip} \Longrightarrow$$

$$(\phi_7 + \theta - \beta_j) \ {}_0^3x_{ip} \geq \sum_{j=1,j\neq p}^{n} \beta_j \ {}_0^3x_{ij}^u \Longrightarrow$$

$$(\phi_7 + \theta - \beta_j) \ {}_0^3x_{ip} \geq \sum_{j=1,j\neq p}^{n} \beta_j \ {}_0^3x_{ij} \Longrightarrow$$

$$\sum_{j=1}^{n} \beta_j \ {}_0^3x_{ij}^u \leq (\phi_7 + \theta) \ {}_0^3x_{ip}$$

therfor $\sum_{j=1,j\neq p}^{n} \alpha_j \ {}_0^4x_{ij}^u + \beta_j\_0^4x_{ip}^l \leq (\phi_8 + \theta) \ {}_0^3x_{ip}^l \Longrightarrow \sum_{j=1,j\neq p}^{n} \alpha_j \ {}_0^4x_{ij}^u \leq (\phi_8 + \theta - \alpha_j) \ {}_0^4x_{ip}^l \Longrightarrow$

$$\sum_{j=1,j\neq p}^{n} \alpha_j \ {}_0^4x_{ij}^u \leq (\phi_8 + \theta - \alpha_j) \ {}_0^4x_{ip} \Longrightarrow$$

$$(\phi_7 + \theta - \beta_j) \ {}_0^4x_{ip} \geq \sum_{j=1,j\neq p}^{n} \alpha_j \ {}_0^4x_{ij}^u \Longrightarrow$$

$$(\phi_8 + \theta - \alpha_j) \ {}_0^4x_{ip} \geq \sum_{j=1,j\neq p}^{n} \alpha_j \ {}_0^4x_{ij} \Longrightarrow$$

$$\sum_{j=1}^{n} \alpha_j \ {}_0^4x_{ij}^u \leq (\phi_8 + \theta) \ {}_0^4x_{ip}$$

therfor $\sum_{j=1,j\neq p}^{n} \lambda_j \ {}_1^2y_{rj}^l + \lambda_j \ {}_1^2y_{rp}^u \geq \phi_9 \sum_{j=1}^{n} \lambda_j \ {}_1^2y_{rp}^u \Longrightarrow (\phi_9 - \alpha_j) \ {}_1^2y_{rp}^u \leq \sum_{j=1,j\neq p}^{n} \lambda_j \ {}_1^2y_{rj}^l$

$$\Longrightarrow (\phi_9 - \lambda_j) \ {}_1^2y_{rp}^u \leq \sum_{j=1,j\neq p}^{n} \lambda_j \ {}_1^2y_{rj} \Longrightarrow$$

$$\sum_{j=1,j\neq p}^{n} \lambda_j \ {}_1^2y_{rj} \geq (\phi_9 - \lambda_j) \ {}_1^2y_{rp}$$

$$\Longrightarrow \sum_{j=1,j\neq p}^{n} \lambda_j \ {}_1^2y_{rj} \geq (\phi_9 - \lambda_j) \ {}_1^2y_{rp} \Longrightarrow$$

$$\sum_{j=1}^{n} \lambda_j \; {}^2_1y_{rj} \geq \phi_9 \; {}^2_1y_{rp}$$

therfor $\sum_{j=1, j\neq p}^{n} \gamma_j \; {}^2_1y^l_{rj} + \gamma_j \; {}^2_1y^u_{rp} \; \leq \; \phi_{10} \sum_{j=1}^{n} \lambda_j \; {}^2_1y^u_{rp} \Longrightarrow (\phi_{10} - \alpha_j) \; {}^2_1y^u_{rp} \; \geq$ $\sum_{j=1, j\neq p}^{n} \gamma_j \; {}^2_1y^l_{rj}$

$$\Longrightarrow (\phi_{10} - \gamma_j) \; {}^2_1y^u_{rp} \geq \sum_{j=1, j\neq p}^{n} \gamma_j \; {}^2_1y_{rj} \Longrightarrow$$

$$\sum_{j=1, j\neq p}^{n} \gamma_j \; {}^2_1y_{rj} \leq (\phi_{10} - \gamma_j) \; {}^2_1y_{rp}$$

$$\Longrightarrow \sum_{j=1, j\neq p}^{n} \gamma_j \; {}^2_1y_{rj} \leq (\phi_{10} - \gamma_j) \; {}^2_1y_{rp} \Longrightarrow$$

$$\sum_{j=1}^{n} \gamma_j \; {}^2_1y_{rj} \leq \phi_{10} \; {}^2_1y_{rp}$$

therfor $\sum_{j=1, j\neq p}^{n} \gamma_j \; {}^3_2y^l_{rj} + \gamma_j \; {}^3_2y^u_{rp} \; \geq \; \phi_{11} \sum_{j=1}^{n} \gamma_j \; {}^3_2y^u_{rp} \Longrightarrow (\phi_{11} - \gamma_j)\beta_j \; {}^3_2y^u_{rp} \; \leq$ $\sum_{j=1, j\neq p}^{n} \beta_j \; {}^3_2y^l_{rj}$

$$\Longrightarrow (\phi_{11} - \beta_j) \; {}^3_2y^u_{rp} \leq \sum_{j=1, j\neq p}^{n} \beta_j \; {}^3_2y_{rj} \Longrightarrow$$

$$\sum_{j=1, j\neq p}^{n} \gamma_j \; {}^3_2y_{rj} \geq (\phi_{11} - \gamma_j) \; {}^3_2y_{rp}$$

$$\Longrightarrow \sum_{j=1,j\neq p}^{n} \beta_j \ {}_2^3y_{rj} \geq (\phi_{11} - \gamma_j) \ {}_3^5y_{rp} \Longrightarrow$$

$$\sum_{j=1}^{n} \gamma_j \ {}_2^3y_{rj} \geq \phi_{11} \ {}_2^3y_{rp}$$

therfor $\sum_{j=1,j\neq p}^{n} \beta_j \ {}_2^3y_{rj}^l + \beta_j \ {}_2^3y_{rp}^u \leq \phi_{12} \sum_{j=1}^{n} \beta_j \ {}_2^3y_{rp}^u \Longrightarrow (\phi_{12} - \beta_j)\beta_j \ {}_2^3y_{rp}^u \geq \sum_{j=1,j\neq p}^{n} \beta_j \ {}_2^3y_{rj}^l$

$$\Longrightarrow (\phi_{12} - \beta_j) \ {}_2^3y_{rp}^u \geq \sum_{j=1,j\neq p}^{n} \beta_j \ {}_2^3y_{rj} \Longrightarrow$$

$$\sum_{j=1,j\neq p}^{n} \beta_j \ {}_2^3y_{rj} \leq (\phi_{12} - \beta_j) \ {}_2^3y_{rp}$$

$$\Longrightarrow \sum_{j=1,j\neq p}^{n} \beta_j \ {}_2^3y_{rj} \leq (\phi_{12} - \beta_j) \ {}_3^5y_{rp} \Longrightarrow$$

$$\sum_{j=1}^{n} \beta_j \ {}_2^3y_{rj} \leq \phi_{12} \ {}_2^3y_{rp}$$

therfor $\sum_{j=1,j\neq p}^{n} \beta_j \ {}_3^4y_{rj}^l + \beta_j \ {}_3^4y_{rp}^u \geq \phi_{13} \sum_{j=1}^{n} \beta_j \ {}_3^4y_{rp}^u \Longrightarrow (\phi_{13} - \beta_j)\beta_j \ {}_3^4y_{rp}^u \leq \sum_{j=1,j\neq p}^{n} \beta_j \ {}_3^4y_{rj}^l$

$$\Longrightarrow (\phi_{13} - \beta_j)\beta_j \ {}_3^4y_{rp}^u \leq \sum_{j=1,j\neq p}^{n} \beta_j \ {}_3^4y_{rj} \Longrightarrow$$

$$\sum_{j=1,j\neq p}^{n} \beta_j \ {}_3^4y_{rj} \geq (\phi_{13}-\beta_j)\beta_j \ {}_3^4y_{rp}$$
$$\Longrightarrow \sum_{j=1,j\neq p}^{n} \beta_j \ {}_3^4y_{rj} \geq (\phi_{13}-\beta_j)\lambda_j \ {}_3^4y_{rp} \Longrightarrow$$
$$\sum_{j=1}^{n} \beta_j \ {}_3^4y_{rj} \geq \phi_{13} \ {}_3^4y_{rp}$$

now ,

therfor $\sum_{j=1,j\neq p}^{n} \alpha_j \ {}_3^4y_{rj}^u + \beta_j \ {}_3^4y_{rp}^l \leq \phi_{14} \sum_{j=1}^{n} \alpha_j \ {}_3^4y_{rp}^l \Longrightarrow (\phi_{13}-\beta_j)\alpha_j \ {}_3^4y_{rp}^u \geq \sum_{j=1,j\neq p}^{n} \alpha_j \ {}_3^4y_{rj}^l$

$$\Longrightarrow (\phi_{14}-\alpha_j)\beta_j \ {}_3^4y_{rp}^u \leq \sum_{j=1,j\neq p}^{n} \alpha_j \ {}_3^4y_{rj} \Longrightarrow$$
$$\sum_{j=1,j\neq p}^{n} \alpha_j \ {}_3^4y_{rj} \geq (\phi_{14}-\alpha_j)\alpha_j \ {}_3^4y_{rp}$$
$$\Longrightarrow \sum_{j=1,j\neq p}^{n} \alpha_j \ {}_3^4y_{rj} \geq (\phi_{14}-\alpha_j)\lambda_j \ {}_3^4y_{rp} \Longrightarrow$$
$$\sum_{j=1}^{n} \alpha_j \ {}_3^4y_{rj} \geq \phi_{14} \ {}_3^4y_{rp}$$

Thus, we conclued from (1) to (19) that:$(\lambda,\theta)\varepsilon s_3$then $s_3 \subset s_1$the amount of objective function wont be better i.e,$\theta^* \leq \theta^{u*}$.

now we proof that $s_1 \subset s_2$, Assume$(\lambda,\theta)\varepsilon s_1$ so ;

$$\sum_{j=1,j\neq p}^{n} \lambda_j \ {}_1^5y_{rj}^u \geq (\phi_1-\lambda_j) \ {}_1^5y_{rp}^l \Longrightarrow \sum_{j=1,j\neq p}^{n} \lambda_j \ {}_1^5y_{rj}^u + \lambda_j \ {}_1^5y_{rp}^l \geq \phi_1 \ {}_1^5y_{rp}^l$$

$$\sum_{j=1,j\neq p}^{n} \gamma_j \ {}_2^5y_{rj}^u \geq (\phi_2 - \gamma_j) \ {}_2^5y_{rp}^l \Longrightarrow \sum_{j=1,j\neq p}^{n} \gamma_j \ {}_2^5y_{rj}^u + \gamma_j \ {}_2^5y_{rp}^l \geq \phi_2 \ {}_2^5y_{rp}^l$$

$$\sum_{j=1,j\neq p}^{n} \beta_j \ {}_3^5y_{rj}^u \geq (\phi_3 - \beta_j) \ {}_3^5y_{rp}^l \Longrightarrow \sum_{j=1,j\neq p}^{n} \beta_j \ {}_3^5y_{rj}^u + \beta_j \ {}_3^5y_{rp}^l \geq \phi_3 \ {}_3^5y_{rp}^l$$

$$\sum_{j=1,j\neq p}^{n} \alpha_j \ {}_4^5y_{rj}^u \geq (\phi_4 - \alpha_j) \ {}_4^5y_{rp}^l \Longrightarrow \sum_{j=1,j\neq p}^{n} \alpha_j \ {}_4^5y_{rj}^u + \alpha_j \ {}_4^5y_{rp}^l \geq \phi_4 \ {}_4^5y_{rp}^l$$

$$\sum_{j=1,j\neq p}^{n} \lambda_j \ {}_0^1x_{ij}^l \leq (\phi_5 + \theta - \lambda_j) \ {}_0^1x_{ip}^u \Longrightarrow \sum_{j=1,j\neq p}^{n} \lambda_j \ {}_0^1x_{ij}^l + \lambda_j \ {}_0^1x_{ip} \leq (\phi_5 + \theta) \ {}_0^1x_{ip}^u$$

$$\sum_{j=1,j\neq p}^{n} \gamma_j \, {}_0^2x_{ij}^l \leq (\phi_6+\theta-\gamma_j) \, {}_0^2x_{ip}^u \Longrightarrow \sum_{j=1,j\neq p}^{n} \gamma_j \, {}_0^2x_{ij}^l + \gamma_j \, {}_0^2x_{ip} \leq (\phi_6+\theta) \, {}_0^2x_{ip}^u$$

$$\sum_{j=1,j\neq p}^{n} \beta_j \, {}_0^3x_{ij}^l \leq (\phi_7+\theta-\beta_j) \, {}_0^3x_{ip}^u \Longrightarrow \sum_{j=1,j\neq p}^{n} \beta_j \, {}_0^3x_{ij}^l + \beta_j \, {}_0^3x_{ip} \leq (\phi_7+\theta) \, {}_0^3x_{ip}^u$$

$$\sum_{j=1,j\neq p}^{n} \alpha_j \, {}_0^4x_{ij}^l \leq (\phi_8+\theta-\alpha_j) \, {}_0^4x_{ip}^u \Longrightarrow \sum_{j=1,j\neq p}^{n} \alpha_j \, {}_0^3x_{ij}^l + \alpha_j \, {}_0^4x_{ip} \leq (\phi_8+\theta) \, {}_0^4x_{ip}^u$$

$$\sum_{j=1,j\neq p}^{n} \lambda_j \, {}_1^2y_{rj}^u \geq (\phi_9 - \lambda_j) \, {}_1^2y_{rp}^l \Longrightarrow \sum_{j=1,j\neq p}^{n} \lambda_j \, {}_1^2y_{rj}^u + \lambda_j \, {}_1^2y_{rp}^l \geq \phi_9 \, {}_1^2y_{rp}^l$$

$$\sum_{j=1,j\neq p}^{n} \gamma_j \, {}_1^2y_{rj}^u \geq (\phi_{10} - \gamma_j) \, {}_1^2y_{rp}^l \Longrightarrow \sum_{j=1,j\neq p}^{n} \gamma_j \, {}_1^2y_{rj}^u + \gamma_j \, {}_1^2y_{rp}^l \geq \phi_{10} \, {}_1^2y_{rp}^l$$

$$\sum_{j=1,j\neq p}^{n} \gamma_j \ {}_2^3y_{rj}^u \geq (\phi_{11} - \gamma_j) \ {}_2^3y_{rp}^l \Longrightarrow \sum_{j=1,j\neq p}^{n} \gamma_j \ {}_2^3y_{rj}^u + \gamma_j \ {}_2^3y_{rp}^l \geq \phi_{11} \ {}_2^3y_{rp}^l$$

$$\sum_{j=1,j\neq p}^{n} \beta_j \ {}_2^3y_{rj}^u \geq (\phi_{12} - \beta_j) \ {}_2^3y_{rp}^l \Longrightarrow \sum_{j=1,j\neq p}^{n} \beta_j \ {}_2^3y_{rj}^u + \beta_j \ {}_2^3y_{rp}^l \geq \phi_{12} \ {}_2^3y_{rp}^l$$

$$\sum_{j=1,j\neq p}^{n} \beta_j \ {}_3^4y_{rj}^u \geq (\phi_{13} - \beta_j) \ {}_3^4y_{rp}^l \Longrightarrow \sum_{j=1,j\neq p}^{n} \beta_j \ {}_3^4y_{rj}^u + \beta_j \ {}_3^4y_{rp}^l \geq \phi_{13} \ {}_3^4y_{rp}^l$$

$$\sum_{j=1,j\neq p}^{n} \alpha_j \ {}_3^4y_{rj}^u \geq (\phi_{14} - \alpha_j) \ {}_3^4y_{rp}^l \Longrightarrow \sum_{j=1,j\neq p}^{n} \alpha_j \ {}_3^4y_{rj}^l + \alpha_j \ {}_3^4y_{rp}^u \geq \phi_{14} \ {}_3^4y_{rp}^u$$

Thus, we conclued from (20) to (33) that:$(\lambda, \theta)\varepsilon s_2$ then$s_2 \subset s_1$. so the amount of objective function wont be better i.e,$\theta^{u*} \leq \theta^{*}$.

we conclued from (B1) and (B2)that,$\theta \in [\theta^{l*}, \theta^{u*}]$

5 conclusion

The BSC-DEA methodology was designed to accommodate uncertain and qualitative data. Since nonfinancial performance measures, which are qualitative measures, become important it is necessary for decision makers to use techniques that can include measures in evaluation process.In this paper we have extended the integrated net DEA and BSC for efficiency indexes of DMUs with interval inputs and outputs. The precise data are the special case of interval data. So, we are allowed to consider all changes Occurred within the prossess of sub- DMUs when the efficiency of a DMU is evaluated . Based on the proved theorems, the efficiency value of DMUs with interval data in network DEA lies in a interval .

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Received: November, 2009