A New Approach to Estimate the Mix Efficiency

in Data Envelopment Analysis

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Abstract

Recently, in Data Envelopment Analysis (DEA) methodology, there are two type of efficiency measurement; radial and non-radial measure represented by the Charnes-Cooper-Rhodes (CCR) and the slack-based measure (SBM) model respectively. There are two characteristics to the target in oriented CCR model; the input minimization (output maximization) plus any further reduction (expansion) indicated by any nonzero slacks while the SBM deals directly with input or output slacks. Both models were important in order to estimate the mix efficiency of production. This study attempts to propose a single DEA model that cope together the characteristics of the CCR and the SBM model. The aim from this proposed model was to provide a new approach for estimating the mix efficiency. We implemented the new approach

on a data set and attained almost similar result of the mix efficiency score compared to the standard method. The validity of our proposed model as a tool for efficiency measurement was proven by the numerical results. We observe that this new approach is computationally advantageous since it only needs to solve a single model instead using the previous approach.

Mathematics Subject Classification: 90

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1   Introduction

Data Envelopment Analysis (DEA) is a mathematical programming model that adapting the linear programming. It has been used to measure the relative efficiency of various observations entities called as decision making units (DMUs). The DMUs under analysis must be comparable, in that they use the same set of inputs to produce the same set of outputs. Hence we measure a particular DMU by comparing it to other entities of its kind. DEA was first introduced by Charnes, Cooper and Rhodes (CCR) (1978) and this approach is known as CCR model. In this paper, we also introduced another important DEA model, which is the slack-based measure (SBM) of Tone (2001).

Basically we have two type of measure in DEA; radial and non-radial. The CCR model was a well known radial measure in DEA literature; hence we used this model to represent the radial one. The objective of the CCR model is to find the minimum (maximum) value that proportionally reduces (expands) the input (output) levels. Next, we need to execute the second phase of the CCR in order to identify the slacks, if any. The drawbacks of this model are the identified slacks are not accounted for the efficiency score and the excess procedure by the second phase. Non-radial model are represented by the slack-based measure (SBM) of Tone (2001). Noted that this type of measure is directly deals with slacks and divided into oriented and non-oriented measure. However, in this paper, we are dealing with the oriented SBM model. For details of comparison between radial and non-radial measure, see Avkiran et.al (2008) along with the shortcoming for both the CCR and the SBM model.

Tone (1998) suggests that the results from both models can be used to evaluate the mix-efficiency. The mix efficiency is a degree of balances of inputs are used (or outputs are produced) together. This study attempts to cope together the characteristics of the CCR and SBM measure into a single DEA model. Furthermore, by applying this proposed model, we wish to establish an alternative method to estimate the mix-efficiency in single measurement instead of Tone's procedure.

This paper unfolds as follows. We introduce the CCR and the SBM model along with their relationship, also the mix efficiency in section 2. In section 3 we proposed

a new model to overcome such shortcomings of the CCR and the SBM, and suggest an alternative method for evaluation of the mix efficiency. We demonstrate a numerical example and compare our approach with the previous method in the next section. Finally, we conclude the paper in the final section.

2 Background

In this section, we introduce the two well-known DEA models; the CCR and the SBM model. The important of these two models are not only to measure the efficiency score of productions, also being used to estimates the mix-efficiency score of a particular DMU. Throughout this paper, we deal with input orientation, in order to keep the consistency between the models.

2.1 Notation and production possibility set

From here onwards, we consider with a set of n DMUs, which early noted for observations set. Each DMU_j ($j=1, 2,\ldots, n$) produces s different amount of outputs $\mathbf{Y}_{rj}=(y_{1j},\ldots,y_{sj})$ utilizing m different amount of inputs $\mathbf{X}_{ij}=(x_{1j},\ldots,x_{mj})$. We assume that the data set is real positive, $\mathbf{X} > \mathbf{0}, \mathbf{Y} > \mathbf{0}$. In order to estimates the set of feasible inputs-outputs combination, Cooper et.al (2006) defined a production possibility set P under constant return-to-scale assumption as follow:

$$P = \{(x_0,y_0) \mid x_0 \geq \mathbf{X}\lambda,\ y_0 \leq \mathbf{Y}\lambda,\ \lambda \geq 0 \} \tag{1}$$

where λ is a semipositive vector in R^n. The efficiency of each DMU_j is evaluates relatively to (1).

2.2 The CCR and the SBM models

Below we briefly explain the CCR and the SBM models and its important aspects on efficiency measurement.

2.2.1 The CCR model

As noted earlier, we deal with the input-oriented CCR (CCR-I) model as formulated as follow:

$$[\text{CCR-I}] \qquad \theta_{CCR}{}^* = \text{Min} \quad \theta_{CCR} \tag{2}$$

subject to

$$\theta_{CCR}x_0 = \mathbf{X}\lambda + s^- \tag{3}$$

$$y_0 = \mathbf{Y}\lambda - \mathbf{s}^+ \tag{4}$$
$$\lambda \geq 0, \mathbf{s}^- \geq 0, \mathbf{s}^+ \geq 0 \tag{5}$$

where $\mathbf{s}^-$, $\mathbf{s}^+$ represents the slacks (input excesses and output shortfalls respectively).

Normally CCR-I was solved in a two-phase process. The first phase was to retrieve the optimal value for θ^*. For the second phase, replacing (2) by 'maximizing $\mathbf{s}^- + \mathbf{s}^+$' as its objective function, followed by the same constraint as first phase. This second phase used to identify the remaining slacks (if any) while keeping θ_{CCR} to its minimum value. Hence we have the following definition.

Definition 2.1 (CCR-I efficient): A DMU_0 $(\mathbf{x_0,y_0})$ is CCR-I efficient, if the optimal objective value θ^* is equal to one and all slacks $\mathbf{s}^-$ and $\mathbf{s}^+$ are zero for every optimal solution of [CCR-I]. However the existence of the nonzero slacks does not reported along the efficiency score.

2.2.2 The SBM model

To keep it coherence with the CCR-I model, we used the input-oriented SBM model as follow:

$$[SBM\text{-}I] \qquad \rho_{in}^* = \min 1 - \frac{1}{m}\sum_{i=1}^{m} t_i^- / x_{i0} \tag{6}$$

subject to

$$x_0 = \mathbf{X}\lambda + t^- \tag{7}$$
$$y_0 = \mathbf{Y}\lambda - t^- \tag{8}$$
$$\lambda \geq 0, \ t^+ \geq 0, t^- \geq 0 \tag{9}$$

The SBM-efficient defined as follow:

Definition 2.2 (SBM-I efficient): A DMU_0 $(\mathbf{x_0,y_0})$ is SBM-I efficient, if the optimal ρ_{in}^* equal to one and all input slacks equal to zero for every optimal solution of [SBM-I].

Note that the SBM ρ_{in}^* was strongly related to the CCR θ^* model, which we have the inequality $\rho_{in}^* \leq \theta^*$. A particular DMU is deemed as SBM-I efficient if and only if it is CCR-efficient. For more details about the SBM model, please refer to Tone (2001). Also see Avkiran et.al (2008) about their comparisons and shortcomings.

2.3 Mix efficiency

The mix-efficiency first introduces by Tone (1998), and was explained later by Cooper et.al (2006). The mix efficiency also based on orientation, which is the input and output mix of efficiency. For the purpose of this paper, we continue with input

orientation of mix efficiency, consistent with both the CCR-I and the SBM-I model in the preceding paragraph. Cooper et.al (2006) defined the input mix efficiency as follow:

Definition 2.3 (Input-mix efficiency): Let the input-oriented CCR and SBM scores of DMU_0 be $\theta_{CCR}*$ and $\rho_{in}*$, respectively. The mix efficiency is defined by

$$[MIX] = \rho_{in}* / \theta_{CCR}* \tag{10}$$

The input mix efficiency is a measure to estimate of how well the set of inputs used together, regarding to the level and mix of inputs in order to efficiently produce the given level of outputs. Noted that the input-mix efficiency satisfies [0, 1] and equals to one if and only if $\rho_{in}* = \theta*$ holds. If the inputs mix efficiency score equal to 1, it shows that the particular DMU has the most efficient combination of inputs, even though they find as technically inefficient. See Herrero et.al (2006).

3   The Proposed Model

In an effort to cope together the characteristics of the CCR-I and the SBM-I measurements, we proposed a linear programming model named by CCRm as follow:

$$[CCRm] \qquad \gamma* = \min \theta - \frac{1}{m}\sum_{i=1}^{m} s_i^-/x_{i0} \tag{11}$$

subject to

$$\theta x_0 = X\lambda + s^- \tag{12}$$
$$y_0 = Y\lambda - s^+ \tag{13}$$
$$0 \leq \theta \leq 1 \tag{14}$$
$$\lambda \geq 0, \ s^- \geq 0, \ s^+ \geq 0 \tag{15}$$

where $\frac{1}{m}\sum_{i=1}^{m} s_i^-/x_{i0}$ is the average of normalized input slacks and the constraint (14) restricted θ to [0,1]. The role of the average of normalized input slacks here is for incorporating the existence of mix input inefficiencies (if any) in reporting the efficiency score.

Noted that the [CCRm] above represent the input orientation. The formula for $\gamma*$ in (11) can be rewritten as follow:

$$\gamma* = \theta - \frac{1}{m}\sum_{i=1}^{m} s_i^-/x_{i0} = \frac{1}{m}\sum_{i=1}^{m}(\theta x_{i0} - s_i^-)/x_{i0} \tag{16}$$

then using (12), the expression in (16) become $\frac{1}{m}\sum_{i=1}^{m} X\lambda/x_{i0}$. Noted that from (1) it gives $x_{i0} \geq X\lambda$, thus we have $0 \leq \gamma* \leq 1$. Let an optimal solution of the above model be

(γ^*, θ^*, λ^*, s^-*, s^+*). Then based on this optimal solution, we have the following definition:

Definition 3.1 (CCRm-efficient): A DMU (x_0,y_0) is CCRm-efficient if and only if $\theta^* = 1$ and all slacks are zero.

This shows that there are no input excesses and output shortfalls were present in its optimal solutions. Therefore, the value of θ^* is necessarily equal to one when a DMU is CCRm-efficient. With $\theta^* = 1$ and all slacks are zero, showing that there is no mix input inefficiencies were present and lead to γ^* equal to 1, reach the full efficiency status. If there is occurrence of nonzero slacks then it leads to the inefficient DMU. Notice that, it is not simply deemed as efficient unit when γ^* equal to 1, since the output shortfalls may occur and not reported in the objective function. Thus the definition 3.1 must be satisfied in order to attain the fully efficient.

Theorem 3.1 If $\theta = 1$, then the CCRm becomes the SBM-I model

Proof: Let θ equal to 1, then substituted onto (11). Obviously the objective function of CCRm becomes the same as of the SBM-I model. The rest of constraints also depict as the SBM-I model, and the restriction of (14) vanished. Thus the CCRm becomes the SBM-I model.

Definition 3.2 (New input mix efficiency): The input mix efficiency of CCRm is defined by

$$\theta_{mix} = \gamma^* / \theta^* \tag{17}$$

which γ^* and θ^* were the optimal solution of CCRm model. The new input mix efficiency satisfies satisfies $0 \leq \theta_{mix} \leq 1$ since we have the relationship of $\gamma^* \leq \theta^*$, based on (11). Hence, we have the score of θ_{mix} equal to one if and only if $\gamma^* = \theta^*$. Therefore the value of θ_{mix} equal to one is not necessary happened only when the DMU under consideration is deemed as technically efficient. We might say that the $\theta_{mix} = 1$ suggests that the combination of inputs being used was in the most efficient state for producing the current outputs level. Previously, in order to estimate the input mix efficiency as suggested by Tone (1998), we need to execute two different DEA model. However, by implementing the CCRm model, we only need to perform a single model to measure its input efficiency score, and at the same time we can estimate the value of input-mix efficiency for a particular DMU. Therefore, this new approach can be using as alternative that advancing the previous approach.

4   Numerical Illustrations

In this paper we illustrate our proposed method with an example data as listed in Table 1. The data set we used consist twelve DMUs, which has two inputs and two outputs each. We performed the CCRm model using this data set and discuss the obtained results as follows.

Table 1: Illustrative data set

DMU	A	B	C	D	E	F	G	H	I	J	K	L
Input 1	20	19	25	27	22	55	33	31	30	50	53	38
Input 2	151	131	160	168	158	255	235	206	244	268	306	284
Output 1	100	150	160	180	94	230	220	152	190	250	260	250
Output 2	90	50	55	72	66	90	88	80	100	100	147	120

Source: Cooper et.al (2006)

Table 2: Results of CCRm-I

DMU	γ^*	θ^*	Excess Input 1	Excess Input 2	Shortage Output 1	Shortage Output 2
A	1	1	0	0	0	0
B	1	1	0	0	0	0
C	0.85217	0.88504	1.64357	0	0	0
D	1	1	0	0	0	0
E	0.75561	0.76609	0.46106	0	0	0
F	0.70376	0.84646	15.69642	0	0	0
G	0.89484	0.90196	0	3.34902	0	0
H	0.77396	0.80439	1.88656	0	0	0
I	0.90464	0.96039	0	27.20627	0	0
J	0.78051	0.88455	10.40386	0	0	0
K	0.86614	0.96357	10.32812	0	0	0
L	0.93602	0.95820	0	12.60062	0	0

Table 2 report the optimal solution of the CCRm model. The information presented by this table showing two different score which are the optimal objective value γ^* the contraction variable θ^* in the second and the third column respectively. The next four columns revealed the inputs and the outputs slacks that the model identified. Observed that the value of γ^* equal to θ^* for efficient DMUs and lesser for inefficient units. This happened for inefficient units due to the existence of the input excesses that affecting its optimal efficiency score. Note that there were no outputs shortage appeared in this table. From this table, we discovered there are only three DMUs are found to be CCRm-efficient. All of these efficient DMUs, which are DMU A, B and

D, satisfy the condition given in Definition 3.1, while the remainders DMUs show some level of inefficiencies. For each inefficient unit, we noticed that there are two sources that lead to inefficiencies; the input contraction by variable θ^*, and the excesses of input slacks. This observation gives a new insight of how the mix efficiency of a production can be discovering by using the CCRm model.

Table 3: The score of CCR, SBM and the input-mix efficiency

DMU	$\theta_{CCR}*$	$\rho_{in}*$	[MIX]	θ_{mix}
A	1	1	1	1
B	1	1	1	1
C	0.88271	0.85217	0.96540	0.96286
D	1	1	1	1
E	0.76350	0.75561	0.98967	0.98632
F	0.83477	0.70376	0.84306	0.83142
G	0.90196	0.89484	0.99211	0.99211
H	0.79633	0.77396	0.97190	0.96217
I	0.96039	0.90464	0.94195	0.94195
J	0.87065	0.78051	0.89647	0.88238
K	0.95510	0.86614	0.90686	0.89889
L	0.95820	0.93602	0.97685	0.97685

Table 3 enumerates the score of the CCR-I and the SBM-I model in the first two columns. These two score represented by $\theta_{CCR}*$ and $\rho_{in}*$ respectively. The next column, labeled with [MIX] stand for the input mix efficiency that being measured as mentioned by Definition 2.3. The last column, labeled with $\theta_{mix}*$ correspond to the new input mix efficiency obtained from the proposed DEA model. By comparing the input mix efficiency score between our new approach, $\theta_{mix}*$ and the former method of [MIX], it show that our method prevailed almost the same as score as [MIX]. Hence, we conclude that this new approach also can be use as measuring tool to estimate the input mix of efficiency. Previously as early suggested as Tone (1998), in order to evaluate the input mix efficiency, we need initially obtained the optimal score of $\theta_{CCR}*$ and $\rho_{in}*$. However, by implementing the CCRm model, we only need to execute a single model in order to evaluate both the technical and the mix efficiency score. This finding surely brings a computationally advantageous compared to the previous approach.

5   Conclusions

This paper establishes a linear programming model that enhanced the solution of the oriented DEA methodology. Our observation show that our proposed model; the

CCRm model overcomes the criticism of two phase in classical CCR solution. By adapting the concept of the slack-based measurements, we incorporated the average of normalized input slack onto the objective function of the CCR model. Thus, we simultaneously coping together both the CCR and the SBM characteristics of efficiency measurement into a single DEA model. In addition, the CCRm method reporting two different source of inefficiency that affecting the slacks value instead of the SBM model that sums up all together the slacks into a single score.

The primary aim of this paper is to provide a new method to evaluate the mix efficiency measurement. We have demonstrated the validity of the new approach by comparing with the result of standard method for estimating the mix efficiency. The principal contribution of this paper is the development of a new single model as alternative to the previously approach. Its proven benefits the computational effort since the new method only need to solve a single measurement instead of conducting two different models in order to obtain the mix efficiency score.

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