

On Simulating Univariate and Multivariate Burr Type III and Type XII Distributions

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Abstract

This paper describes a method for simulating univariate and multivariate Burr Type III and Type XII distributions with specified correlation matrices. The methodology is based on the derivation of the parametric forms of a *pdf* and *cdf* for this family of distributions. The paper shows how shape parameters can be computed for specified values of skew and kurtosis. It is also demonstrated how to compute percentage points and other measures of central tendency such as the mode, median, and trimmed mean. Examples are provided to demonstrate how this Burr family can be used in the context of distribution fitting using real data sets. The results of a Monte Carlo simulation are provided to confirm that the proposed method generates distributions with user specified values of skew, kurtosis, and intercorrelation. Tabled values of shape parameters and boundary values of kurtosis are also provided in the appendices for the user.

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1 Introduction

Burr [4] introduced a system of twelve cumulative distribution functions (cdfs) for the primary purpose of fitting data. Burr [5] and Tadikamalla [37] gave additional attention to the Type III and Type XII distributions because they include a variety of distributions with varying degrees of skew and kurtosis. For example, the Type XII distributions include characteristics of the normal,

lognormal, gamma, logistic, and exponential distributions as well as other characteristics associated with the Pearson family of distributions [34, 37]. Further, the Type III or Type XII distributions have been used in a variety of settings for the purpose of statistical modeling. Some examples include such topics as forestry [8, 24], fracture roughness [28], life testing [42, 43], operational risk [6], option market price distributions [35], meteorology [26], modeling crop prices [38], and reliability [27].

Although the Type III and Type XII distributions have quantile (or percentile) functions available in closed-form, these distributions have received less attention or use in the context of Monte Carlo studies e.g. examining the robustness or power properties of competing parametric or nonparametric statistics. Other competing families of distributions that also possess closed-form quantile functions such as the power method [7, 13, 14], generalized lambda (GLD) [20, 30-32], or Tukey g -and- h distributions [15, 18, 19, 39] appear more often in the applied literature. One reason for this could be attributed to the computational difficulties associated with simulating Burr distributions with a specified correlation structure when juxtaposed to some of the other families of distributions. Specifically, and in terms of the power method, computationally efficient procedures for simulating correlated non-normal distributions have been provided (see [10, 13, 40, 41]). Some examples of where the power method has been used in this context include such topics or techniques as analysis of covariance [9, 11, 29], item response theory [36], logistic regression [16], regression [12], repeated measures [1, 23], and structural equation modeling [17, 33].

An additional concern to be made regarding the Type III and Type XII distributions is the paucity of research that considers these distributions as a single family in the context of simulating univariate and multivariate non-normal distributions. This concern is raised because one of the advantages that the Type III and Type XII family of distributions has over the other competing families of distributions mentioned above, is that it covers a larger region of kurtosis towards the lower end of the skew (α_1) and kurtosis (α_2) boundary i.e. $\alpha_2 \geq \alpha_1^2 - 2$. For example, there are some Type III distributions with values of $\alpha_1 = 1$ and $\alpha_2 < 0$ (see [5]). On the other hand, the power method, GLD, and Tukey g -and- h transformations are unable to produce distributions with such values of α_1 and α_2 .

In view of the above, the present aim here is to develop the methodology for simulating univariate and multivariate Burr Type III and Type XII distributions. Specifically, we first consider the Type III and Type XII distributions together as a single family and derive general parametric forms of a probability density function (pdf) and a cdf for this family. As such, determining shape parameters, measures of central tendency, and fitting pdfs to empirical data can be done in a computationally efficient manner for either type of distri-

bution. The methodology is subsequently presented for extending the Burr family from univariate to multivariate data generation. Numerical examples and the results from a Monte Carlo simulation are provided to demonstrate and confirm the methodology. Tabled values of shape parameters and boundary values of kurtosis are also provided.

2 Univariate Type III and Type XII Burr Distributions

The Burr family of distributions considered herein is based on transformations that produce non-normal distributions with defined mean, variance, skew, and kurtosis. These transformations are computationally efficient because they only require the knowledge of two shape parameters and an algorithm that generates regular uniform pseudo-random deviates. We begin the derivation of the general parametric forms of a pdf and a cdf for this family of distributions with the following definitions.

Definition 2.1 Let U be a uniformly distributed random variable with pdf and cdf expressed as

$$f_U(u) = 1 \tag{1}$$

$$F_U(u) = Pr(U \leq u) = \int_0^u 1 du = u \tag{2}$$

where $0 \leq u \leq 1$. Let $u = (x, y)$ be the auxiliary variable that maps the parametric curves of (1) and (2) as

$$f : u \rightarrow \mathfrak{R}^2 := f_U(u) = f_U(x, y) = f_U(u, f_U(u)) = f_U(u, 1) \tag{3}$$

$$F : u \rightarrow \mathfrak{R}^2 := F_U(u) = F_U(x, y) = F_U(u, F_U(u)) = F_U(u, u). \tag{4}$$

Definition 2.2 Let the analytical forms of the quantile functions (for generating data) associated with Burr Type III and Type XII distributions be defined as

$$q(u) = q(u)_{III} = (u^{-\frac{1}{k}} - 1)^{\frac{1}{c}} \tag{5}$$

$$q(u) = q(u)_{XII} = ((1 - u)^{-\frac{1}{k}} - 1)^{\frac{1}{c}} \tag{6}$$

where $q(u)$ is said to be a strictly increasing monotonic function in u i.e. derivative $q'(u) > 0$ with real valued parameters c and k that control the shape of a distribution. For both Type III and Type XII distributions the value of k is positive. The value of c is negative (positive) for Type III (Type XII) distributions.

The derivatives of (5) and (6) are

$$q'(u) = q'(u)_{\text{III}} = -(u^{-\frac{1}{k}-1})(u^{-\frac{1}{k}} - 1)^{\frac{1}{c}-1}/(ck) \tag{7}$$

$$q'(u) = q'(u)_{\text{XII}} = (1 - u)^{-\frac{1}{k}-1}((1 - u)^{-\frac{1}{k}} - 1)^{\frac{1}{c}-1}/(ck). \tag{8}$$

Proposition 2.1 If the compositions $f \circ q$ and $F \circ q$ map the parametric curves of $f_{q(U)}(q(u))$ and $F_{q(U)}(q(u))$ where $q(u) = q(x, y)$ as

$$f \circ q : q(u) \rightarrow \mathfrak{R}^2 := f_{q(U)}(q(u)) = f_{q(U)}(q(x, y)) = f_{q(U)}(q(u), 1/q'(u)) \tag{9}$$

$$F \circ q : q(u) \rightarrow \mathfrak{R}^2 := F_{q(U)}(q(u)) = F_{q(U)}(q(x, y)) = F_{q(U)}(q(u), u) \tag{10}$$

then $f_{q(U)}(q(u), 1/q'(u))$ and $F_{q(U)}(q(u), u)$ in (9) and (10) are the pdf and cdf associated with the quantile function in either (5) or (6).

Proof: It is first shown that $f_{q(U)}(q(u), 1/q'(u))$ in (9) has the following properties:

Property 2.1 $\int_0^1 f_{q(U)}(q(u), 1/q'(u))du = 1$, and

Property 2.2 $f_{q(U)}(q(u), 1/q'(u)) \geq 0, 0 \leq u \leq 1$.

To prove Property 2.1, let $y = f(x)$ be a function where $\int_0^1 f(x)dx = \int_0^1 ydx$. Thus, given that $x = q(u)$ and $y = 1/q'(u)$ in $f_{q(U)}(q(x, y))$ of (9) we have $\int_0^1 f_{q(U)}(q(u), 1/q'(u))du = \int_0^1 ydx = \int_0^1 (1/q'(u))dq(u) = \int_0^1 (1/q'(u))q'(u) du = \int_0^1 du = 1$ which integrates to one because by Definition 2.1 $f_{(U)}(u) = 1$ is the regular uniform pdf.

To prove Property 2.2, it is given by definition that $q'(u) > 0$. Hence, $f_{q(U)}(q(u), 1/q'(u)) \geq 0$ because $1/q'(u)$ will be nonnegative on the support of u for all $u \in (0, 1)$ provided that $k > 0$ and $c < 0$ ($c > 0$) for Type III (Type XII) distributions. In terms of Type III distributions, the limit conditions are:

- (a) $\lim_{u \rightarrow 0} f_{q(U)}(q(u), 1/q'(u)) = +\infty$, for $0 < k < 1$ and $-1 < ck < 0$,
- (b) $\lim_{u \rightarrow 0} f_{q(U)}(q(u), 1/q'(u)) = 0$, for $0 < k < 1$ and $ck < -1$,
- (c) $\lim_{u \rightarrow 0} f_{q(U)}(q(u), 1/q'(u)) = 0$, for $k > 1$ and $c < -1$,
- (d) $\lim_{u \rightarrow 1} f_{q(U)}(q(u), 1/q'(u)) = 0$, for $k > 0$ and $c < 0$.

In terms of Type XII distributions, the limit conditions are:

- (a) $\lim_{u \rightarrow 0} f_{q(U)}(q(u), 1/q'(u)) = +\infty$, for $k > 0$ and $0 < c < 1$,
- (b) $\lim_{u \rightarrow 0} f_{q(U)}(q(u), 1/q'(u)) = 0$, for $k > 0$ and $c > 1$,
- (c) $\lim_{u \rightarrow 1} f_{q(U)}(q(u), 1/q'(u)) = 0$, for $k > 0$ and $c > 0$. □

See Figure 1 and Figure 2 for examples of the limit conditions for Type III and Type XII distributions, respectively.

A corollary to Property 2.1 is stated as follows

Corollary 2.1 The derivative of the cdf $F_{q(U)}(q(u), u)$ in (10) is the pdf $f_{q(U)}(q(u), 1/q'(u))$ in (9).

Proof: It follows from $x = q(u)$ and $y = u$ in $F_{q(U)}(q(u), u)$ of (10) that $dx = q'(u)du$ and $dy = 1du$. Hence, using the parametric form of the derivative

we have $y = dy/dx = 1/q'(u)$ in $f_{q(U)}(q(x, y))$ of (9). Whence, $F'_{q(U)}(q(u), u) = F'_{q(U)}(q(x, dy/dx)) = f_{q(U)}(q(x, y)) = f_{q(U)}(q(u), 1/q'(u))$. Thus, $f_{q(U)}(q(u), 1/q'(u))$ in (9) and $F_{q(U)}(q(u), u)$ in (10) are the pdf and cdf associated with the quantile functions in (5) and (6). \square

The moments associated with the Type III and Type XII family of distributions can be determined from

$$E[q(u)^r] = \int_0^1 q(u)^r du = \Gamma[(c + r)/c]\Gamma[k - r/c]/\Gamma[k]. \tag{11}$$

In terms of Type III distributions, the r -th moment exists if $c + r < 0$. For Type XII distributions, the r -th moment exists if $ck > r$. Given that the first $r = 4$ moments exist, the measures of skew (α_1) and kurtosis (α_2) can subsequently be obtained from [21]

$$\alpha_1 = \frac{(E[q(u)^3] - 3E[q(u)^2]E[q(u)] + 2(E[q(u)])^3)/}{(E[q(u)^2] - (E[q(u)])^2)^{\frac{3}{2}}} \tag{12}$$

$$\alpha_2 = \frac{(E[q(u)^4] - 4E[q(u)^3]E[q(u)] - 3(E[q(u)^2])^2 + 12E[q(u)^2] \times (E[q(u)])^2 - 6(E[q(u)])^4)/(E[q(u)^2] - (E[q(u)])^2)^2.}{(E[q(u)^2] - (E[q(u)])^2)^2} \tag{13}$$

Using (11), (12), and (13), the formulae for the mean, variance, skew, and kurtosis for the family of Burr distributions are

$$\mu = (\Gamma[(c + 1)/c]\Gamma[k - 1/c])/ \Gamma[k] \tag{14}$$

$$\sigma^2 = \Gamma[k]^{-2}(\Gamma[(2 + c)/c]\Gamma[k]\Gamma[k - 2/c] - \Gamma[1 + 1/c]^2\Gamma[k - 1/c]^2) \tag{15}$$

$$\alpha_1 = \{1/(\Gamma[(2 + c)/c]\Gamma[k]\Gamma[k - 2/c] - \Gamma[1 + 1/c]^2\Gamma[k - 1/c]^2)\}^{\frac{3}{2}} \times \{ \Gamma[(3 + c)/c]\Gamma[k]^2\Gamma[k - 3/c] - c^{-2}(6\Gamma[1/c]\Gamma[2/c]\Gamma[k]\Gamma[k - 2/c]\Gamma[k - 1/c]) + 2\Gamma[1 + 1/c]^3\Gamma[k - 1/c]^3 \} \tag{16}$$

$$\alpha_2 = \{ \Gamma[(4 + c)/c]\Gamma[k]^3\Gamma[k - 4/c] - c^{-3}(3\Gamma[k - 1/c](4c\Gamma[1/c]\Gamma[3/c]\Gamma[k]^2\Gamma[k - 3/c] - 4\Gamma[1/c]^2\Gamma[2/c]\Gamma[k]\Gamma[k - 2/c]\Gamma[k - 1/c] + c^3\Gamma[1 + 1/c]^4 \times \Gamma[k - 1/c]^3)) \} / \{ \Gamma[(2 + c)/c]\Gamma[k]\Gamma[k - 2/c] - \Gamma[1 + 1/c]^2\Gamma[k - 1/c]^2 \}^2 - 3. \tag{17}$$

Thus, given specified values of α_1 and α_2 associated with either Type III or Type XII distributions, equations (16) and (17) are used to simultaneously solve for the shape parameters of c and k . The solved values of c and k can then be used to evaluate (14) and (15) to determine the mean and variance. Appendix A gives solved values of c and k for various combinations of α_1 and α_2 . Provided in Appendix B are (approximate) lower boundary (LB) and upper boundary (UB) values of α_2 for the given values of α_1 in Appendix A.

The other measures of central tendency considered are the mode(s), median, and trimmed mean. Specifically, if a mode associated with the pdf in (9) exists then it is located at $f_U(q(\tilde{u}), 1/q'(\tilde{u}))$, where $u = \tilde{u}$ is a critical number that solves $dy/du = d(1/q'(u))/du = 0$ and maximizes (either locally or globally) $y = 1/q'(\tilde{u})$ at $x = q(\tilde{u})$. Type III distributions are unimodal if $c < 0$, $k > -1/c$, and $\tilde{u}_{\text{III}} = ((1 + ck)/(c + ck))^k$. Type XII distributions are unimodal if $c > 1$, $k > 0$, and $\tilde{u}_{\text{XII}} = 1 - ((1 + ck)/(c + ck))^k$. Substituting \tilde{u}_{III} into (5) and \tilde{u}_{XII} into (6) and simplifying locates the mode for either type of distribution at the point where $q(\tilde{u}) = ((c - 1)/(ck + 1))^{\frac{1}{c}}$.

The median associated with (9) is located at $q(u = 0.50) = (2^{\frac{1}{k}} - 1)^{\frac{1}{c}}$. This can be shown by letting $x_{0.50} = q(u)$ and $y_{0.50} = F_U(u) = Pr(U \leq u)$ denote the coordinates of the cdf in (10) that are associated with the 50th percentile. In general, we must have $u = 0.50$ such that $y_{0.50} = 0.50 = F_U(0.50) = Pr(U \leq 0.50)$ holds in (10) for the regular uniform distribution. As such, the limit of the quantile function locates the median at $\lim_{u \rightarrow 0.50} q(u) = (2^{\frac{1}{k}} - 1)^{\frac{1}{c}}$.

The 100γ percent symmetric trimmed mean (TM) can be obtained from (11) (with $r = 1$) and from the definition of a TM as ([2] p. 401)

$$\text{TM} = (1 - 2\gamma)^{-1} \int_{\gamma}^{1-\gamma} q(u) du. \quad (18)$$

As $\gamma \rightarrow 0$ the TM will converge to the mean in (14). Conversely, as $\gamma \rightarrow 0.50$ then the TM will converge to the median.

To demonstrate the use of the methodology above, presented in Figure 1 and Figure 2 are pdfs and cdfs from the Burr Type III and Type XII families, respectively. The values and graphs in these figures were obtained using various *Mathematica* [44] functions. More specifically, the values of c and k were obtained by solving equations (16) and (17) using the function FindRoot and the graphs of the pdfs and cdfs were determined using equations (9) and (10) and the function ParametricPlot. The modes were determined by evaluating $q(\tilde{u}) = ((c - 1)/(ck + 1))^{\frac{1}{c}}$ for the solved values of c and k . The heights of the pdfs were subsequently obtained by solving for the values of \tilde{u} that yielded the modes and then evaluating for the heights using $1/q'(\tilde{u})$ in (9). The values that yielded the probabilities of obtaining values of $q(u)$ in the upper 5% of the tail regions were determined by evaluating the quantile functions in (5) and (6) for $u = 0.95$.

3 Fitting Burr Distributions to Data

Presented in Figure 3 are Burr Type XII pdfs superimposed on histograms of circumference measures (in centimeters) taken from the abdomen, chest, forearm, and knee of 252 adult males. Inspection of Figure 3 indicates that

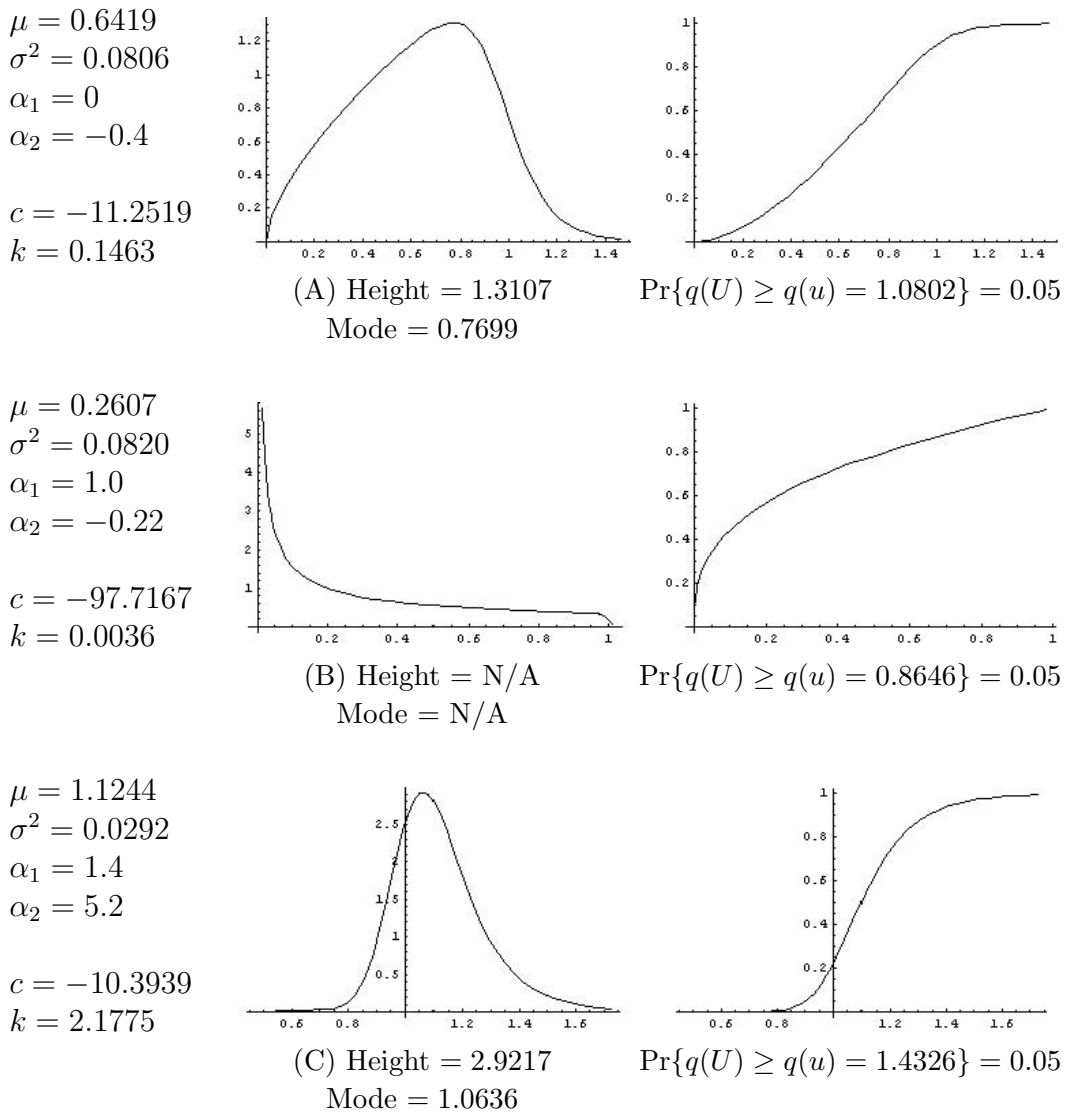
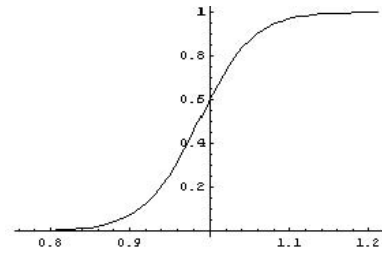
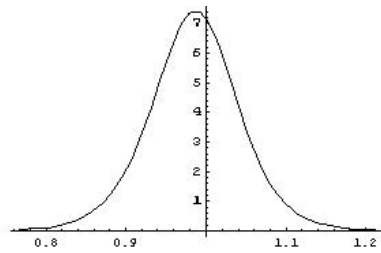


Figure 1: Examples of c and k parameters for Type III distributions and their associated pdfs and cdfs. These distributions can also be empirically simulated by using the quantile function in Equation (5).

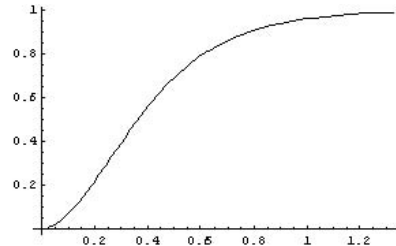
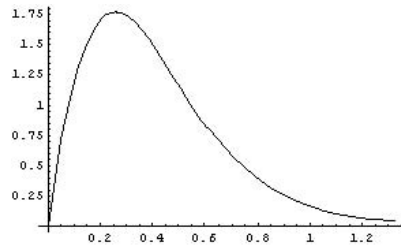
$\mu = 0.9858$
 $\sigma^2 = 0.0036$
 $\alpha_1 = 0$
 $\alpha_2 = 1$
 $c = 27.0729$
 $k = 1.3257$



(A) Height = 7.411
Mode = 0.9873

$\Pr\{q(U) \geq q(u) = 1.0826\} = 0.05$

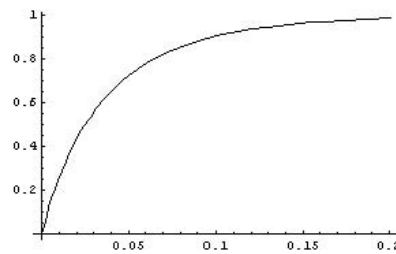
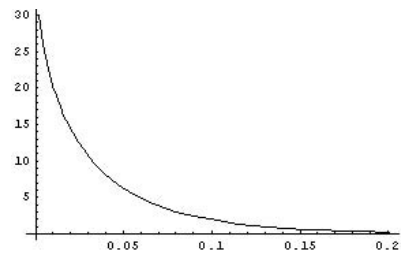
$\mu = 0.4186$
 $\sigma^2 = 0.0772$
 $\alpha_1 = 1.5$
 $\alpha_2 = 4.5$
 $c = 1.8149$
 $k = 4.6909$



(B) Height = 1.769
Mode = 0.2582

$\Pr\{q(U) \geq q(u) = 0.9401\} = 0.05$

$\mu = 0.0408$
 $\sigma^2 = 0.0023$
 $\alpha_1 = 2.75$
 $\alpha_2 = 13$
 $c = 0.9066$
 $k = 20.0039$



(C) Height = N/A
Mode = N/A

$\Pr\{q(U) \geq q(u) = 0.1339\} = 0.05$

Figure 2: Examples of c and k parameters for Type XII distributions and their associated pdfs and cdfs. These distributions can also be empirically simulated by using the quantile function in Equation (6).

the Burr pdfs provide good approximations to the empirical data. We note that to fit the Burr pdfs to the data, the following transformation had to be imposed on the quantile function $q(u) : (M\sigma - \mu S + Sq(u))/\sigma$. The values of the means (M, μ) and standard deviations (S, σ) for the data and Burr pdfs are given in Figure 3.

One way of determining how well a Burr pdf models a set of data is to compute a chi-square goodness of fit statistic. For example, listed in Table 1 below are the cumulative percentages and class intervals based on the Burr pdf for the chest data in Panel B of Figure 3. The asymptotic value of $p = 0.290$ indicates that the Burr pdf provides a good fit to the data. We note that the degrees of freedom for this test were computed as $df = 5 = 10(\text{class intervals}) - 4(\text{parameter estimates}) - 1(\text{sample size})$. Further, the Burr TMs given in Table 2 also indicate a good fit as the TMs are all within the 95% bootstrap confidence intervals based on the data. The confidence intervals are based on 25000 bootstrap samples.

Cumulative %	Burr Class Intervals	Observed Freq	Expected Freq
5	< 88.52	10	12.6
10	88.52 – 90.67	14	12.6
15	90.67 – 92.26	11	12.6
30	92.26 – 95.90	38	37.8
50	95.90 – 100.02	57	50.4
70	100.02 – 104.52	48	50.4
85	104.52 – 109.35	38	37.8
90	109.35 – 111.85	9	12.6
95	111.85 – 115.83	13	12.6
100	> 115.83	13	12.6

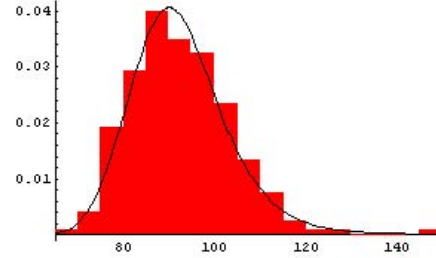
$$\chi^2 = 2.938 \quad \Pr\{\chi_5^2 \leq 2.938\} = 0.292 \quad n = 252$$

Table 1: Observed and expected frequencies and chi-square test based on the Burr Type XII approximation to the chest data in Figure 3.

Empirical Distribution	20% TM	Burr TM
Abdomen	91.808 (91.030, 92.605)	91.721
Chest	100.128 (99.546, 100.736)	100.172
Forearm	28.699 (28.539, 28.845)	28.608
Knee	38.491 (38.320, 38.637)	38.472

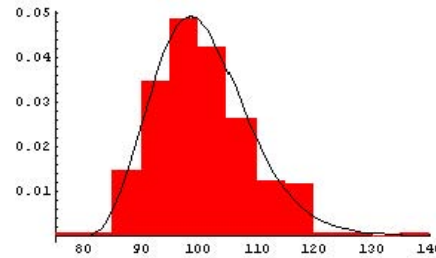
Table 2: Examples of Burr Type XII trimmed means (TMs). Each TM is based on a sample size of $n = 152$ and has a 95% bootstrap confidence interval enclosed in parentheses.

$$\begin{aligned}
 M &= 92.5560 & \mu &= 0.793968 \\
 S &= 10.7831 & \sigma &= 0.287478 \\
 \alpha_1 &= 0.833419 & c &= 3.872133 \\
 \alpha_2 &= 2.18074 & k &= 2.264652
 \end{aligned}$$



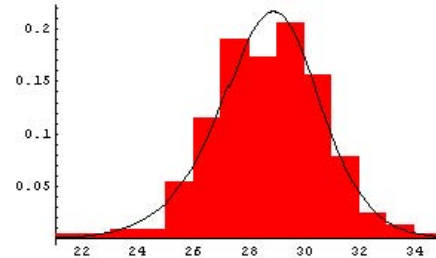
(A) Abdomen

$$\begin{aligned}
 M &= 100.824 & \mu &= 0.559304 \\
 S &= 8.43048 & \sigma &= 0.238712 \\
 \alpha_1 &= 0.677492 & c &= 2.867086 \\
 \alpha_2 &= 0.944087 & k &= 4.468442
 \end{aligned}$$



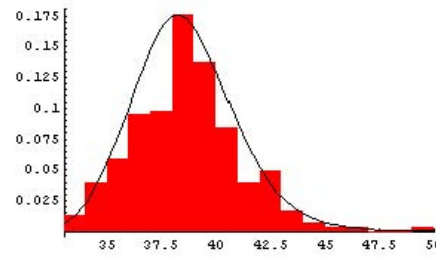
(B) Chest

$$\begin{aligned}
 M &= 28.6639 & \mu &= 0.966818 \\
 S &= 2.02069 & \sigma &= 0.0626464 \\
 \alpha_1 &= -0.218025 & c &= 23.543917 \\
 \alpha_2 &= 0.825501 & k &= 1.7765786
 \end{aligned}$$



(C) Forearm

$$\begin{aligned}
 M &= 38.5905 & \mu &= 0.781809 \\
 S &= 2.41180 & \sigma &= 0.239507 \\
 \alpha_1 &= 0.513663 & c &= 4.446853 \\
 \alpha_2 &= 1.01687 & k &= 2.576122
 \end{aligned}$$



(D) Knee

Data Burr Distribution

Figure 3: Examples of Burr Type XII pdfs' approximations to empirical data using measures of circumference (in centimeters) taken from $n = 252$ males. To fit the Burr distributions to the data the quantile functions $q(u)$ were transformed as $(M\sigma - \mu S + Sq(u))/\sigma$.

4 Multivariate Data Generation

The family of Burr distributions we are considering can be extended from univariate to multivariate data generation by specifying T quantile functions $q(u)$ of the form of either (5) or (6). Specifically, let Z_1, \dots, Z_T denote standard normal variables where the distribution functions and bivariate density function associated with Z_j and Z_k are expressed as

$$\Phi(z_j) = \Pr\{Z_j \leq z_j\} = \int_{-\infty}^{z_j} (2\pi)^{-\frac{1}{2}} \exp\{-u_j^2/2\} du_j \tag{19}$$

$$\Phi(z_k) = \Pr\{Z_k \leq z_k\} = \int_{-\infty}^{z_k} (2\pi)^{-\frac{1}{2}} \exp\{-u_k^2/2\} du_k \tag{20}$$

$$f_{jk} := f_{z_j z_k}(z_j, z_k, \rho_{z_j z_k}) = \left(2\pi(1 - \rho_{z_j z_k}^2)^{\frac{1}{2}}\right)^{-1} \times \exp\left\{-\left(2(1 - \rho_{z_j z_k}^2)\right)^{-1} (z_j^2 + z_k^2 - 2z_j z_k \rho_{z_j z_k})\right\}. \tag{21}$$

Using (19), it follows that a quantile function of the form of either (5) or (6) can be expressed as $q_j(\Phi(z_j))$ because $\Phi(z_j) \sim U[0, 1]$. As such, the bivariate correlation between two standardized Burr distributions, denoted as $x_j(q_j(\Phi(z_j)))$ and $x_k(q_k(\Phi(z_k)))$, can be determined as

$$\rho_{x_j(q_j(\Phi(z_j))), x_k(q_k(\Phi(z_k)))} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x_j(q_j(\Phi(z_j))) x_k(q_k(\Phi(z_k))) f_{jk} dz_j dz_k \tag{22}$$

for $j \neq k$ and where the correlation $\rho_{z_j z_k}$ in f_{jk} is referred to as an intermediate correlation. Thus, the objective is to use equation (22) to solve for the values of the $T(T - 1)/2$ intermediate correlations $\rho_{z_j z_k}$ such that T specified Burr distributions also have a specified correlation matrix.

To generate multivariate Burr distributions, the process begins by assembling the solved intermediate correlations into a $T \times T$ matrix and subsequently factoring this matrix (e.g., a Cholesky factorization). The results from the factorization are then used to produce standard normal deviates that are correlated at the intermediate levels as follows

$$\begin{aligned} Z_1 &= a_{11}V_1 \\ Z_2 &= a_{12}V_1 + a_{22}V_2 \\ &\vdots \\ Z_j &= a_{1j}V_1 + a_{2j}V_2 + \dots + a_{ij}V_i + \dots + a_{jj}V_j \\ &\vdots \\ Z_T &= a_{1T}V_1 + a_{2T}V_2 + \dots + a_{iT}V_i + \dots + a_{TT}V_T \end{aligned} \tag{23}$$

where V_1, \dots, V_T are independent standard normal random deviates and where a_{ij} represents the element in the i -th row and j -th column of the matrix associated with the Cholesky factorization.

To generate the regular uniform deviates required for the quantile functions in (5) or (6), it is suggested that the following series expansion for the unit normal cdf [25] be used on the Z_j in (23) as

$$\Phi(z_j) = .5 + \varphi(z_j) \{ z_j + z_j^3/3 + z_j^5/(3 \cdot 5) + z_j^7/(3 \cdot 5 \cdot 7) + z_j^9/(3 \cdot 5 \cdot 7 \cdot 9) + \dots \} \quad (24)$$

where $\varphi(z_j)$ is the standard normal pdf and where the absolute error associated with (24) is less than 8×10^{-16} .

5 Numerical Example and Monte Carlo Simulation

Suppose we desire to generate correlated data associated with the following Type III and Type XII Burr distributions: (1) Panel A in Figure 1, (2) Panel A in Figure 2, and (3) Panel B in Figure 2 with the specified correlation matrix given in Table 3. Figure 4 gives *Mathematica* source code for numerically solving equation (22) for the three intermediate correlations that are given in Table 4. The results of a Cholesky factorization on the intermediate correlation matrix are given in Table 5. These results are then used in (23) to create Z_1 , Z_2 , and Z_3 with the specified intermediate correlations.

	$q_1(\Phi(Z_1))$	$q_2(\Phi(Z_2))$	$q_3(\Phi(Z_3))$
$q_1(\Phi(Z_1))$	1	0.50	0.60
$q_2(\Phi(Z_2))$	0.50	1	0.70
$q_3(\Phi(Z_3))$	0.60	0.70	1

Table 3: Specified correlations between three Burr distributions. See Figure 1, Panel A (q_1); Figure 2, Panels A and B (q_2 and q_3).

	Z_1	Z_2	Z_3
Z_1	1	0.504372	0.633945
Z_2	0.504372	1	0.736946
Z_3	0.633945	0.736946	1

Table 4: Solved intermediate correlations.

To empirically demonstrate the procedure described above, the three selected Burr distributions in Figures 1 and 2 were simulated in accordance to

```

(* c and k Parameters *)
c1 = -11.25186;
k1 = 0.146295;
c2 = 27.072953;
k2 = 1.325711;
c3 = 1.814856;
k3 = 4.690922;

(* Intermediate Correlations *)
ρ12 = .504372;
ρ13 = .633945;
ρ23 = .736946;

(* Standard Normal CDFs *)
Φ1 = ∫-∞x1 Exp[-u12/2] / Sqrt[2 * Pi] du1;
Φ2 = ∫-∞x2 Exp[-u22/2] / Sqrt[2 * Pi] du2;
Φ3 = ∫-∞x3 Exp[-u32/2] / Sqrt[2 * Pi] du3;

(* Quantile Functions *)
q1 = ((Φ1) ^ (-1/k1) - 1) ^ (1/c1);
q2 = ((1 - Φ2) ^ (-1/k2) - 1) ^ (1/c2);
q3 = ((1 - Φ3) ^ (-1/k3) - 1) ^ (1/c3);

(* Means and Standard Deviations *)
m1 = 0.641872;
m2 = 0.985833;
m3 = 0.418600;
s1 = 0.283920;
s2 = 0.059991;
s3 = 0.277863;

(* Standardized Quantile Functions *)
x1 = (q1 - m1) / s1;
x2 = (q2 - m2) / s2;
x3 = (q3 - m3) / s3;

(* Standard Normal Bivariate PDFs *)
f12 = (Exp[(-1 / (2 * (1 - ρ122))) * ((z12) - 2 * ρ12 * z1 * z2 + z22)] /
(2 * Pi * Sqrt[1 - (ρ122)]);
f13 = (Exp[(-1 / (2 * (1 - ρ132))) * ((z12) - 2 * ρ13 * z1 * z3 + z32)] /
(2 * Pi * Sqrt[1 - (ρ132)]);
f23 = (Exp[(-1 / (2 * (1 - ρ232))) * ((z22) - 2 * ρ23 * z2 * z3 + z32)] /
(2 * Pi * Sqrt[1 - (ρ232)]);

(* Integrals to Compute the Specified Correlations. See Equation 22 *)
int1 = NIntegrate[x1 * x2 * f12, {z1, -8, 8}, {z2, -8, 8}, Method → Trapezoidal]
int2 = NIntegrate[x1 * x3 * f13, {z1, -8, 8}, {z3, -8, 8}, Method → Trapezoidal]
int3 = NIntegrate[x2 * x3 * f23, {z2, -8, 8}, {z3, -8, 8}, Method → Trapezoidal]

```

Figure 4: Mathematica code for computing the intermediate correlations listed in Table 4.

$a_{11} = 1$	$a_{12} = 0.504372$	$a_{13} = 0.633945$
0	$a_{22} = 0.863486$	$a_{23} = 0.483160$
0	0	$a_{33} = 0.603878$

Table 5: Cholesky factorization on the intermediate correlations in Table 4.

the specified correlation matrix in Table 3 using an algorithm coded in Fortran. The algorithm employed the use of subroutines UNI1 and NORMB1 [3] to generate pseudo-random uniform and standard normal deviates. Samples of size $N = 10, 250$, and 1000000 were drawn for each of the three selected distributions using the specified values of c and k given in Figures 1 and 2.

For the samples of size $N = 10$ and 250 the empirical estimates of the correlations, $\hat{\alpha}_1$ and $\hat{\alpha}_2$ were computed using an averaging procedure across 25000 replications and are reported in Tables 6 and 7 along with their root mean squared errors (RMSEs). In terms of the samples of size $N = 1000000$, the statistics were computed directly on these single samples and are reported in Table 8. Inspection of Tables 6-8 indicates that the procedure produced excellent agreement between the empirical estimates and the specified parameters even for samples sizes as small as $N = 10$.

	$q_1(\Phi(Z_1))$	$q_2(\Phi(Z_2))$	$\hat{\alpha}_1$	$\hat{\alpha}_2$
$q_1(\Phi(Z_1))$	1		0.006(1.157)	-0.389(4.200)
$q_2(\Phi(Z_2))$	0.504(0.359)	1	-0.002(1.896)	1.031(7.495)
$q_3(\Phi(Z_3))$	0.601(0.402)	0.702(0.475)	1.500(5.972)	4.700(68.93)

Table 6: Estimated values of correlation, skew ($\hat{\alpha}_1$), and kurtosis ($\hat{\alpha}_2$) between three Burr distributions. See Figure 1, Panel A (q_1); Figure 2, Panels A, B (q_2, q_3). The estimates are based on 25,000 samples of size 10. The RMSEs are in parentheses.

	$q_1(\Phi(Z_1))$	$q_2(\Phi(Z_2))$	$\hat{\alpha}_1$	$\hat{\alpha}_2$
$q_1(\Phi(Z_1))$	1		0.001(0.229)	-0.395(0.848)
$q_2(\Phi(Z_2))$	0.500(0.071)	1	0.000(0.377)	1.002(1.487)
$q_3(\Phi(Z_3))$	0.601(0.080)	0.700(0.095)	1.501(1.102)	4.510(12.81)

Table 7: Estimated values of correlation, skew ($\hat{\alpha}_1$), and kurtosis ($\hat{\alpha}_2$) between three Burr distributions. See Figure 1, Panel A (q_1); Figure 2, Panels A, B (q_2, q_3). The estimates are based on 25,000 samples of size 250. The RMSEs are in parentheses.

	$q_1(\Phi(Z_1))$	$q_2(\Phi(Z_2))$	$\hat{\alpha}_1$	$\hat{\alpha}_2$
$q_1(\Phi(Z_1))$	1		0	-0.4
$q_2(\Phi(Z_2))$	0.5	1	-0.005	1.013
$q_3(\Phi(Z_3))$	0.6	0.7	1.493	4.444

Table 8: Estimated values of correlation, skew ($\hat{\alpha}_1$), and kurtosis ($\hat{\alpha}_2$) between three Burr distributions. See Figure 1, Panel A (q_1); Figure 2, Panels A, B (q_2, q_3). The estimates are based on single samples of size 1,000,000.

6 Comments

Monte Carlo and simulation techniques are widely used in statistical research. Since real-world data sets can often be radically non-normal, it is essential that statisticians have a variety of techniques available for univariate or multivariate non-normal data generation. As mentioned above, the Burr distributions have not been as popular as some other competing methods such as power method polynomials [7, 10, 13, 40] for simulating multivariate non-normal distributions. More specifically, Kotz et al. [22] noted “this [power] method does provide a way of generating bivariate non-normal random variables. Simple extensions of other univariate methods are not available yet” (p. 37). However, the extension of the Burr family of distributions from univariate to multivariate data generation, as presented above, makes this family a viable competitor to the power method because of its simplicity and ease of execution.

Provided in Appendix A are values of the shape parameters c and k to assist the user of the methodology. These values can be used directly or can be used as initial guesses in an equation solver if other shape parameters are desired but not listed in the Tables. It is also important to point out that the solutions to the shape parameters are not unique and that the Type III and Type XII distributions are not mutually exclusive. For example, consider the Type III distribution depicted in Panel C of Figure 1. Given the values of $\alpha_1 = 1.4$ and $\alpha_2 = 5.2$ there also exists a Type XII distribution with these values of skew and kurtosis. The values of the shape parameters are $c = 2.62418$ and $k = 2.57947$.

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$\alpha_1 = 4.00$		
α_2	c	k
LB(III)	LB(III)	LB(III)
16.60	-101.529	0.000491
16.80	-20.2311	0.002526
17.00	-14.7786	0.003542
18.00	-8.78235	0.006606
19.00	-7.27466	0.008652
20.00	-6.54084	0.010293
21.00	-6.09648	0.011693
22.00	-5.79485	0.012926
23.00	-5.57502	0.014034
24.00	-5.40684	0.015042
25.00	-5.27351	0.015969
26.00	-5.16492	0.016828
27.00	-5.07457	0.017628
27.20	-5.05828	0.017782
27.40	-5.04251	0.017933
27.50	-5.03482	0.018009
LB(XII)	LB(XII)	LB(XII)
27.60	0.647667	32722.8
28.00	0.662552	109.723
30.00	0.731156	20.5905
40.00	0.971179	6.11723
50.00	1.11608	4.52013
75.00	1.32217	3.39810
100.00	1.44083	3.00308
125.00	1.52235	2.78914
150.00	1.58347	2.65106
175.00	1.63175	2.55293
200.00	1.67125	2.47883
225.00	1.70441	2.42045
250.00	1.73278	2.37303
275.00	1.75743	2.33358
300.00	1.77910	2.30014
325.00	1.79836	2.27137
350.00	1.81561	2.24631
375.00	1.83119	2.22424
UB(XII)	UB(XII)	UB(XII)
400.00	-4.08421	0.984940
800.00	-4.04064	0.867108
1200.00	-4.02678	0.832512
1600.00	-4.01997	0.815969
2000.00	-4.01592	0.806270
2400.00	-4.01323	0.799897
2800.00	-4.01132	0.795389
3200.00	-4.00989	0.792031
3250.00	-4.00974	0.791671
3264.00	-4.00970	0.791572
3264.50	-4.00970	0.791569
UB(III)	UB(III)	UB(III)

c	k	α_1	α_2
-370.586	0.01	-1.00	0.528799
-434.960	0.01	-1.10	0.881051
-516.170	0.01	-1.20	1.27132
-621.380	0.01	-1.30	1.70128
-762.580	0.01	-1.40	2.17292
-961.320	0.01	-1.50	2.68823
-44.7270	0.10	-1.00	0.794885
-52.6910	0.10	-1.10	1.14702
-62.9910	0.10	-1.20	1.53922
-76.7670	0.10	-1.30	1.97308
-96.0540	0.10	-1.40	2.45047
-124.864	0.10	-1.50	2.97331
-172.395	0.10	-1.60	3.54395
-265.280	0.10	-1.70	4.16467
-526.600	0.10	-1.80	4.83829
-981.124	0.10	-1.85	5.19580
-35.48897	0.20	-1.00	1.37555
-43.0768	0.20	-1.10	1.73755
-53.8449	0.20	-1.20	2.14357
-70.2401	0.20	-1.30	2.59526
-98.0950	0.20	-1.40	3.09453
-155.590	0.20	-1.50	3.64354
-342.590	0.20	-1.60	4.24468
-795.800	0.20	-1.65	4.56560
-47.3200	0.30	-1.00	1.99488
-64.6020	0.30	-1.10	2.38207
-98.6650	0.30	-1.20	2.81772
-196.279	0.30	-1.30	3.30369
-137.629	0.40	-1.00	2.53710
-852.500	0.40	-1.10	2.95980

Appendix B. Approximate values of the lower boundary (LB) and upper boundary (UB) of kurtosis for the Type III and Type XII distributions in the tables of Appendix A.

$\alpha_1 = .00$			
LB(III)	-1.1999975259810600	-1531.8534192235800	0.0006527558000171320
LB(XII)	-0.2831389179221800	3.6023494257189600	Infinity
UB(XII)	1.20	Infinity	1.00
UB(III)	1.2903020932270200	-27.658759663905600	0.7299814725364500
$\alpha_1 = .25$			
LB(III)	-1.1879182994534800	-947.4623508491720	0.0007962750911985100
LB(XII)	-0.2304184621255770	2.7656299151425700	Infinity
UB(XII)	1.3547424358032000	35.02294611136490	1.00
UB(III)	1.5245793593464300	-13.965101416892600	0.6455984580349740
$\alpha_1 = .50$			
LB(III)	-1.0177772742865700	-808.3772140762210	0.0007136504223431020
LB(XII)	0.028004457141406600	2.2155977834006700	Infinity
UB(XII)	1.8277601041089300	17.803614853388400	1.00
UB(III)	2.129137179211670	-9.641023962104130	0.5663797972947810
$\alpha_1 = .75$			
LB(III)	-0.6950059039032010	-667.5648449484920	0.0006700971502532650
LB(XII)	0.49057487329392900	1.8358807672751200	Infinity
UB(XII)	2.6461715472751800	12.183618755201600	1.00
UB(III)	3.169694470780740	-7.579428249677900	0.4946967105670540
$\alpha_1 = 1.00$			
LB(III)	-0.22333305667913900	-690.3619670115560	0.0005096167182371810
LB(XII)	1.1591374468706200	1.5639140222157900	Infinity
UB(XII)	3.8578691537503500	9.453794577377300	1.00
UB(III)	4.763927949783530	-6.405051359585	0.4318300397486560
$\alpha_1 = 1.25$			
LB(III)	0.3947123929110210	-475.62575464988600	0.0005901795019437870
LB(XII)	2.037979246154570	1.3631779212065600	Infinity
UB(XII)	5.536310510524430	7.871273342311370	1.00
UB(III)	7.114148592982970	-5.664766013962820	0.37798203243016600
$\alpha_1 = 1.50$			
LB(III)	1.1572353047855600	-827.9884050348280	0.00027429248849887800
LB(XII)	3.133024205697250	1.2111243388484900	Infinity
UB(XII)	7.788906706405670	6.85514445182749	1.00
UB(III)	10.575801330332400	-5.165722051604490	0.3325672598826120
$\alpha_1 = 1.75$			
LB(III)	2.0632368940042100	-654.1336322729040	0.00028474067199035700
LB(XII)	4.451204300183300	1.093264316483310	Infinity
UB(XII)	10.771284908462000	6.156982123655090	1.00
UB(III)	15.813927862040900	-4.812339958056170	0.2945770585821110

$\alpha_1 = 2.00$			
LB(III)	3.1117840905154200	-631.2751999800900	0.00024506902859725400
LB(XII)	6.00	1.00	Infinity
UB(XII)	14.711705533161400	5.65324308585641	1.00
UB(III)	24.210744519704400	-4.5522678822382500	0.26287302904354400
$\alpha_1 = 2.25$			
LB(III)	4.302352538827770	-958.0222463622800	0.00013571709899864300
LB(XII)	7.787122675680750	0.9248122199484170	Infinity
UB(XII)	19.954142797604500	5.275920734424100	1.00
UB(III)	39.16019495539240	-4.354742125310470	0.23636729281034500
$\alpha_1 = 2.50$			
LB(III)	5.634613629258550	-980.5369429410990	0.00011266067114394700
LB(XII)	9.820305552719350	0.8631745147305940	Infinity
UB(XII)	27.038114932743800	4.984734578248810	1.00
UB(III)	71.77261503501210	-4.200707419395350	0.21410637463456500
$\alpha_1 = 2.75$			
LB(III)	7.108672950382420	-741.9032648514840	0.0001277659831352040
LB(XII)	12.10717250475590	0.8118803542680700	Infinity
UB(XII)	36.856976109595600	4.7544616520397700	1.00
UB(III)	191.50555613913700	-4.077854737873690	0.1952728383175950
$\alpha_1 = 3.00$			
LB(III)	8.727749898778970	-764.871419098459	0.00010727293239711600
LB(XII)	14.655160175116700	0.7686155182291970	Infinity
UB(XII)	51.002147158594800	4.5685900000	1.00
UB(III)	689.7543310989990	-4.0200	0.11983600
$\alpha_1 = 3.25$			
LB(III)	10.478914284126800	-822.7001385732750	0.0000871059778453576
LB(XII)	17.471475424275700	0.7316804290877360	Infinity
UB(XII)	72.600484570407	4.415950000155090	1.00
UB(III)	925.0837323167740	-4.0150	0.08577150
$\alpha_1 = 3.50$			
LB(III)	12.375945811929000	-753.91130071296800	0.00008361063093333150
LB(XII)	20.563075547423600	0.6998063983865790	Infinity
UB(XII)	108.7746658007350	4.288688202415080	1.00
UB(III)	1447.8404258791200	-4.0100	0.06813000
$\alpha_1 = 3.75$			
LB(III)	14.413795891067100	-758.9880659741690	0.00007355319868765170
LB(XII)	23.936662792761300	0.672031865058582	Infinity
UB(XII)	179.9557272424410	4.181207576606010	1.00
UB(III)	2042.6955026479100	-4.0075	0.05621950

$\alpha_1 = 4.00$			
LB(III)	16.592364537808100	-828.0185325255550	0.00006008319629168
LB(XII)	27.59868759438910	0.6476176430783680	Infinity
UB(XII)	378.32930662991700	4.089391793019120	1.00
UB(III)	3264.859278230980	-4.0050	0.04764655
$\alpha_1 = -.25$			
LB(III)	-1.04635820408785	-824.9152274072160	0.001629763950085830
LB(XII)	-0.12414849605625500	4.970536608883070	Infinity
UB(XII)	1.1152818251350400	220.00042557851500	1.336396505942700
UB(III)	1.396185175186670	-1120.436598051390	0.7998370746964720
$\alpha_1 = -.50$			
LB(III)	-0.7171725247634950	-1293.457607799400	0.0014199814712539400
LB(XII)	0.25810926173229600	7.493531535546860	Infinity
UB(XII)	1.218028708942510	220.00005597875700	1.8558949823153600
UB(III)	1.7221365709068400	-1826.7756965590200	0.6569791225338680
$\alpha_1 = -.75$			
LB(III)	-0.19857011961895000	-1025.7380032091900	0.002499999840538190
LB(XII)	0.8818665657334750	13.388171864910700	Infinity
UB(XII)	1.5038040546867900	220.000	3.0522772727272700
UB(III)	2.1758862296024700	-3460.086639163930	0.5416587051792580
$\alpha_1 = -1.00$			
LB(III)	0.5259438684990860	-1479.4010245307800	0.0024999982616121600
LB(XII)	1.7732728048672000	40.74306711415020	Infinity
UB(XII)	1.9870253273082700	220.0000016645500	9.449545370622180
UB(III)	2.7399934037154	-3413.482202884210	0.4420805584421760

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