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Weighted Composition Operators between Weighted Bergman-Nevanlinna and Bloch-Type Spaces

Ajay K. Sharma

School of Mathematics, Shri Mata Vaishno Devi University
Kakryal, Katra- 182301, J & K, India
aksju_76@yahoo.com

Zaheer Abbas

Department of Applied Mathematics
Baba Gulam Shah Bad Shah University, Rajouri, J & K, India
az11292000@yahoo.co.in

Abstract. In this paper, we characterize the boundedness and compactness of weighted composition operators $W_{\varphi,\psi}(f) = \psi(f \, o \, \varphi)$ acting between Bergman-Nevanlinna and Bloch type spaces of holomorphic functions on the open unit disk $\mathbb{D}$.

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Introduction

Let G be a non-empty set, X a topological vector space , $F(G,X)$ a topological vector space of functions from G to X with point - wise vector space operations and $\varphi : G \longrightarrow G$ be a function such that $fo\varphi \in F(G,X)$ for all $f \in F(G,X)$. Then the linear transformation $C_\varphi : F(G,X) \longrightarrow F(G,X)$ defined as $C_\varphi(f) = fo\varphi$ for all $f \in F(G,X)$, is called a composition transformation, induced by φ, on the space $F(G,X)$. If C_φ is continuous, then it is called a composition operator or substitution operator, induced by φ, on the space $F(G,X)$. For more about composition operators we refer to [1] and [10]. Further if $\psi: G \longrightarrow \mathbb{C}$ be such that $\psi(fo\varphi) \in F(G,X)$ for all $f \in F(G,X)$, then the linear transformation $W_{\varphi,\psi} : F(G,X) \longrightarrow F(G,X)$ defined as $W_{\varphi,\psi}(f) = \psi(fo\varphi)$ is called a weighted composition transformation, induced by φ

and ψ, on $F(G,X)$. If $W_{\varphi,\psi}$ is continuous, then $W_{\varphi,\psi}$ is called a weighted composition operator, induced by φ and ψ, on $F(G,X)$. When $\psi = 1$, then $W_{\varphi,\psi} = C_\varphi$ and when $\varphi(x) = x$, then $W_{\varphi,\psi}$ is called multiplication operator or transformation (depending upon whether it is continuous or not) and is denoted by M_ψ.

These operators appear naturally in the study of classical operators like cesaro and Hilbert-Schmidt operators and they play an important role in the study of Composition operators on the Hardy spaces of half plane (see [7] for details).

Recently several authors have studied weighted composition operators on different spaces of analytic functions. (see for example [2], [3], [4], [5], [8], [9], [11], [12] and references there in for more details).

Let $\mathbb{D}$ be the open unit disc in the complex plane $\mathbb{C}$ and $\lambda \in (-1,\infty)$. Then weighted Bergman-Nevanlinna space $\mathcal{A}^0_\lambda(\mathbb{D})$ consists of all complex valued holomorphic functions on $\mathbb{D}$ such that

$$\|f\|_{\mathcal{A}^0_\lambda(\mathbb{D})} = \int_{\mathbb{D}} log(1+|f(z)|)d\vartheta_\lambda(z) < \infty,$$

where,

$$d\vartheta_\lambda(z) = (\lambda+1)(1-|z|^2)^\lambda dA(z), \qquad dA(z) = \frac{dxdy}{\pi}.$$

In fact, $\|f\|_{\mathcal{A}^0_\lambda(\mathbb{D})}$ fails to be a norm on $\mathcal{A}^0_\lambda(\mathbb{D})$, but $(f,g) \longrightarrow \|f-g\|_{\mathcal{A}^0_\lambda}$ defines a translation invariant metric on $\mathcal{A}^0_\lambda(\mathbb{D})$ and this turns $\mathcal{A}^0_\lambda(\mathbb{D})$ in to a complete metric space. Also by the subharmonicity of $log(1+|f(z)|)$ we have

$$log(1+|f(z)|) \le c_\lambda \frac{\|f\|_{\mathcal{A}^0_\lambda(\mathbb{D})}}{(1-|z|^2)^{\lambda+2}}, \qquad (1.1)$$

for all $f \in \mathcal{A}^0_\lambda(\mathbb{D})$. In particular, (1.1) tells us that if $f_n \longrightarrow f$ in $\mathcal{A}^0_\lambda(\mathbb{D})$ then $f_n \longrightarrow f$ locally uniformly. Here, locally uniform convergence refers to the uniform convergence on every compact subset of $\mathbb{D}$.

Let α be any positive real number. Then the generalized Bloch space $\mathcal{B}^\alpha(\mathbb{D})$ of the unit disc $\mathbb{D}$ consists of analytic functions $f : \mathbb{D} \longrightarrow \mathbb{C}$ such that

$$\|f\|_{\mathcal{B}^\alpha(\mathbb{D})} = sup\{(1-|z|^2)^\alpha |f'(z)| : z \in \mathbb{D}\} < \infty.$$

Note that $\mathcal{B}^1(\mathbb{D})$ is the usual Bloch space.

For $f \in \mathcal{B}^\alpha(\mathbb{D})$, define

$$\|f\| = |f(0)| + sup\{(1-|z|^2)^\alpha |f'(z)| : z \in \mathbb{D}\}.$$

With this norm $\mathcal{B}^\alpha(\mathbb{D})$ is a Banach space. The little Bloch space of the unit disc $\mathbb{D}$, denoted by $\mathcal{B}^\alpha_0(\mathbb{D})$, is the closed subspace of $\mathcal{B}^\alpha(\mathbb{D})$ consisting of all analytic functions $f : \mathbb{D} \longrightarrow \mathbb{C}$ with

$$(1-|z|^2)^\alpha |f'(z)| \longrightarrow 0, \qquad |z| \longrightarrow 1^-.$$

That is,

$$\mathcal{B}^\alpha_0(\mathbb{D}) = \left\{f \in \mathcal{B}^\alpha(\mathbb{D}) : \lim_{|z|\longrightarrow 1^-}(1-|z|^2)^\alpha |f'(z)| = 0\right\}.$$

For more details of all these spaces introduced above we refer [6]. In this paper, we characterize weighted composition operators between weighted Bergman-Nevanlinna and Bloch type spaces. The weighted composition operators between Bergman spaces and Bloch spaces are considered by Sharma and Kumari [12], Theorems 3.1, 3.2, 4.1 and 4.2 of Sharma and Kumari assert that there do exists bounded weighted composition operators between Bergman spaces and Bloch spaces. In this paper we prove that every bounded weighted composition operators from weighted Bergman-Nevanlinna to Bloch type spaces is compact.

2. Weighted composition operators between $\mathcal{A}_\beta^0(\mathbb{D})$ and $\mathcal{B}^\alpha(\mathbb{D})$

In this section we characterize the boundedness and compactness of weighted composition operators from $\mathcal{A}_\beta^0(\mathbb{D})$ and $\mathcal{B}^\alpha(\mathbb{D})$.
The following criterion for compactness is a useful tool for our purpose and follows from some standard arguments, for example, to those outlined in proposition 3.11 of [5].
Lemma 2.1 : *Let $\lambda \in (-1,\infty)$. Then $W_{\varphi,\psi} : \mathcal{A}_\lambda^0(\mathbb{D}) \longrightarrow \mathcal{B}^\alpha(\mathbb{D})$ is compact if and only if for any sequence $\{f_n\}$ in $\mathcal{A}_\lambda^0(\mathbb{D})$ with $sup\left\{\|f_n\|_{\mathcal{A}^0_{\lambda(\mathbb{D})}} : n \geq 1\right\} = M < \infty$ and which converges to zero locally uniformly on $\mathbb{D}$, we have*

$$\lim_{n\to\infty}\left\|W_{\varphi,\psi}\, f_n\right\|_{\mathcal{B}^\alpha(\mathbb{D})} \longrightarrow 0.$$

Theorem 2.2: *Let $\alpha > 0$, $\beta > -1$, $\psi \in H(\mathbb{D})$ and φ be a holomorphic self-map on $\mathbb{D}$. Then the following are equivalent:*

(i) $W_{\varphi,\psi}$ *maps* $\mathcal{A}_\beta^0(\mathbb{D})$ *boundedly in to* $\mathcal{B}^\alpha(\mathbb{D})$.

(ii) $W_{\varphi,\psi}$ *maps* $\mathcal{A}_\beta^0(\mathbb{D})$ *compactly in to* $\mathcal{B}^\alpha(\mathbb{D})$

(iii) *For all* $c > 0$,

$$\lim_{|\varphi(z)|\to 1}(1-|z|^2)^\alpha\, |\psi'(z)|\, exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\beta+2}}\right] = 0,$$

$$\lim_{|\varphi(z)|\to 1}\frac{(1-|z|^2)^\alpha\, |\psi(z)\varphi'(z)|}{1-|\varphi(z)|^2}\, exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\beta+2}}\right] = 0,$$

$\psi \in \mathcal{B}^\alpha(\mathbb{D})$ *and* $\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha\, |\psi(z)\varphi'(z)| < \infty.$

Proof: (i) ⇒(iii) Suppose (i) holds. By taking $f(z) = 1$, the constant function in $\mathcal{A}_\beta^0(\mathbb{D})$, we get $\psi \in \mathcal{B}^\alpha(\mathbb{D})$. Again by taking $f(z) = z$,in $\mathcal{A}_\beta^0(\mathbb{D})$ we get

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha\, |\varphi(z)\psi'(z)+\psi(z)\varphi'(z)| < \infty.$$

Since $\psi \in \mathcal{B}^\alpha(\mathbb{D})$ and $|\varphi(z)| < 1$, we get

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha\, |\psi(z)\varphi'(z)| < \infty.$$

For any $c > 0$ and $\in \mathbb{D}$, consider the function

$$f_\lambda(z) = \left\{\frac{(1-|\varphi(\lambda)|^2)^{2(\beta+2)}}{3\left(1-\overline{\varphi(\lambda)}\, z\right)^{3(\beta+2)}} - \frac{1}{2}\left(\frac{1-|\varphi(\lambda)|^2}{\left(1-\overline{\varphi(\lambda)}\, z\right)^2}\right)^{\beta+2}\right\}$$
$$\times\, exp\left[6c\left\{\frac{1}{2}\left(\frac{1-|\varphi(\lambda)|^2}{\left(1-\overline{\varphi(\lambda)}\, z\right)^2}\right)^{\beta+2} - \frac{(1-|\varphi(\lambda)|^2)^{2(\beta+2)}}{3\left(1-\overline{\varphi(\lambda)}\, z\right)^{3(\beta+2)}}\right\}\right].$$

Using the elementary inequalities $log(1+xy) \leq log(1+x) + log(1+y)$, $log(1+x) \leq 1 + log^+(x)$ and $log(1+x) \leq x$, which holds for all $x, y \geq 0$, we have

$$
\begin{aligned}
&log(1+|f_\lambda(z)|) \\
&\leq log\left[1 + \frac{(1-|\varphi(\lambda)|^2)^{2(\beta+2)}}{3\left|1-\overline{\varphi(\lambda)}\, z\right|^{3(\beta+2)}} + \frac{1}{2}\left(\frac{1-|\varphi(\lambda)|^2}{\left|1-\overline{\varphi(\lambda)}\, z\right|^2}\right)^{\beta+2} + 1 \right. \\
&\left. +6c\left\{\frac{1}{2}\frac{(1-|\varphi(\lambda)|^2)^{\beta+2}}{\left|1-\overline{\varphi(\lambda)}\, z\right|^{2(\beta+2)}} + \frac{(1-|\varphi(\lambda)|^2)^{2(\beta+2)}}{3\left|1-\overline{\varphi(\lambda)}\, z\right|^{3(\beta+2)}}\right\}\right] \\
&\leq 1 + (1+6c)\left\{\frac{1}{2}\frac{(1-|\varphi(\lambda)|^2)^{\beta+2}}{\left|1-\overline{\varphi(\lambda)}\, z\right|^{2(\beta+2)}} + \frac{(1-|\varphi(\lambda)|^2)^{2(\beta+2)}}{3\left|1-\overline{\varphi(\lambda)}\, z\right|^{3(\beta+2)}}\right\}
\end{aligned}
$$

and so

$$
\|f_\lambda\|_{\mathcal{A}^0_\beta(\mathbb{D})} \leq (2+6c)\left\{\frac{1}{2}\frac{(1-|\varphi(\lambda)|^2)^{\beta+2}}{\left|1-\overline{\varphi(\lambda)}\, z\right|^{2(\beta+2)}} + \frac{(1-|\varphi(\lambda)|^2)^{2(\beta+2)}}{3\left|1-\overline{\varphi(\lambda)}\, z\right|^{3(\beta+2)}}\right\} d\nu_\beta(z).
$$

Thus we have

$$
\begin{aligned}
\|f_\lambda\|_{\mathcal{A}^0_\beta(\mathbb{D})} &\leq 1 + (1+6c)\left(\frac{1}{2} + \frac{2^{\beta+2}}{3}\right)\int_{\mathbb{D}} \frac{(1-|\varphi(\lambda)|^2)^{\beta+2}}{\left|1-\overline{\varphi(\lambda)}\, z\right|^{2(\beta+2)}} d\nu_\beta(z) \\
&\leq 1 + (1+6c)\left(3+2^{\beta+3}\right)c \; = m(say).
\end{aligned}
$$

Moreover,

$$
f_\lambda(\varphi(z)) = \frac{-1}{6}\frac{1}{(1-|\varphi(\lambda)|^2)^{\beta+2}}\; exp\left[\frac{c}{(1-|\varphi(\lambda)|^2)^{\beta+2}}\right].
$$

Also,

$$
\begin{aligned}
f'_\lambda(z) = &\left[\left\{\frac{(1-|\varphi(\lambda)|^2)^{2(\beta+2)}}{\left(1-\overline{\varphi(\lambda)}\, z\right)^{3(\beta+2)+1}} - \frac{(1-|\varphi(\lambda)|^2)^{\beta+2}}{\left(1-\overline{\varphi(\lambda)}\, z\right)^{2(\beta+2)+1}}\right\}\right. \\
&\times (\beta+2)\overline{\varphi(\lambda)} + 6c(\beta+2)\overline{\varphi(\lambda)} \\
&\times \left\{\frac{(1-|\varphi(\lambda)|^2)^{(\beta+2)}}{\left(1-\overline{\varphi(\lambda)}\, z\right)^{2(\beta+2)+1}} - \frac{(1-|\varphi(\lambda)|^2)^{2(\beta+2)}}{\left(1-\overline{\varphi(\lambda)}\, z\right)^{3(\beta+2)+1}}\right\} \\
&\left.\times \left\{\frac{(1-|\varphi(\lambda)|^2)^{2(\beta+2)}}{3\left|1-\overline{\varphi(\lambda)}\, z\right|^{3(\beta+2)}} - \frac{1}{2}\left(\frac{1-|\varphi(\lambda)|^2}{\left(1-\overline{\varphi(\lambda)}\, z\right)^2}\right)^{\beta+2}\right\}\right] \\
&\times exp\left[6c\left\{\left(\frac{1-|\varphi(\lambda)|^2}{\left(1-\overline{\varphi(\lambda)}\, z\right)^2}\right)^{\beta+2} - \frac{1}{3}\frac{(1-|\varphi(\lambda)|^2)^{2(\beta+2)}}{\left(1-\overline{\varphi(\lambda)}\, z\right)^{3(\beta+2)}}\right\}\right].
\end{aligned}
$$

Moreover,

$$
f'_\lambda(\varphi(\lambda)) = 0.
$$

Since $W_{\varphi,\psi}$ maps $\mathcal{A}^0_\beta(\mathbb{D})$ boundedly in to $\mathcal{B}^\alpha(\mathbb{D})$, we can find some $m_0 > 0$ such that

$$m_0 \geq (1-|\lambda|^2)^\alpha \left|\psi'(\lambda)f\big(\varphi(\lambda)\big)+\psi(\lambda)\varphi'(\lambda)\, f'\big(\varphi(\lambda)\big)\right|$$
$$= \frac{(1-|\lambda|^2)^2|\psi'(\lambda)|}{6(1-|\varphi(\lambda)|^2)^{\beta+2}}\, exp\left[\frac{c}{(1-|\varphi(\lambda)|^2)^{\beta+2}}\right]$$

and so

$$(1-|\lambda|^2)^\alpha\, |\psi'(\lambda)| exp\left[\frac{c}{(1-|\varphi(\lambda)|^2)^{\beta+2}}\right] \leq 6m_0(1-|\varphi(\lambda)|^2)^{\beta+2}.$$

Taking limit as $|\varphi(\lambda)| \longrightarrow 1$, on both sides of the inequality, we get

$$\lim_{|\varphi(\lambda)|\to 1}(1-|\lambda|^2)^\alpha|\psi'(\lambda)|\, exp\left[\frac{c}{(1-|\varphi(\lambda)|^2)^{\beta+2}}\right] = 0.$$

Once again, consider the function

$$f_\lambda(z) = \big(z-\varphi(\lambda)\big)\left(\frac{1-|\varphi(\lambda)|^2}{\big(1-\overline{\varphi(\lambda)}\, z\big)^2}\right)^{\beta+2}$$
$$\times\, exp\left[6c\left\{\frac{1}{2}\left(\frac{1-|\varphi(\lambda)|^2}{\big(1-\overline{\varphi(\lambda)}\, z\big)^2}\right)^{\beta+2} - \frac{(1-|\varphi(\lambda)|^2)^{2(\beta+2)}}{3\big(1-\overline{\varphi(\lambda)}\, z\big)^{3(\beta+2)}}\right\}\right].$$

Then it can be shown that $f_\lambda \in \mathcal{A}^0_\beta(\mathbb{D})$ and $\|f_\lambda\| \leq m$ for some $m > 0$. Moreover ,

$$f_\lambda\big(\varphi(\lambda)\big) = 0.$$

Also,

$$f'_\lambda(z) = \left[\left(\frac{1-|\varphi(\lambda)|^2}{\big(1-\overline{\varphi(\lambda)}\, z\big)^2}\right)^{\beta+2} + \big(z-\varphi(\lambda)\big)(2\beta+4)\,\overline{\varphi(\lambda)}\right.$$
$$\times\, \frac{(1-|\varphi(\lambda)|^2)^{(\beta+2)}}{\big(1-\overline{\varphi(\lambda)}\, z\big)^{2(\beta+2)+1}} + \frac{6c\big(z-\varphi(\lambda)\big)(\beta+2)(1-|\varphi(\lambda)|^2)^{\beta+2}}{\big(1-\overline{\varphi(\lambda)}\, z\big)^{2(\beta+2)}}\overline{\varphi(\lambda)}$$
$$\left.\times\left\{\frac{(1-|\varphi(\lambda)|^2)^{(\beta+2)}}{3\big(1-\overline{\varphi(\lambda)}\, z\big)^{2(\beta+2)+1}} - \frac{(1-|\varphi(\lambda)|^2)^{2(\beta+2)}}{\big(1-\overline{\varphi(\lambda)}\, z\big)^{3(\beta+2)}}\right\}\right]$$
$$\times\, exp\left[6c\left\{\left(\frac{1-|\varphi(\lambda)|^2}{\big(1-\overline{\varphi(\lambda)}\, z\big)^2}\right)^{\beta+2} - \frac{(1-|\varphi(\lambda)|^2)^{2(\beta+2)}}{3\big(1-\overline{\varphi(\lambda)}\, z\big)^{3(\beta+2)}}\right\}\right].$$

Also,

$$f'_\lambda\big(\varphi(\lambda)\big) = \frac{1}{(1-|\varphi(\lambda)|^2)^{(\beta+2)}}\, exp\left[\frac{c}{(1-|\varphi(\lambda)|^2)^{\beta+2}}\right].$$

Since ψC_φ maps $\mathcal{A}^0_\beta(\mathbb{D})$ boundedly in to $\mathcal{B}^\alpha(\mathbb{D})$, we can find some $m_1 > 0$ such that

$$m_1 \geq (1-|\lambda|^2)^\alpha \left|\psi'(\lambda)f\big(\varphi(\lambda)\big)+\psi(\lambda)\varphi'(\lambda)\, f'\big(\varphi(\lambda)\big)\right|$$
$$= \frac{(1-|\lambda|^2)|\psi(\lambda)||\varphi'(\lambda)|}{(1-|\varphi(\lambda)|^2)^{\beta+2}}\, exp\left[\frac{c}{(1-|\varphi(\lambda)|^2)^{\beta+2}}\right]$$

and so

$$\frac{(1-|\lambda|^2)|\psi(\lambda)||\varphi'(\lambda)|}{(1-|\varphi(\lambda)|^2)} exp\left[\frac{c}{(1-|\varphi(\lambda)|^2)^{\beta+2}}\right] \leq \; m_1(1-|\varphi(\lambda)|^2)^{\beta+1}.$$

Taking limit as $|\varphi(\lambda)| \longrightarrow 1$ on both sides of the above inequality, we get

$$\lim_{|\varphi(\lambda)|\longrightarrow 1} \frac{(1-|\lambda|^2)|\psi(\lambda)||\varphi'(\lambda)|}{(1-|\varphi(\lambda)|^2)} exp\left[\frac{c}{(1-|\varphi(\lambda)|^2)^{\beta+2}}\right] = 0.$$

(3) ⇒ (2). Assume that conditions in (3) are valid for all positive reals c. Note that if $f \in \mathcal{A}_\beta^0(\mathbb{D})$, then by Cauchy's integral formula for derivatives

$$(1-|z|^2)|f'(z)| \leq \frac{2}{\pi} \int_{\partial\mathbb{D}} \left| f\left(z + \tfrac{1}{2}(1-|z|)\xi\right)\right| |d\xi|$$
$$\leq exp\left[\frac{4^{2+\beta}\, m_0 \|f_\lambda\|_{\mathcal{A}_\beta^0}}{(1-|z|^2)^{\beta+2}}\right].$$

Choose a sequence $\{f_n\}$ in $\mathcal{A}_\beta^0(\mathbb{D})$ such that $\|f_\lambda\|_{\mathcal{A}_\beta^0(\mathbb{D})} \leq m$ and $f_n \longrightarrow 0$ uniformly on compact subsets of $\mathbb{D}$. Then for each $r \in (0,1)$,

$$\sup_{|\varphi(z)| \leq r} (1-|z|^2)^\alpha \left|\psi'(z)f\big(\varphi(z)\big) + \psi(z)\varphi'(z)\, f'\big(\varphi(z)\big)\right|$$
$$\leq A \sup_{|\varphi(z)| \leq r} \left|f_n\big(\varphi(z)\big)\right| + B \sup_{|\varphi(z)| \leq r} \left|f_n'\big(\varphi(z)\big)\right|$$
$$\longrightarrow 0 \;\; as\, n \longrightarrow \infty,$$

where

$$A = \sup_{z \in \mathbb{D}} (1-|z|^2)^\alpha\, |\psi'(z)| < \infty$$

and

$$B = \sup_{z \in \mathbb{D}} (1-|z|^2)^\alpha\, |\psi(z)\varphi'(z)| < \infty.$$

On the other hand, whenever $r \longrightarrow 1$, we have

$$\sup_{|\varphi(z)| > r} (1-|z|^2)^\alpha \left|\big(W_{\varphi,\psi}f_n\big)'(z)\right|$$
$$\leq \sup_{|\varphi(z)| > r} (1-|z|^2)^\alpha \left|\psi'(z)f_n\big(\varphi(z)\big)\right|$$
$$+ \sup_{|\varphi(z)| > r} (1-|z|^2)^\alpha \left|\psi(z)\varphi'(z)\, f'\big(\varphi(z)\big)\right|$$
$$\leq \sup_{|\varphi(z)| > r} (1-|z|^2)^\alpha |\psi'(z)| exp\left[\frac{mm'}{(1-|\varphi(z)|^2)^{\beta+2}}\right]$$
$$+ \sup_{|\varphi(z)| > r} \frac{(1-|z|^2)^\alpha |\psi(z)\varphi'(z)\,|}{1-|\varphi(z)|^2} exp\left[\frac{4^{2+\beta}\, m_0 m_1}{(1-|\varphi(z)|^2)^{\beta+2}}\right]$$
$$\longrightarrow 0 \;\; as \;\; r \;\longrightarrow 1.$$

Combining the above estimates, we see that $\left\|\psi C_\varphi f_n\right\|_{\mathcal{A}_\beta^0\,(\mathbb{D})} \to 0 \; as\, n \to \infty$. Thus (2) follows. As (2) ⇒ (i) is obvious, the result follows. ∎

Corollary 2.3: *Let* > 0 , $\beta > -1$, *and* φ *be a holomorphic self-map on* $\mathbb{D}$. *Then the following are equivalent:*

(i) C_φ *maps* $\mathcal{A}_\beta^0(\mathbb{D})$ *boundedly in to* $\mathcal{B}^\alpha(\mathbb{D})$.

(ii) C_φ *maps* $\mathcal{A}_\beta^0(\mathbb{D})$ *compactly in to* $\mathcal{B}^\alpha(\mathbb{D})$

(iii) *For all* $c > 0$,

$$\lim_{|\varphi(z)|\longrightarrow 1}(1-|z|^2)^\alpha\,|\varphi'(z)|\,exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\beta+2}}\right]=0.$$

Corollary 2.4: *Let* $\alpha > 0$, $\beta > -1$ *and* $\psi \in H(\mathbb{D})$. *Then the following are equivalent:*

(i) M_ψ *maps* $\mathcal{A}_\beta^0(\mathbb{D})$ *boundedly in to* $\mathcal{B}^\alpha(\mathbb{D})$.

(ii) M_ψ *maps* $\mathcal{A}_\beta^0(\mathbb{D})$ *compactly in to* $\mathcal{B}^\alpha(\mathbb{D})$

(iii) $\psi \equiv 0$.

3. Weighted composition operators between $\mathcal{A}_\beta^0(\mathbb{D})$ and $\mathcal{B}_0^\alpha(\mathbb{D})$

In this section we characterize the boundedness and compactness of weighted composition operators from $\mathcal{A}_\beta^0(\mathbb{D})$ and $\mathcal{B}_0^\alpha(\mathbb{D})$. Once again prove that boundedness and compactness of weighted composition operators from $\mathcal{A}_\beta^0(\mathbb{D})$ and $\mathcal{B}_0^\alpha(\mathbb{D})$ are equivalent.

Lemma 3.1: *Let* > 0 , $\beta > -1$, $\psi \in H(\mathbb{D})$ *and* φ *be a holomorphic self-map on* $\mathbb{D}$. *Then the following are equivalent:*

(i) *For all* $c > 0$,

$$\lim_{|z|\longrightarrow 1}(1-|z|^2)^\alpha\,|\psi'(z)|\,exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\beta+2}}\right]=0.$$

(ii) *For all* $c > 0$,

$$\lim_{|\varphi(z)|\longrightarrow 1}(1-|z|^2)^\alpha\,|\psi'(z)|\,exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\beta+2}}\right]=0$$

and $\psi \in \mathcal{B}_0^\alpha(\mathbb{D})$.

Proof: Suppose that (i) holds for all $c > 0$. Then

$$(1-|z|^2)^\alpha\,|\psi'(z)| \le c(1-|z|^2)^\alpha\,|\psi'(z)|\,exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\beta+2}}\right]$$

$\longrightarrow 0$ *as* $|z| \longrightarrow 1$.

Again if $|\varphi(z)| \longrightarrow 1$, then $|z| \longrightarrow 1$, from which it follows that

$$\lim_{|\varphi(z)|\longrightarrow 1}(1-|z|^2)^\alpha\,|\psi'(z)|\,exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\beta+2}}\right]=0.$$

Conversely, suppose that (ii) holds but (i) is not true for some $c > 0$. Then there are c_0 and ϵ_0 and a sequence $\{z_n\}$ tending to $\partial\mathbb{D}$ such that

$$(1-|z_n|^2)^\alpha\,|\psi'(z_n)|\,exp\left[\frac{c}{(1-|\varphi(z_n)|^2)^{\beta+2}}\right]\ge \epsilon_0 \qquad (3.1).$$

Since $\psi \in \mathcal{B}_0^\alpha$, (3.1) indicates that $\{z_n\}$ has a subsequence $\{z_{n_k}\}$ with $|\varphi(z_{n_k})| \longrightarrow 1$. Thus (ii) produces the following limit:

$$\left(1-\left|z_{n_k}\right|^2\right)^{\alpha}\ \left|\psi'\left(z_{n_k}\right)\right| exp\left[\frac{c}{\left(1-\left|\varphi\left(z_{n_k}\right)\right|^2\right)^{\beta+2}}\right]\longrightarrow 0,$$

which contradicts (3.1). Hence we are done. ■

Lemma 3.2: *Let* $\alpha>0$, $\beta>-1$, $\psi\in H(\mathbb{D})$ *and* φ *be a holomorphic self map on* $\mathbb{D}$. *Then the following are equivalent:*

(i) *For all* $c>0$,

$$\lim_{|z|\longrightarrow 1}\frac{(1-|z|^2)^{\alpha}}{1-|\varphi(z)|^2}\ |\psi(z)\varphi'(z)|\ exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\beta+2}}\right]=0.$$

(ii) *For all* $c>0$,

$$\lim_{|\varphi(z)|\longrightarrow 1}\frac{(1-|z|^2)^{\alpha}}{1-|\varphi(z)|^2}\ |\psi(z)\varphi'(z)|\ exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\beta+2}}\right]=0$$

and

$$\lim_{|z|\longrightarrow 1}(1-|z|^2)^{\alpha}\,|\psi(z)\varphi'(z)|=0.$$

Proof: Suppose that (i) holds for all $c>0$. Then

$$(1-|z|^2)^{\alpha}\,|\psi(z)\varphi'(z)|$$
$$\leq\frac{(1-|z|^2)^{\alpha}}{1-|\varphi(z)|^2}\ |\psi(z)\varphi'(z)|\ exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\beta+2}}\right]$$

$\longrightarrow 0\ as\ |z|\longrightarrow 1$.

Again, if $|\varphi(z)|\longrightarrow 1$, then $|z|\longrightarrow 1$ and so, we have

$$\lim_{|\varphi(z)|\longrightarrow 1}\frac{(1-|z|^2)^{\alpha}}{1-|\varphi(z)|^2}\ |\psi(z)\varphi'(z)|\ exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\beta+2}}\right]=0.$$

Conversely, suppose that (ii) holds but (i) is not true for some $c>0$. Then there exists c_0 and ϵ_0 and a sequence $\{z_n\}$ tending to $\partial\mathbb{D}$ such that

$$\frac{(1-|z_n|^2)^{\alpha}}{1-|\varphi(z_n)|^2}\ |\psi(z_n)\varphi'(z_n)|\ exp\left[\frac{c}{(1-|\varphi(z_n)|^2)^{\beta+2}}\right]\geq\epsilon_0 \qquad (3.2).$$

Since

$$\lim_{|z|\longrightarrow 1}(1-|z|^2)^{\alpha}\,|\psi(z)\varphi'(z)|=0,$$

(3.2) indicates that $\{z_n\}$ has a subsequence $\{z_{n_k}\}$ with $\left|\varphi\left(z_{n_k}\right)\right|\longrightarrow 1$. Thus (ii) produces the following limit:

$$\frac{\left(1-\left|z_{n_k}\right|^2\right)^{\alpha}}{1-\left|\varphi\left(z_{n_k}\right)\right|^2}\ \left|\psi\left(z_{n_k}\right)\varphi'\left(z_{n_k}\right)\right| exp\left[\frac{c}{\left(1-\left|\varphi\left(z_{n_k}\right)\right|^2\right)^{\beta+2}}\right]\longrightarrow 0,$$

which contradicts (3.2). Hence we are done. ■

Theorem 3.3: *Let* $\alpha>0$, $\beta>-1$, $\psi\in H(\mathbb{D})$ *and* φ *be a holomorphic self map on* $\mathbb{D}$. *Then the following are equivalent:*

(i) $W_{\varphi,\psi}$ *maps* $\mathcal{A}_{\beta}^{0}(\mathbb{D})$ *boundedly in to* $\mathcal{B}_0^{\alpha}(\mathbb{D})$.

(ii) $W_{\varphi,\psi}$ *maps* $\mathcal{A}_{\beta}^{0}(\mathbb{D})$ *compactly in to* $\mathcal{B}_0^{\alpha}(\mathbb{D})$

(iii) *For all* $c>0$,

$$\lim_{|z|\to 1}(1-|z|^2)^{\alpha}\,|\psi'(z)|\,exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\beta+2}}\right]=0$$

and

$$\lim_{|z|\to 1}\frac{(1-|z|^2)^{\alpha}}{1-|\varphi(z)|^2}\,|\psi(z)\varphi'(z)|\,exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\beta+2}}\right]=0.$$

Proof: (ii) implies (i) is obvious. Let us prove (i) implies (iii). Suppose (i) holds. Taking $f(z)=1$ in $\mathcal{A}_{\beta}^{0}(\mathbb{D})$, we get

$$\psi\in\mathcal{B}_0^{\alpha}. \tag{3.3}$$

Again taking $f(z)=z$ in $\mathcal{A}_{\beta}^{0}$, we have

$$\lim_{|z|\to 1}(1-|z|^2)^{\alpha}\,|\psi(z)\varphi'(z)|=0 \tag{3.4}$$

Also by choosing the same test function as in the theorem 4.3.1 , we get

$$\lim_{|\varphi(z)|\to 1}(1-|z|^2)^{\alpha}\,|\psi'(z)|\,exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\beta+2}}\right]=0 \tag{3.5}$$

and

$$\lim_{|\varphi(z)|\to 1}\frac{(1-|z|^2)^{\alpha}}{1-|\varphi(z)|^2}\,|\psi(z)\varphi'(z)|\,exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\beta+2}}\right]=0 \tag{3.6}$$

Combining (3.3) and (3.5), by Lemma 3.1, we get

$$\lim_{|z|\to 1}(1-|z|^2)^{\alpha}\,|\psi'(z)|\,exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\beta+2}}\right]=0$$

Again combining (3.4) and (3.6), by Lemma 3.2, we get

$$\lim_{|z|\to 1}\frac{(1-|z|^2)^{\alpha}}{1-|\varphi(z)|^2}\,|\psi(z)\varphi'(z)|\,exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\beta+2}}\right]=0.$$

Now suppose that (iii) holds. We can prove (ii) on the same lines as in the Theorem 2.1. we omit the details. ■

Corollary 3.4: *Let* >0 , $\beta>-1$, *and* φ *be a holomorphic self-map on* $\mathbb{D}$. *Then the following are equivalent:*

(i) C_{φ} *maps* $\mathcal{A}_{\beta}^{0}(\mathbb{D})$ *boundedly in to* $\mathcal{B}_0^{\alpha}(\mathbb{D})$.

(ii) C_{φ} *maps* $\mathcal{A}_{\beta}^{0}(\mathbb{D})$ *compactly in to* $\mathcal{B}_0^{\alpha}(\mathbb{D})$

(iii) *For all* $c>0$,

$$\lim_{|z|\to 1}(1-|z|^2)^{\alpha}\,|\varphi'(z)|\,exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\beta+2}}\right]=0.$$

Corollary 3.5: *Let* $\alpha>0$, $\beta>-1$, $\psi\in H(\mathbb{D})$ *Then the following are equivalent:*

(i) M_{ψ} *maps* $\mathcal{A}_{\beta}^{0}(\mathbb{D})$ *boundedly in to* $\mathcal{B}_0^{\alpha}(\mathbb{D})$.

(ii) M_{ψ} *maps* $\mathcal{A}_{\beta}^{0}(\mathbb{D})$ *compactly in to* $\mathcal{B}_0^{\alpha}(\mathbb{D})$

(iii) $\psi\equiv 0$.

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