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On Statistically Convergent Double Sequences in Intuitionistic Fuzzy 2-Normed Spaces

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Abstract

Quite recently, the concept of intuitionistic fuzzy 2-normed spaces was introduced by Mursaleen and Lohani [6]. Also, Mursaleen and Mohiuddine [7] have studied the notion of statistical convergence of double sequences in intuitionistic fuzzy normed spaces. In this paper, we study the concept of statistically convergent and statistically Cauchy double sequences in intuitionistic fuzzy 2-normed spaces.

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1 Introduction

The idea of statistical convergence was introduced by Fast [2] and Steinhaus [17] independently and later on studied by various authors. Active researches on this topic were started after the papers of Salat [14] and Fridy [3].

Definition 1.1 *Let K be a subset of $\mathbb{N}$. A number sequence $x = (x_k)$ is said to be statistically convergent to the number ℓ if for each $\epsilon > 0$, the set $K(\epsilon) = \{k \leq n : |x_k - \ell| > \epsilon\}$ has asymptotic density zero, i.e.*

$$\lim_n \frac{1}{n}|\{k \leq n : |x_k - \ell| > \epsilon\}| = 0.$$

In this case we write $st\text{-}\lim x = \ell$.

Notice that every convergent sequence is statistically convergent to the same limit, but its converse need not be true.

Definition 1.2 *A double sequence $x = (x_{jk})$ is said to be $Pringsheim's$ convergent (or P-convergent) if for given $\epsilon > 0$ there exists an integer N such that $|x_{jk} - \ell| < \epsilon$ whenever $j, k > N$. We shall write this as*

$$\lim_{j,k\to\infty} x_{jk} = \ell,$$

where j and k tending to infinity independent of each other [9].

Definition 1.3 *Let $K \subseteq \mathbb{N} \times \mathbb{N}$ be a two-dimensional set of positive integers and let $K(m,n)$ be the numbers of (j,k) in K such that $j \leq m$ and $k \leq n$. Then the two-dimensional analogue of natural density can be defined as follows [5].*
The lower asymptotic density of the set $K \subseteq \mathbb{N} \times \mathbb{N}$ is defined as

$$\underline{\delta_2}(K) = \liminf_{m,n} \frac{K(m,n)}{mn}.$$

In case the sequence $K(m,n)/mn$ has a limit in Pringsheim's sense then we say that K has a double natural density and is defined as

$$\lim_{m,n} \frac{K(m,n)}{mn} = \delta_2(K).$$

Example 1.1 *Let $K = \{(i^2, j^2) : i, j \in \mathbb{N}\}$. Then*

$$\delta_2(K) = \lim_{m,n} \frac{K(m,n)}{mn} \leq \lim_{m,n} \frac{\sqrt{m}\sqrt{n}}{mn} = 0,$$

i.e. the set K has double natural density zero, while the set $\{(i, 2j) : i, j \in \mathbb{N}\}$ has double natural density $1/2$.

Note that, if we set $m = n$, we have a two dimensional natural density due two Christopher [4].

Statistical convergence for double sequences $x = (x_{jk})$ of real numbers was introduced and studied by Mursaleen and Edely [15] and for double sequences of fuzzy real numbers by Savaş and Mursaleen [25].

Definition 1.4 *A real double sequence $x = (x_{jk})$ is said to be statistically convergent to the number ℓ if for each $\epsilon > 0$, the set*

$$\{(j,k), j \leq m \text{ and } k \leq n : | x_{jk} - \ell | \geq \epsilon\}$$

has double natural density zero. In this case we write $st_2\text{-}\lim_{j,k} x_{jk} = \ell$.

The notion of fuzzy sets was first introduced by Zadeh [18] in 1965. The concept and the theory of intuitionistic fuzzy metric space was studied by a lot of researchers [8], [10], [11], [12], [13]. The idea of statistically convergence of single sequences in IFNS was studied by Karakus et al. [4] and for double sequences in IFNS by Mursaleen and Mohiuuddine [7]. Recently, Mursaleen and Mohiuddine [7] studied the concept of statistical convergence of double sequences in intuitionistic fuzzy normed spaces. In this paper we shall study the concept of statistical convergence of double sequences in intuitionistic fuzzy 2-normed spaces.

2 Preliminaries

We now recall some notations and basic definitions that we need throghout this paper.

Definition 2.1 *[16] A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is said to be a continuous t-norm if it satisfies the following conditions:*

1. *$*$ is associative and commutative,*
2. *$*$ is continuous,*
3. *$a * 1 = a$ for all $a \in [0,1]$,*
4. *$a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for each $a, b, c, d \in [0,1]$.*

Definition 2.2 *[16] A binary operation $\diamondsuit : [0,1] \times [0,1] \to [0,1]$ is said to be a continuous t-conorm if it satisfies the following conditions:*

1. *$\diamondsuit$ is associative and commutative,*
2. *$\diamondsuit$ is continuous,*

3. $a\diamondsuit 0 = a$ *for all* $a \in [0,1]$,

4. $a\diamondsuit b \leq c\diamondsuit d$ *whenever* $a \leq c$ *and* $b \leq d$ *for each* $a,b,c,d \in [0,1]$.

Definition 2.3 *Let* V *be a real vector space of dimension* d, *where* $2 \leq d < \infty$. *A* 2*-norm on* V *is a funtion* $\|.,.\| : V \times V \to \mathbb{R}$ *which satisfies,*

1. $\|x,y\| = 0$ *if and only if* x *and* y *are linearly dependent;*

2. $\|x,y\| = \|y,x\|$;

3. $\|\alpha x,y\| = |\alpha|\|x,y\|$;

4. $\|x,y+z\| \leq \|x,y\| + \|x,z\|$.

The pair $(V, \|.,.\|)$ is then called a 2-*normed space*.

Example 2.1 *As an example of a 2-normed space take* $V = \mathbb{R}^2$ *being equipped with the 2-norm* $\|x,y\|$:= *the area of the parallelogram spanned by the vectors* x *and* y, *which may be given explicitly by the formula,*

$$\|x,y\| = |x_1y_2 - x_2y_1|, \quad x = (x_1,x_2), \quad y = (y_1,y_2).$$

Using the continuous t-norm and t-conorm, Saadati and Park [11] introduced the concept of intuitionistic fuzzy normed space while the concept of intuitionistic fuzzy 2-normed space has been introduced and studied by Mursaleen and Lohani [6] as follows:

Definition 2.4 *The five-tuple* $(V,\mu,\nu,*,\diamondsuit)$ *is said to be an intuitionistic fuzzy 2- normed space (for short, IF2NS) if* V *is a vector space,* $*$ *is a continuous* t*-norm,* $\diamondsuit$ *is a continuous* t*-conorm, and* μ,ν *are fuzzy sets on* $V \times V \times (0,\infty)$ *satisfying the following conditions for every* $x,y,z \in V$, *and* $s,t > 0$

1. $\mu(x,y;t) + \nu(x,y;t) \leq 1$,

2. $\mu(x,y;t) > 0$,

3. $\mu(x,y;t) = 1$ *if and only if* x *and* y *are linearly dependent,*

4. $\mu(\alpha x,y;t) = \mu(x,y;\frac{t}{|\alpha|})$ *for each* $\alpha \neq 0$,

5. $\mu(x,y;t) * \mu(x,z;s) \leq \mu(x,y+z;t+s)$,

6. $\mu(x,y;\cdot):(0,\infty)\to[0,1]$ *is continuous,*
7. $\lim_{t\to\infty}\mu(x,y;t)=1$ *and* $\lim_{t\to 0}\mu(x,y;t)=0$,
8. $\mu(x,y;t)=\mu(y,x;t)$,
9. $\nu(x,y;t)<1$,
10. $\nu(x,y;t)=0$ *if and only if* x *and* y *are linearly dependent,*
11. $\nu(\alpha x,y;t)=\nu(x,y;\frac{t}{|\alpha|})$ *for each* $\alpha\neq 0$,
12. $\nu(x,y;t)\Diamond\nu(x,z;s)\geq\nu(x,y+z;t+s)$,
13. $\nu(x,y;\cdot):(0,\infty)\to[0,1]$ *is continuous,*
14. $\lim_{t\to\infty}\nu(x,y;t)=0$ *and* $\lim_{t\to 0}\nu(x,y;t)=1$,
15. $\nu(x,y;t)=\nu(y,x;t)$

In this case (μ,ν) *is called an intuitionistic fuzzy 2-norm on* V*, and we denote it by* $(\mu,\nu)_2$.

Example 2.2 *Let* $(V,\|.,.\|)$ *be a 2-normed space, and let* $a*b=ab$ *and* $a\Diamond b=min\{a+b,1\}$ *for all* $a,b\in[0,1]$. *For all* $x\in V$ *and every* $t>0$, *consider*

$$\mu(x,y;t):=\frac{t}{t+\|x,y\|}\qquad \text{and}\qquad \nu(x,y;t):=\frac{\|x,y\|}{t+\|x,y\|}.$$

Then $(X,\mu,\nu,*,\Diamond)$ *is an intuitionistic fuzzy 2-normed space.*

The concepts of convergence and Cauchy sequences in an intuitionistic fuzzy normed space are studied in [11].

Definition 2.5 *Let* $(X,\mu,\nu,*,\Diamond)$ *be an IFNS. Then, a sequence* $x=(x_k)$ *is said to be convergent to* $L\in X$ *with respect to the intuitionistic fuzzy norm* (μ,ν) *if, for every* $\epsilon>0$ *and* $t>0$, *there exists* $k_0\in\mathbb{N}$ *such that* $\mu(x_k-L,t)>1-\epsilon$ *and* $\nu(x_k-L,t)<\epsilon$ *for all* $k\geq k_0$. *In this case we write* $(\mu,\nu)\text{-}\lim x=L$ *or* $x_k\xrightarrow{(\mu,\nu)}L$ *as* $k\to\infty$.

Definition 2.6 *Let* $(X, \mu, \nu, *, \diamondsuit)$ *be an IFNS. Then,* $x = (x_k)$ *is said to be Cauchy sequence with respect to the intuitionistic fuzzy norm* (μ, ν) *if, for every* $\epsilon > 0$ *and* $t > 0$*, there exists* $k_0 \in \mathbb{N}$ *such that* $\mu(x_k - x_\ell, t) > 1 - \epsilon$ *and* $\nu(x_k - x_\ell, t) < \epsilon$ *for all* $k, \ell \geq k_0$*.*

3 On statistically convergent double sequences in IF2NS

We define the following

Definition 3.1 *Let* $(X, \mu, \nu, *, \diamondsuit)$ *be an IF2NS. Then, a double sequence* $x = (x_{jk})$ *is said to be statistically convergent to* $L \in X$ *with respect to the intuitionistic fuzzy 2-norm* $(\mu, \nu)_2$ *provided that, for every* $\epsilon > 0$ *and* $t > 0$*,*

$$\delta_2\{(j,k) \in \mathbb{N} \times \mathbb{N} : \mu(x_{jk} - L, y; t) \leq 1 - \epsilon \quad or \quad \nu(x_{jk} - L, y; t) \geq \epsilon\} = 0, \tag{1}$$

or equivalently

$$\lim_{m,n} \frac{1}{mn} |\{j \leq m, k \leq n : \mu(x_{jk} - L, y; t) \leq 1 - \epsilon \quad or \quad \nu(x_{jk} - L, y; t) \geq \epsilon\}| = 0.$$

In this case we write $st_2^{(\mu,\nu)_2}\text{-}\lim x = L$. Using (1) and well known properties of the double natural density, we easily get the following lemma.

Lemma 3.1 *Let* $(X, \mu, \nu, *, \diamondsuit)$ *be an IF2NS. Then, for every* $\epsilon > 0$ *and* $t > 0$*, the following statements are equivalent:*

1. $st_2^{(\mu,\nu)_2}\text{-}\lim x = L$.
2. $\delta_2\{(j,k) \in \mathbb{N} \times \mathbb{N} : \mu(x_{jk} - L, y; t) \leq 1 - \epsilon\} = \delta_2\{(j,k) \in \mathbb{N} \times \mathbb{N} : \nu(x_{jk} - L, y; t) \geq \epsilon\} = 0$.
3. $\delta_2\{(j,k) \in \mathbb{N} \times \mathbb{N} : \mu(x_{jk} - L, y; t) > 1 - \epsilon \quad and \quad \nu(x_{jk} - L, y; t) < \epsilon\} = 1$.
4. $\delta_2\{(j,k) \in \mathbb{N} \times \mathbb{N} : \mu(x_{jk} - L, y; t) > 1 - \epsilon\} = \delta_2\{(j,k) \in \mathbb{N} \times \mathbb{N} : \nu(x_{jk} - L, y; t) < \epsilon\} = 1$.
5. $st_2\text{-}\lim \mu(x_{jk} - L, y; t) = 1 \quad and \quad st_2\text{-}\lim \nu(x_{jk} - L, y; t) = 0$.

Theorem 3.1 *Let $(X, \mu, \nu, *, \diamondsuit)$ be an IF2NS. If a double sequence $x = (x_{jk})$ is statistically convergent with respect to the intuitionistic fuzzy 2-norm $(\mu, \nu)_2$, then $st_2^{(\mu,\nu)_2}$-limit is unique.*

Proof. Suppose that $st_2^{(\mu,\nu)_2}\text{-}\lim x = L_1$ and $st_2^{(\mu,\nu)_2}\text{-}\lim x = L_2$. For a given $\epsilon > 0$, choose $r > 0$ such that $(1-r) * (1-r) > 1-\epsilon$ and $r \diamondsuit r < \epsilon$. For any $t > 0$, define the following sets:

$$K_{\mu,1}(r,t) = \{(j,k) \in \mathbb{N} \times \mathbb{N} : \mu(x_{jk} - L_1, y; t) \leq 1 - r\},$$

$$K_{\mu,2}(r,t) = \{(j,k) \in \mathbb{N} \times \mathbb{N} : \mu(x_{jk} - L_2, y; t) \leq 1 - r\},$$

$$K_{\nu,1}(r,t) = \{(j,k) \in \mathbb{N} \times \mathbb{N} : \nu(x_{jk} - L_1, y; t) \geq r\},$$

$$K_{\nu,2}(r,t) = \{(j,k) \in \mathbb{N} \times \mathbb{N} : \nu(x_{jk} - L_2, y; t) \geq r\}.$$

Since $st_2^{(\mu,\nu)_2}\text{-}\lim x = L_1$ and $st_2^{(\mu,\nu)_2}\text{-}\lim x = L_2$, we have

$$\delta_2(K_{\mu,1}(\epsilon,t)) = 0 = \delta_2(K_{\nu,1}(\epsilon,t)) \text{ for all } t > 0,$$

$$\delta_2(K_{\mu,2}(\epsilon,t)) = 0 = \delta_2(K_{\nu,2}(\epsilon,t)) \text{ for all } t > 0.$$

Now let $K_{\mu,\nu}(\epsilon,t) = (K_{\mu,1}(\epsilon,t) \cup K_{\mu,2}(\epsilon,t)) \cap (K_{\nu,1}(\epsilon,t) \cup K_{\nu,2}(\epsilon,t))$. Then $\delta_2(K_{\mu,\nu}(\epsilon,t)) = 0$, which implies that $\delta_2(\mathbb{N} \times \mathbb{N}/K_{\mu,\nu}(\epsilon,t)) = 1$. If $(j,k) \in \mathbb{N} \times \mathbb{N}/K_{\mu,\nu}(\epsilon,t)$, then we have two possible cases. The first case is that $(j,k) \in \mathbb{N} \times \mathbb{N}/(K_{\mu,1}(\epsilon,t) \cup K_{\mu,2}(\epsilon,t))$; and the second is $(j,k) \in \mathbb{N} \times \mathbb{N}/(K_{\nu,1}(\epsilon,t) \cup K_{\nu,2}(\epsilon,t))$. We first consider that $(j,k) \in \mathbb{N} \times \mathbb{N}/(K_{\mu,1}(\epsilon,t) \cup K_{\mu,2}(\epsilon,t))$. Then we have

$$\mu(L_1 - L_2, y; t) \geq \mu\left(x_{jk} - L_1, y; \frac{t}{2}\right) * \mu\left(x_{jk} - L_2, y; \frac{t}{2}\right) > (1-r) * (1-r) > 1 - \epsilon. \tag{2}$$

Since $\epsilon > 0$ was arbitrary, by (2) we get $\mu(L_1 - L_2, y; t) = 1$ for all $t > 0$, which yields $L_1 = L_2$. On the other hand, if $(j,k) \in \mathbb{N} \times \mathbb{N}/(K_{\nu,1}(\epsilon,t) \cup K_{\nu,2}(\epsilon,t))$, then

$$\nu(L_1 - L_2, y; t) \leq \nu\left(x_{jk} - L_1, y; \frac{t}{2}\right) \diamondsuit \nu\left(x_{jk} - L_2, y; \frac{t}{2}\right) < r \diamondsuit r < \epsilon.$$

Again, since $\epsilon > 0$ was arbitrary, we have $\nu(L_1 - L_2, y; t) = 0$ for all $t > 0$, which implies $L_1 = L_2$. Therefore, in all cases, we conclude that $st_2^{(\mu,\nu)_2}$-limit is unique. This completes the proof of the theorem.

Theorem 3.2 *Let $(X, \mu, \nu, *, \diamondsuit)$ be an IF2NS. If $(\mu,\nu)_2$-$\lim x = L$ then $st_2^{(\mu,\nu)_2}$-$\lim x = L$. But converse need not hold.*

Proof. Let $(\mu,\nu)_2$-$\lim x = L$. Then for every $\epsilon > 0$ and $t > 0$, there is a number $k_0 \in$ such that

$$\mu(x_{jk} - L, y; t) > 1 - \epsilon \ \text{ and } \ \nu(x_{jk} - L, y; t) < \epsilon$$

for all $j, k \geq k_0$. Hence the set

$$\{(j,k) \in \mathbb{N} \times \mathbb{N} : \mu(x_{jk} - L, y; t) \leq 1 - \epsilon \ \text{ and } \ \nu(x_{jk} - L, y; t) \geq \epsilon\}$$

has atmost finitely many terms. Since every finite subset of $\mathbb{N} \times \mathbb{N}$ has density zero,

$$\delta_2(\{k \leq n : \mu(x_{jk} - L, y; t) \leq 1 - \epsilon \ \text{ and } \ \nu(x_{jk} - L, y; t) \geq \epsilon\}) = 0.$$

This completes the proof of the theorem.

The following example shows that the converse of Theorem 3.2 need not be true.

Example 3.1 *Let $X = \mathbb{R}^2$ with $\|x, y\| = |x_1y_2 - x_2y_1|$, $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{R}^2$, and let $a * b = ab$ and $ab = \min\{a+b, 1\}$ for all $a, b \in [0,1]$. For all $x \in X$ and every $t > 0$, consider*

$$\mu(x, y; t) := \frac{t}{t + \| x, y \|} \quad \textit{and} \quad \nu(x, y; t) := \frac{\| x, y \|}{t + \| x, y \|},$$

Then $(X, \mu, \nu, *, \diamondsuit)$ is an IF2NS. Now we define a double sequence $x = (x_{jk})$ by

$$x_{jk} = \begin{cases} (jk, 0) \ ; & j = m^2 \ , \ k = n^2; \ m, n \in \mathbb{N} \\ (0,0) \ ; & \text{otherwise.} \end{cases}$$

Then in this case

$$K_{m,n}(\epsilon, t) = \{j \leq m, k \leq n; x_{jk} = \sqrt{jk}\}$$

and we get

$$\frac{1}{mn}|K_{m,n}(\epsilon, t)| \leq \frac{1}{mn}|\{j \leq m, k \leq n : \ j = m^2 \ , \ k = n^2; \ m, n \in \mathbb{N}\}|$$

$$\leq \frac{\sqrt{m}\sqrt{n}}{mn} \to 0 \text{ as } m, n \to \infty,$$

that is, $st_2^{(\mu,\nu)_2}\text{-}\lim x = 0$. *However* $x = (x_{jk})$ *is not convergent in* $(X, \|, \|)$ *and hence* $(\mu, \nu)_2$*-limit does not exist.*

This completes the proof of the theorem.

Theorem 3.3 *Let* $(X, \mu, \nu, *, \diamondsuit)$ *be an IF2NS. Then,* $st_2^{(\mu,\nu)_2}\text{-}\lim x = L$ *if and only if there exists a subset* $K = \{(j,k)\} \subseteq \mathbb{N} \times \mathbb{N};\ \ j,k = 1,2,\cdots$ *such that* $\delta_2(K) = 1$ *and* $(\mu,\nu)_2\text{-}\lim_{\substack{m,n\to\infty \\ (m,n)\in K}} x_{mn} = L$.

Proof. Necessity. Suppose that $st_2^{(\mu,\nu)_2}\text{-}\lim x = L$. Let for any $t > 0$ and $r = 1, 2, \cdots$

$$M_{\mu,\nu}(r,t) = \left\{(j,k) \in \mathbb{N} \times \mathbb{N} : \mu(x_{jk} - L, y; t) > 1 - \frac{1}{r} \ \text{ and } \ \nu(x_{jk} - L, y; t) < \frac{1}{r}\right\},$$

$$K_{\mu,\nu}(r,t) = \left\{(j,k) \in \mathbb{N} \times \mathbb{N} : \mu(x_{jk} - L, y; t) \leq 1 - \frac{1}{r} \ \text{ or } \ \nu(x_{jk} - L, y; t) \geq \frac{1}{r}\right\}.$$

Then $\delta_2(K_{\mu,\nu}(r,t)) = 0$ and for $t > 0$ and $r = 1, 2, \cdots$, we have

$$M_{\mu,\nu}(r,t) \supset M_{\mu,\nu}(r+1,t), \tag{3}$$

$$\delta_2(M_{\mu,\nu}(r,t)) = 1 \tag{4}$$

Now we have to show that for $(j,k) \in M_{\mu,\nu}(r,t)$, $x_{jk} \xrightarrow{(\mu,\nu)_2} L$. Suppose that $x_{jk} \not\xrightarrow{(\mu,\nu)_2} L$. Therefore there is some $\lambda > 0$ such that

$$\left\{(j,k) \in \mathbb{N} \times \mathbb{N} : \mu(x_{jk} - L, y; t) \leq 1 - \lambda \ \text{ or } \ \nu(x_{jk} - L, y; t) \geq \lambda\right\}$$

for infinitely many terms x_{jk}. For $\lambda > \frac{1}{r}$, $r = 1, 2, \cdots$, let

$$M_{\mu,\nu}(\lambda,t) = \left\{(j,k) \in \mathbb{N} \times \mathbb{N} : \mu(x_{jk} - L, y; t) > 1 - \lambda \ \text{ and } \ \nu(x_{jk} - L, y; t) < \lambda\right\}$$

Hence

$$\delta_2(M_{\mu,\nu}(\lambda, t)) = 0,$$

and using (3), $M_{\mu,\nu}(r, t) \subset M_{\mu,\nu}(\lambda, t)$. Hence $\delta_2(M_{\mu,\nu}(r, t)) = 0$ which contradicts (4). Therefore $x_{jk} \xrightarrow{(\mu,\nu)_2} L$.

Sufficiency. Suppose that there exists a set $K = \{(j, k) : j, k = 1, 2, \cdots\} \subseteq \mathbb{N} \times \mathbb{N}$ such that $\delta_2(K) = 1$ and $(\mu, \nu)_2$- $\lim_{\substack{j,k\to\infty \\ (j,k)\in K}} x_{jk} = L$, i.e. there exists $N \in \mathbb{N}$ such that for every $\lambda > 0$ and $t > 0$

$$\mu(x_{jk} - L, y; t) > 1 - \lambda \text{ and } \nu(x_{jk} - L, y; t) < \lambda.$$

Now

$$K_{\mu,\nu}(\lambda, t) := \{(j, k) \in \mathbb{N} \times \mathbb{N} : \mu(x_{jk} - L, y; t) \leq 1 - \lambda \text{ and } \nu(x_{jk} - L, y; t) \geq \lambda\}$$

$$\subseteq \mathbb{N} \times \mathbb{N} - \{(j_{N+1}, k_{N+1}), (j_{N+2}, k_{N+2}), \cdots\}.$$

Therefore $\delta_2(K_{\mu,\nu}(\lambda, t)) \leq 1-1 = 0$. Hence $st_2^{(\mu,\nu)_2}$- $\lim x = L$. This completes the proof of the theorem.

4 On statistically Cauchy double sequences in IF2NS

Recently, Mursaleen and Mohiuddine [7] defined and studied statistically Cauchy double sequences in IFNS. Now we define this concept in IF2NS as follows.

Definition 4.1 *Let $(X, \mu, \nu, *, \diamondsuit)$ be an IF2NS. Then, a double sequence $x = (x_{jk})$ is said to be statistically Cauchy with respect to the intuitionistic fuzzy 2-norm $(\mu, \nu)_2$ if for every $\epsilon > 0$ and $t > 0$, there exist $N = N(\epsilon)$ and $M = M(\epsilon)$ such that for all $j, p \geq N$; $k, q \geq M$*

$$\delta_2\{(j, k) \in \mathbb{N} \times \mathbb{N} : \mu(x_{jk} - x_{pq}, y; t) \leq 1 - \epsilon \;\; or \;\; \nu(x_{jk} - x_{pq}, y; t) \geq \epsilon\} = 0.$$

Theorem 4.1 *Let* $(X, \mu, \nu, *, \diamondsuit)$ *be an IF2NS. A double sequence* $x = (x_{jk})$ *is statistically convergent with respect to* $(\mu, \nu)_2$ *if and only if it is statistically Cauchy with respect to* $(\mu, \nu)_2$.

Proof. Let $x = (x_{jk})$ be statistically convergent to L with respect to $(\mu, \nu)_2$, i.e., $st_2^{(\mu,\nu)_2}\text{-}\lim x = L$. Then

$$\delta_2(\{(j,k) \in \mathbb{N} \times \mathbb{N} : \mu(x_{jk} - L, y; \frac{t}{2}) \leq 1 - \epsilon \ \text{ or } \ \nu(x_{jk} - L, y; \frac{t}{2}) \geq \epsilon\}) = 0.$$

In particular, for $j = M,\ k = N$

$$\delta_2(\{(j,k) \in \mathbb{N} \times \mathbb{N} : \mu(x_{MN} - L, y; \frac{t}{2}) \leq 1 - \epsilon \ \text{ or } \ \nu(x_{MN} - L, y; \frac{t}{2}) \geq \epsilon\}) = 0.$$

Since

$$\mu(x_{jk} - x_{MN}, y; t) = \mu(x_{jk} - L - x_{MN} + L, y; \frac{t}{2} + \frac{t}{2})$$

$$\geq \mu(x_{jk} - L, y; \frac{t}{2}) * \mu(x_{MN} - L, y; \frac{t}{2})$$

and since

$$\nu(x_{jk} - x_{MN}, y; t) \leq \nu(x_{jk} - L, y; \frac{t}{2}) \diamondsuit \nu(x_{MN} - L, y; \frac{t}{2}),$$

we have

$$\delta_2(\{(j,k) \in \mathbb{N} \times \mathbb{N} : \mu(x_{jk} - x_{MN}, y; t) \leq 1 - \epsilon \ \text{ or } \ \nu(x_{jk} - x_{MN}, y; t) \geq \epsilon\}) = 0,$$

that is, x is statistically Cauchy with respect to $(\mu, \nu)_2$.

Conversely, let $x = (x_{jk})$ be statistically Cauchy but not statistically convergent with respect to the intuitionistic fuzzy 2-norm $(\mu, \nu)_2$. Then there exist N and M such that the sets $\delta_2(A(\epsilon, t)) = 0$ and $\delta_2(B(\epsilon, t)) = 0$, where

$$A(\epsilon, t) = \{(j,k) \in \mathbb{N} \times \mathbb{N} : \mu(x_{jk} - x_{NM}, y; t) \leq 1 - \epsilon \ \text{ or } \ \nu(x_{jk} - x_{NM}, y; t) \geq \epsilon\},$$

$$B(\epsilon,t)=\{(j,k)\in\mathbb{N}\times\mathbb{N}:\mu(x_{jk}-L,y;\frac{t}{2})>1-\epsilon\ \text{ and }\ \nu(x_{jk}-L,y;\frac{t}{2})<\epsilon\}.$$

Since

$$\mu(x_{jk}-x_{NM},y;t)\geq 2\mu(x_{jk}-L,y;\frac{t}{2})>1-\epsilon,$$

and

$$\nu(x_{jk}-x_{NM},y;t)\leq 2\nu(x_{jk}-L,y;\frac{t}{2})<\epsilon,$$

if $\mu(x_{jk}-L,y;\frac{t}{2})>(1-\epsilon)/2$ and $\nu(x_{jk}-L,y;\frac{t}{2})<\epsilon/2$. Therefore

$$\delta_2(\{(j,k)\in\mathbb{N}\times\mathbb{N}:\mu(x_{jk}-x_{NM},y;t)>1-\epsilon\ \text{ and }\ \nu(x_{jk}-x_{NM},y;t)<\epsilon\})=0,$$

that is, $\delta_2(A^C(\epsilon,t))=0$ and hence $\delta_2(A(\epsilon,t))=1$, which leads to a contradiction. Hence x must be statistically convergent with respect to the intuitionistic fuzzy 2-norm $(\mu,\nu)_2$. This completes the proof of the theorem.

Using Theorem 3.3 and Theorem 4.1, we can state the following:

Theorem 4.2 *Let $(X,\mu,\nu,*,\diamondsuit)$ be an IF2NS, and $x=(x_{jk})$ be a double sequence in X. Then, the following conditions are equivalent:*

1. *x is a statistically convergent with respect to the intuitionistic fuzzy 2-norm $(\mu,\nu)_2$.*
2. *x is a statistically Cauchy with respect to the intuitionistic fuzzy 2-norm $(\mu,\nu)_2$.*
3. *There exists an increasing index sequence $K=\{(j_n,k_n)\}\subseteq\mathbb{N}\times\mathbb{N}$ such that $\delta_2(K)=1$ and the subsequence (x_{j_n,k_n}) is a statistically Cauchy with respect to the intuitionistic fuzzy 2-norm $(\mu,\nu)_2$.*

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