Steady State Analysis of an $M^X/\begin{pmatrix} G_{1A} & G_{2A} \\ G_{1B} & G_{2B} \end{pmatrix}/1$ Queue with Restricted Admissibility of Arriving Batches and Modified Bernoulli Schedule Server Vacations Based on a Single Vacation Policy

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Abstract

We study a batch arrival queue with a single server providing two- stages of heterogeneous service with each customer having the option to choose one of the two types of first stage service followed by one of the two types of second stage service. In addition, after completion of the two stages of service in succession to each customer, the server has the option to take a vacation of a random length with probability p or to continue staying in the system with probability 1-p. Further, the batches arriving at the system have restricted admissibility into the system. In addition, the policy of restriction also differs when the server is available in the system and when he is away on vacation. We derive the steady state queue size distribution at a random epoch and some important performance measures for this model. Moreover, attempts have been made to unify several classes of related batch queueing systems.

Mathematics Subject Classification: Primary 60K25, Secondary 60 J15

Keywords: $M^X / \begin{pmatrix} G_{1A} & G_{2A} \\ G_{1B} & G_{2B} \end{pmatrix} /1$ queue, two-stage sequential service, Bernoulli schedule server vacations, restricted admissibility , queue size distribution

1. Introduction

Madan and Choudhury [16] studied a batch arrival queue $M^X /(G_1, G_2)/1$ with restricted admissibility of arriving batches and modified server vacations under a single vacation policy. This single server queueing system provides a two-stage heterogeneous sequential service to customers. In the present paper, we study a more generalized queueing system $M^X / \begin{pmatrix} G_{1A} & G_{2A} \\ G_{1B} & G_{2B} \end{pmatrix} /1$ with restricted admissibility of arriving batches and modified server vacations under a single vacation policy. In this system too, the server provides two stages of sequential service. However, each stage of service has two options and a customer has the option to choose either of the two types first stage service followed by either of the two types of the second stage service.

The restricted admissibility of arriving batches was first introduced by Madan and Abu-Dayyeah [14, 15] followed by Madan and Choudhury [16]. One finds quite a few papers in earlier literature on different control models of queueing systems including control of servers, control of service rates, control of admission of customers and control of queue discipline. For such papers, the reader is referred to Crabill, Gross and Magazine [6], Rue and Rosenberg [20], Stidham [21], Neuts [17] and Huang and Mc-Donald [8]. For a few detailed examples of restricted admissibility models, refer to Madan and Choudhury [16].

The two-stage sequential service was first studied by Madan [12, 13] in his two papers. Bernoulli vacations were studied by many authors including Keilson and Servi [9], Ramaswamy and Servi [18], Doshi [7] and Takagi [22]. Numerous researchers, including Baba [1], Choudhury [3], Choudhury and Borthakur [4], Lee and Srinivasan [10], Lee et-al [11], Rosenberg and Yechiali [19] and Teghem [23] and many others have studied batch arrival queueing systems under different vacation policies.

2. The Mathematical Model

We consider a batch arrival queueing system, where arrivals occur according to a compound Poisson process with the batch size random variable '*X*'. Following Madan

and Choudhury [16], let c_1 $(0 \leq c_1 \leq 1)$ be the probability that an arriving batch will be allowed to join the system during the period of time when the server is busy and c_2 $(0 \leq c_2 \leq 1)$ be the probability that an arriving batch will be allowed to join the system during the period of time when the server is on vacation. The server provides two stages of heterogeneous service in succession. The first stage service (FSS) has two options FSSA or FSSB. The second stage service (SSS) again has two options SSSA or SSSSB. The service discipline is assumed to be first come, first served (*FCFS*) for both stages of service. When a customer's turn for service comes, he chooses FSSA with probability r_1 or FSSB with probability $1 - r_1$. After completion of the first stage service FSSA or FSSB, a customer enters second stage service and he may choose SSSA with probability r_2 or SSSB with probability $1 - r_2$. Further, it is assumed that the service time S_{1A} for FSSA and the service times S_{1B} for FSSB follows a general probability distribution with respective distribution functions $S_{1A}(x)$ and $S_{1B}(x)$, respective Laplace-Stieltjes transform (*LST*) $S_{1A}^*(\theta)$ and $S_{1A}^*(\theta)$ and respective finite moments $\mathrm{E}(S_{1A}^k)$ and $\mathrm{E}(S_{1B}^k)$ $k \geq 1$. Similarly, the second stage service times S_{2A} for SSS(A) and S_{2B} for SSS(B) follow a general probability distribution with respective distribution functions $S_{2A}(x)$ and $S_{2B}(x)$, respective Laplace-Stieltjes transforms (*LST*) $S_{2A}^*(\theta)$ and $S_{2B}^*(\theta)$ and respective finite moments $\mathrm{E}(S_{2A}^k)$ and $\mathrm{E}(S_{2B}^k)$ $k \geq 1$. As soon as the second stage service SSSA or SSSB of a customer is completed, the server may go for a vacation of random length V with probability p $(0 \leq p \leq 1)$ or he may continue serving the next customer, if any, or may stay idle and wait for a customer to arrive at the system. Next, we assume that the vacation time V of the server follows a general probability distribution with distribution function $V(x)$, *LST* $V^*(\theta)$ and finite moments $E(V^k)$, k=1, 2 and is independent of the service times S_{1A}, S_{1B}, S_{2A} and S_{2B} and the arrival process. Further, it is also assumed that if, after returning from a vacation, the server does not find any customers in the system, even then he joins the system without taking any further vacations and this policy is termed as single vacation (V_S) with Bernoulli schedule (*BS*).

Notationally, our model may be denoted by $M^X / \left(\begin{pmatrix} G_{1A} \\ G_{1B} \end{pmatrix}, \begin{pmatrix} G_{2A} \\ G_{2B} \end{pmatrix} \right) / 1 / BS / SV$ *Policy*

Queue. Thus, for this system the time required by a unit to complete a service cycle, which may be called as modified service time, is given by

$B = S_{1A} + S_{2A} + V$, with probability $r_1 r_2 p$,

$= S_{1A} + S_{2B} + V$, with probability $r_1(1 - r_2)p$,

$= S_{1B} + S_{2A} + V$, with probability $(1 - r_1)r_2 p$,

$= S_{1B} + S_{2B} + V$, with probability $(1 - r_1)(1 - r_2)p$,

$$= S_{1A} + S_{2A}, \text{ with probability } r_1 r_2 (1 - p),$$
$$= S_{1A} + S_{2B}, \text{ with probability } r_1 (1 - r_2)(1 - p),$$
$$= S_{1B} + S_{2A}, \text{ with probability } (1 - r_1) r_2 (1 - p),$$
$$= S_{1B} + S_{2B}, \text{ with probability } (1 - r_1)(1 - r_2)(1 - p).$$

3. Queue Size Distribution at a Random Epoch

In this section, we shall first setup the system state equations for the queue size distribution at a random point of time by treating the elapsed FSSA time, the elapsed FSSB time, the elapsed SSSA time, the elapsed SSSB time and the elapsed vacation time V as supplementary variables. Then we solve these equations and derive the Probability Generating Function (*PGF*) for the queue size distribution at a random epoch.

Assuming that the system is in the steady state, we define $\lambda = $ batch arrival rate, $X = $ batch size (a random variable), $a_k = $ Prob [$X = k$],

$$X(z) = \sum_{k=1}^{\infty} z^k a_k, \text{ the PGF of } X, \ E[X_{[k]}] = E[X(X-1)...(X-k+1)], \text{ the } k \text{ th factorial}$$

moment of X.

Further, since $S_{1A}(x)$, $S_{1B}(x)$, $S_{2A}(x)$, $S_{2B}(x)$ and $V(x)$ are distribution functions, therefore, we have we $S_{1A}(0) = 0$, $S_{1A}(\infty) = 1$, $S_{1B}(0) = 0$, $S_{1B}(\infty) = 1$, $S_{2A}(0) = 0$, $S_{2A}(\infty) = 1$, $S_{2B}(0) = 0$, $S_{2B}(\infty) = 1$, and $V(\infty) = 1$, and since the distributions are

continuous at $x = 0$, therefore, $\mu_{1A}(x)dx = \dfrac{dS_{1A}(x)}{1 - S_{1A}(x)}$, $\mu_{1B}(x)dx = \dfrac{dS_{1B}(x)}{1 - S_{1B}(x)}$,

$\mu_{2A}(x)dx = \dfrac{dS_{2A}(x)}{1 - S_{2A}(x)}$, $\mu_{2B}(x)dx = \dfrac{dS_{2B}(x)}{1 - S_{2B}(x)}$ and $v(x)dx = \dfrac{dV(x)}{1 - V(x)}$ are the first order

differential functions (hazard rates).

Let $N_Q(t)$ be the queue size (including one being served, if any) at time 't', $S_{1A}^0(t)$ be the elapsed FSSA time at time 't', $S_{1B}^0(t)$ be the elapsed FSSB time at time 't', $S_{2A}^0(t)$ be the elapsed SSSA time at time 't', $S_{2B}^0(t)$ be the elapsed SSSB time at time 't' and $V^0(t)$ be the elapsed vacation time at time 't'. For further development of this model, we introduce the random variable $Y(t)$ as follows:

At time 't',

$Y(t) = 0$, if the server is idle,

    $= 1A$, if the server is busy with FSSA,

    $= 1B$, if the server is busy with FSSB,

    $= 2A$, if the server is busy with SSSA,

    $= 2B$, if the server is busy with SSSB a,

    $= 3$, if the server is on vacation.

Thus the supplementary variables $S_{1A}^0(t)$, $S_{1B}^0(t)$, $S_{2A}^0(t)$, $S_{2B}^0(t)$ and $V^0(t)$ are introduced in order to obtain a bivariate Markov process $\{N_Q(t), L(t)\}$, where $L(t) = 0$ if $Y(t) = 0$, $L(t) = S_{1A}^0(t)$ if $Y(t) = 1A$, $L(t) = S_{1B}^0(t)$ if $Y(t) = 1B$, $L(t) = S_{2A}^0(t)$ if $Y(t) = 2A$, $L(t) = S_{2B}^0(t)$ if $Y(t) = 2B$ and $L(t) = V^0(t)$ if $Y(t) = 3$ and define the following probabilities:

$R_0(t) = \Pr ob \ [\ N_Q(t) = 0, \ L(t) = 0\ ]$,

$P_{1A,n}(x;t)dx = \Pr ob \ [\ N_Q(t) = n, \ L(t) = S_{1A}^0(t); \ x < S_{1A}^0(t) \le x + dx\ ], \ x > 0, \ n \ge 1$,

$P_{1B,n}(x;t)dx = \Pr ob \ [\ N_Q(t) = n, \ L(t) = S_{1B}^0(t); \ x < S_{1B}^0(t) \le x + dx\ ], \ x > 0, \ n \ge 1$,

$P_{2A,n}(x;t)dx = \Pr ob \ [\ N_Q(t) = n, \ L(t) = S_{2A}^0(t); \ x < S_{2A}^0(t) \le x + dx\ ], \ x > 0, \ n \ge 1$,

$P_{2B,n}(x;t)dx = \Pr ob \ [\ N_Q(t) = n, \ L(t) = S_{2B}^0(t); \ x < S_{2B}^0(t) \le x + dx\ ], \ x > 0, \ n \ge 1$,

$Q_n(x;t)dx = \Pr ob \ [\ N_Q(t) = n, \ L(t) = V^{(0)}(t); \ x < V^0(t) \le x + dx\ ], \ x > 0, \ n \ge 0$.

Then, assuming that $R_0 = \underset{t \to \infty}{Lim} \ R_0(t)$, $P_{1A,n}(x)dx = \underset{t \to \infty}{Lim} \ P_{1A,n}(x;t)dx$, $P_{1B,n}(x)dx = \underset{t \to \infty}{Lim} P_{1B,n}(x;t)dx$, $P_{2A,n}(x)dx = \underset{t \to \infty}{Lim} \ P_{2A,n}(x;t)dx$, $P_{2B,n}(x)dx = \underset{t \to \infty}{Lim} P_{2B,n}(x;t)dx$, and $Q_n(x)dx = \underset{t \to \infty}{Lim} \ Q_n(x;t)dx$, $x > 0, \ n \ge 1$ exist and are independent of the initial state, we utilize the argument of Cox [5] and obtain the following Kolmogorov forward equations under the steady state conditions:

$$\frac{d}{dx}P_{1A,n}(x) + [\lambda + \mu_{1A}(x)]P_{1A,n}(x) = \lambda(1-c_1) + \lambda c_1 \sum_{k=1}^{n} a_k P_{1A,n-k}(x), \ x > 0, \ n \ge 1, \qquad (3.1)$$

$$\frac{d}{dx}P_{1B,n}(x) + [\lambda + \mu_{1B}(x)]P_{1B,n}(x) = \lambda(1-c_1) + \lambda c_1 \sum_{k=1}^{n} a_k P_{1B,n-k}(x), \ x > 0, \ n \ge 1, \qquad (3.2)$$

$$\frac{d}{dx}P_{2A,n}(x) + [\lambda + \mu_{2A}(x)]P_{2A,n}(x) = \lambda(1-c_1) + \lambda c_1 \sum_{k=1}^{n} a_k P_{2A,n-k}(x), \ n \ge 1, \qquad (3.3)$$

$$\frac{d}{dx}P_{2B,n}(x) + [\lambda + \mu_{2B}(x)]P_{2B,n}(x) = \lambda(1-c_1) + \lambda c_1 \sum_{k=1}^{n} a_k P_{2B,n-k}(x), \ n \ge 1, \qquad (3.4)$$

$$\frac{d}{dx}Q_n(x) + [\lambda + v(x)]Q_n(x) = \lambda(1 - c_2) + \lambda c_2 \sum_{k=1}^{n} a_k Q_{n-k}(x), \ x > 0, \ n \geq 1, \tag{3.5}$$

$$\frac{d}{dx}Q_0(x) + [\lambda + v(x)]Q_0(x) = \lambda(1 - c_2)Q_0(x) \quad x > 0, \tag{3.6}$$

$$\lambda R_0 = \lambda(1 - c_1)R_0 + (1 - p)\int_0^\infty \mu_{2A}(x)P_{2A,1}(x)dx + (1 - p)\int_0^\infty \mu_{2B}(x)P_{2B,1}(x)dx + \int_0^\infty v(x)Q_0(x)dx$$
$$, \quad x > 0, \tag{3.7}$$

The above set of equations is to be solved under the following boundary conditions at $x = 0$:

$$P_{1A,n}(0) = \lambda c_1 r_1 a_n R_0 + (1 - p)r_1 \int_0^\infty \mu_{2A}(x)P_{2A,n+1}(x)dx + (1 - p)r_1 \int_0^\infty \mu_{2B}(x)P_{2B,n+1}(x)dx$$
$$+ r_1 \int_0^\infty v(x)Q_n(x)dx \ n \geq 1, \tag{3.8}$$

$$P_{1B,n}(0) = \lambda c_1(1 - r_1)a_n R_0 + (1 - p)(1 - r_1)\int_0^\infty \mu_{2A}(x)P_{2A,n+1}(x)dx + (1 - p)(1 - r_1)\int_0^\infty \mu_{2B}(x)P_{2B,n+1}(x)dx$$
$$+ (1 - r_1)\int_0^\infty v(x)Q_n(x)dx \ n \geq 1, \tag{3.9}$$

$$P_{2A,n}(0) = r_2 \int_0^\infty \mu_{1A}(x)P_{1A,n+1}(x)dx + r_2 \int_0^\infty \mu_{1B}(x)P_{1B,n+1}(x)dx, \qquad n \geq 1, \tag{3.10}$$

$$P_{2B,n}(0) = (1 - r_2)\int_0^\infty \mu_{1A}(x)P_{1A,n+1}(x)dx + (1 - r_2)\int_0^\infty \mu_{1B}(x)P_{1B,n+1}(x)dx, \quad n \geq 1, \tag{3.11}$$

$$Q_n(0) = p\int_0^\infty \mu_{2A}(x)P_{2A,n+1}(x)dx + p\int_0^\infty \mu_{2B}(x)P_{aB,n+1}(x)dx \ n \geq 0, \tag{3.12}$$

and the normalizing condition

$$R_0 + \sum_{n=1}^\infty \int_0^\infty P_{1A,n}(x)dx + \sum_{n=1}^\infty \int_0^\infty P_{1B,n}(x)dx + \sum_{n=1}^\infty \int_0^\infty P_{2A,n}(x)dx$$
$$+ \sum_{n=1}^\infty \int_0^\infty P_{2B,n}(x)dx + \sum_{n=0}^\infty \int_0^\infty Q_n(x)dx = 1 \tag{3.13}$$

Next, we define the following *PGF*s:

$$P_{1A}(x;z) = \sum_{n=1}^{\infty} z^n P_{1A,n}(x),\ x{>}0,\ |z| \le 1;\ P_{1B}(x;z) = \sum_{n=1}^{\infty} z^n P_{1B,n}(x),\ x{>}0,\ |z| \le 1, \quad (3.14a)$$

$$P_{2A}(x;z) = \sum_{n=1}^{\infty} z^n P_{2A,n}(x),\ x{>}0,\ |z| \le 1;\ P_{2B}(x;z) = \sum_{n=1}^{\infty} z^n P_{2B,n}(x),\ x{>}0,\ |z| \le 1, \quad (3.14b)$$

$$P_{1A}(0;z) = \sum_{n=1}^{\infty} z^n P_{1A,n}(0),\ x{>}0,\ |z| \le 1;\ P_{1B}(0;z) = \sum_{n=1}^{\infty} z^n P_{1B,n}(0),\ x{>}0,\ |z| \le 1, \quad (3.14c)$$

$$P_{2A}(0;z) = \sum_{n=1}^{\infty} z^n P_{2A,n}(0),\ x{>}0,\ |z| \le 1;\ P_{2B}(0;z) = \sum_{n=1}^{\infty} z^n P_{2B,n}(0),\ x{>}0,\ |z| \le 1, \quad (3.14d)$$

$$Q(x;z) = \sum_{n=0}^{\infty} z^n Q_n(x),\ x{>}0,\ |z| \le 1;\ Q(0;z) = \sum_{n=0}^{\infty} z^n Q_n(0),\ |z| \le 1. \quad (3.14e)$$

Proceeding in the usual manner with the equations (3.1) – (3.6), we obtain

$$P_{1A}(x;z) = P_{1A}(0;z)[1 - S_{1A}(x)]e^{-\lambda c_1(1-X(z))x},\ x{>}0, \tag{3.15}$$

$$P_{1B}(x;z) = P_{1B}(0;z)[1 - S_{1B}(x)]e^{-\lambda c_1(1-X(z))x},\ x{>}0, \tag{3.16}$$

$$P_{2A}(x;z) = P_{2A}(0;z)[1 - S_{2A}(x)]e^{-\lambda c_1(1-X(z))x},\ x{>}0, \tag{3.17}$$

$$P_{2B}(x;z) = P_{2B}(0;z)[1 - S_{21B}(x)]e^{-\lambda c_1(1-X(z))x},\ x{>}0, \tag{3.18}$$

$$Q(x;z) = Q(0;z)[1 - V(x)]e^{-\lambda c_2(1-X(z))x},\ x{>}0. \tag{3.19}$$

Next, we multiply equations (3.8)-(3.12) by appropriate powers of z^n and then take summation over all possible values of *n* and use (3.13). Thus we get on simplification

$$zP_{1A}(0,z) = \lambda c_1 r_1 z[X(z) - 1]R_0 + r_1(1-p)S_{2A}^*\big(\lambda c_1(1-X(z))\big)P_{2A}(0,z)$$
$$r_1(1-p)S_{2B}^*\big(\lambda c_1(1-X(z))\big)P_{2B}(0,z) + zr_1 V^*\big(\lambda c_2(1-X(z))\big)Q(0,z), \quad (3.20)$$

$$zP_{1B}(0,z) = \lambda c_1(1-r_1)z[X(z) - 1]R_0 + (1-r_1)(1-p)S_{2A}^*\big(\lambda c_1(1-X(z))\big)P_{2A}(0,z)$$
$$(1-r_1)(1-p)S_{2B}^*\big(\lambda c_1(1-X(z))\big)P_{2B}(0,z)$$
$$+ z(1-r_1)V^*\big(\lambda c_2(1-X(z))\big)Q(0,z), \quad (3.21)$$

$$P_{2A}(0,z) = r_2 S_{1A}^*\big(\lambda c_1(1-X(z))\big)P_{1A}(0,z) + r_2 S_{1B}^*\big(\lambda c_1(1-X(z)P_{1B}(0,z)\big), \tag{3.22}$$

$$P_{2B}(0,z) = (1-r_2)S_{1A}^*\big(\lambda c_1(1-X(z))\big)P_{1A}(0,z) + (1-r_2)S_{1B}^*\big(\lambda c_1(1-X(z))\big)P_{1B}(0,z), \tag{3.23}$$

$$zQ(0,z) = p\, S_{2A}^{*}\big(\lambda c_1(1 - X(z))\big)P_{2A}(0,z) + pS_{2B}^{*}\big(\lambda c_1((1 - X(z))\big)P_{2B}(0,z)\,, \qquad (3.24)$$

where

$$S_{1A}^{*}\big(\lambda c_1(1 - X(z))\big) = \int_0^{\infty} e^{-\lambda c_1(1-X(z))x}\, dS_{1A}(x) \ \text{ is the } z\text{-transform of } S_{1A}\,,$$

$$S_{1B}^{*}\big(\lambda c_1(1 - X(z))\big) = \int_0^{\infty} e^{-\lambda c_1(1-X(z))x}\, dS_{1B}(x) \ \text{ is the } z\text{-transform of } S_{1B}\,,$$

$$S_{2A}^{*}\big(\lambda c_1(1 - X(z))\big) = \int_0^{\infty} e^{-\lambda c_1(1-X(z))x}\, dS_{2A}(x) \ \text{ is the } z\text{-transform of } S_{2A}\,,$$

$$S_{2B}^{*}\big(\lambda c_1(1 - X(z))\big) = \int_0^{\infty} e^{-\lambda c_1(1-X(z))x}\, dS_{2B}(x) \ \text{ is the } z\text{-transform of } S_{2B}\,, \ \text{ and}$$

$$V^{*}\big(\lambda c_2(1 - X(z))\big) = \int_0^{\infty} e^{-\lambda c_2(1-X(z))x}\, dV(x) \ \text{ is the } z\text{-transform of } V.$$

Solving the set of equations (3.20) to (3.24) for $P_{1A}(0,z),\ P_{1B}(0,z), P_{2A}(0,z),\ \ P_{2B}(0,z)$ and $Q(0,z)$ we obtain on
simplifying

$$P_{1A}(0,z) = \frac{[\lambda c_1 z[X(z)-1]R_0]\left\langle \begin{array}{l} r_1\big[z - (1-r_1)r_2[(1-p)+pV^{*}]S_{2A}^{*}S_{1B}^{*} - (1-r_1)(1-r_2)[(1-p)+pV^{*}]S_{1B}^{*}S_{2B}^{*}\big] \\ -(1-r_1)\big[1 - r_1 r_2[(1-p)+pV^{*}]S_{2A}^{*}S_{1B}^{*} - r_1(1-r_2)[(1-p)+pV^{*}]S_{1A}^{*}S_{2B}^{*}\big] \end{array} \right\rangle}{z\left\langle\, 1 - \big[(1-p)+pV^{*}\big]\big[r_1 r_2 S_{1A}^{*}S_{2A}^{*} + r_1(1-r_2)S_{1A}^{*}S_{2B}^{*} + (1-r_1)r_2 S_{2A}^{*}S_{1B}^{*} + (1-r_1)(1-r_2)S_{2A}^{*}S_{2B}^{*}\big]\,\right\rangle}$$

$$(3.25)$$

$$P_{1B}(0,z) = \frac{[\lambda c_1 z[X(z)-1]R_0]\left\langle \begin{array}{l} r_1\big[(1-r_1)(1-r_2)[(1-p)+pV^{*}]S_{1A}^{*}S_{2B}^{*} - (1-r_1)r_2[(1-p)+pV^{*}]S_{1A}^{*}S_{2A}^{*}\big] \\ +(1-r_1)\big[z + r_1 r_2[(1-p)+pV^{*}]S_{1A}^{*}S_{2A}^{*} - (1-r_1)(1-r_2)[(1-p)+pV^{*}]S_{1A}^{*}S_{2B}^{*}\big] \end{array} \right\rangle}{z\left\langle\, 1 - \big[(1-p)+pV^{*}\big]\big[r_1 r_2 S_{1A}^{*}S_{2A}^{*} + r_1(1-r_2)S_{1A}^{*}S_{2B}^{*} + (1-r_1)r_2 S_{2A}^{*}S_{1B}^{*} + (1-r_1)(1-r_2)S_{2A}^{*}S_{2B}^{*}\big]\,\right\rangle}$$

$$(3.26)$$

$$P_{2A}(0,z) = \frac{\left[\lambda c_1 z [X(z)-1] R_0\right] \left[r_1 r_2 S_{1A}^* + (1-r_1)r_2 S_{1B}^*\right]}{\left\langle 1 - \left[(1-p)+pV^*\right]\left[r_1 r_2 S_{1A}^* S_{2A}^* + r_1(1-r_2)S_{1A}^* S_{2B}^* + (1-r_1)r_2 S_{2A}^* S_{1B}^* + (1-r_1)(1-r_2)S_{2A}^* S_{2B}^*\right] \right\rangle}$$

$$(3.27)$$

$$P_{2B}(0,z) = \frac{\left[\lambda c_1 z [X(z)-1] R_0\right] \left[(1-r_1)(1-r_2)S_{1B}^*\right]}{\left\langle 1 - \left[(1-p)+pV^*\right]\left[r_1 r_2 S_{1A}^* S_{2A}^* + r_1(1-r_2)S_{1A}^* S_{2B}^* + (1-r_1)r_2 S_{2A}^* S_{1B}^* + (1-r_1)(1-r_2)S_{2A}^* S_{2B}^*\right] \right\rangle}$$

$$(3.28)$$

$$Q(0,z) = \frac{p\left[\lambda c_1 z [X(z)-1] R_0\right] \left\langle \left[r_1 r_2 S_{1A}^* + (1-r_1)r_2 S_{1B}^*\right] S_{2A}^* + \left[(1-r_1)(1-r_2)S_{1B}^*\right] S_{2B}^* \right\rangle}{\left\langle 1 - \left[(1-p)+pV^*\right]\left[r_1 r_2 S_{1A}^* S_{2A}^* + r_1(1-r_2)S_{1A}^* S_{2B}^* + (1-r_1)r_2 S_{2A}^* S_{1B}^* + (1-r_1)(1-r_2)S_{2A}^* S_{2B}^*\right] \right\rangle}$$

$$(3.29)$$

where S_{1A}^*, S_{1B}^*, S_{2A}^*, S_{2B}^* and V^* in equations (3.25) to (3.29) represent

$S_{1A}^*\left(\lambda c_1(1-X(Z))\right)$, $S_{1B}^*\left(\lambda c_1(1-X(Z))\right)$, $S_{2A}^*\left(\lambda c_1(1-X(Z))\right)$, $S_{2B}^*\left(\lambda c_1(1-X(Z))\right)$ and

$V^*\left(\lambda c_2(1-X(Z))\right)$ respectively.

Now, from equation (3.15) and (3.25), we obtain

$$P_{1A}(z) = \int_0^\infty P_{1A}(x,z)dx$$

$$\frac{\left\langle \begin{array}{l} r_1\left[z-(1-r_1)r_2[(1-p)+pV^*]S_{2A}^*S_{1B}^*-(1-r_1)(1-r_2)[(1-p)+pV^*]S_{1B}^*S_{2B}^*\right] \\ -(1-r_1)\left[1-r_1r_2[(1-p)+pV^*]S_{2A}^*S_{1B}^*-r_1(1-r_2)[(1-p)+pV^*]S_{1A}^*S_{2B}^*\right] \end{array}\right\rangle \left(S_{1A}^*-1\right)R_0}{\left\langle 1-\left[(1-p)+pV^*\right]\left[r_1r_2S_{1A}^*S_{2A}^*+r_1(1-r_2)S_{1A}^*S_{2B}^*+(1-r_1)r_2S_{2A}^*S_{1B}^*+(1-r_1)(1-r_2)S_{2A}^*S_{2B}^*\right]\right\rangle}$$

$$(3.30)$$

Similarly from equations (3.16) and (3.26), we get

$$P_{1B}(z)=\int_0^\infty P_{1B}(x,z)dx$$

$$=\frac{\left\langle \begin{array}{l} r_1\left[(1-r_1)(1-r_2)[(1-p)+pV^*]S_{1A}^*S_{2B}^*-(1-r_1)r_2[(1-p)+pV^*]S_{1A}^*S_{2A}^*\right] \\ +(1-r_1)\left[z+r_1r_2[(1-p)+pV^*]S_{1A}^*S_{2A}^*-(1-r_1)(1-r_2)[(1-p)+pV^*]S_{1A}^*S_{2B}^*\right] \end{array}\right\rangle \left(S_{1B}^*-1\right)R_0}{\left\langle 1-\left[(1-p)+pV^*\right]\left[r_1r_2S_{1A}^*S_{2A}^*+r_1(1-r_2)S_{1A}^*S_{2B}^*+(1-r_1)r_2S_{2A}^*S_{1B}^*+(1-r_1)(1-r_2)S_{2A}^*S_{2B}^*\right]\right\rangle}$$

$$(3.31)$$

Then from equations (3.17) and (3.27), we get

$$P_{2A}(z)=\int_0^\infty P_{2A}(x,z)dx$$

$$= =\frac{\left[r_1 r_2 S_{1A}^*+(1-r_1)r_2S_{1B}^*\right]\left(S_{2A}^*-1\right)R_0}{\left\langle 1-\left[(1-p)+pV^*\right]\left[r_1 r_2S_{1A}^*S_{2A}^*+r_1(1-r_2)S_{1A}^*S_{2B}^*+(1-r_1)r_2S_{2A}^*S_{1B}^*+(1-r_1)(1-r_2)S_{2A}^*S_{2B}^*\right]\right\rangle}$$

$$(3.32)$$

Yet again, from equations (3.18) and (3.28), we get

$$P_{2B}(z) = \int_0^\infty P_{2B}(x,z)dx$$

$$= \frac{\left[(1-r_1)(1-r_2)S_{1B}^*\right]\left[S_{2B}^* - 1\right]R_0}{\left\langle 1 - \left[(1-p)+pV^*\right]\left[r_1\,r_2 S_{1A}^*\,S_{2A}^* + r_1\,(1-r_2)\,S_{1A}^* S_{2B}^* + (1-r_1)\,r_2\,S_{2A}^* S_{1B}^* + (1-r_1)\,(1-r_2)\,S_{2A}^* S_{2B}^*\right]\right\rangle}$$

(3.33)

Finally from equations (3.19) and (3.29), we get

$$Q(z) = \int_0^\infty Q(x,z)dx$$

$$= \frac{p\left\langle \left[r_1 r_2\,S_{1A}^* + (1-r_1)\,r_2 S_{1B}^*\right]S_{2A}^* + \left[(1-r_1)(1-r_2)S_{1B}^*\right]S_{2B}^*\right\rangle \left\langle 1 - V^*(\lambda c_2(1-X(z)))\right\rangle R_0}{z\left\langle 1 - \left[(1-p)+pV^*\right]\left[r_{1}r_2 S_{1A}^* S_{2A}^* + r_1\,(1-r_2)\,S_{1A}^* S_{2B}^* + (1-r_1)\,r_2 S_{2A}^* S_{1B}^* + (1-r_1)\,(1-r_2)\,S_{2A}^*\,S_{2B}^*\right]\right\rangle}$$

(3.34)

Now, the unknown constant R_0 can be determined by using the normalizing condition (3.13), which is equivalent to $R_0 + P_{1A}(1) + P_{1B}(1) + P_{2A}(1) + P_{2B}(1) + Q(1) = 1$. Thus we get

$$R_0 = (1-\rho),$$

(3.35)

where

$$\rho = (1-p)[\lambda E(X)c_1\{r_1 E(S_{1A}) + (1-r_1)E(S_{1B}) + r_2 E(S_{2A}) + (1-r_2)E(S_{2B})\}]$$
$$+ p[\lambda E(X)c_1\{r_1 E(S_{1A}) + (1-r_1)E(S_{1B}) + r_2 E(S_2) + (1-r_2)E(S_{2B})\} + c_2 \lambda E(X)E(V)]$$

$$= \lambda E(X)\left[c_1\left(r_1 E(S_1) + (1-r_1)E(S_{1B}) + E(r_2 S_{2A} + (1-r_2)S_{2B})\right) + c_2 pE(V)\right] \ < 1$$

is the utilization factor of this system.

Note that equation (3.35) gives the steady state probability that the server is idle but available in the system and it gives the stability condition under which the steady state solution exists. Further, utilizing the value of R_0 from (3.35) into equations (3.30)-(3.34), we have now completely and explicitly determined the queue size *PGF*s $P_{1A}(z)$, $P_{1B}(z)$, $P_{2A}(z)$, $P_{2B}(z)$ and $Q(z)$. In addition, various system state probabilities can also be obtained from equations (3.30)-(3.34) on putting $z = 1$. Thus we have

Prob [the server is busy with FSS(A)]$= P_{1A}(1) = \lambda c_1 E(X) r_1 E(S_{1A})$,

Prob [the server is busy with FSS(B)]$= P_{1B}(1) = \lambda c_1 E(X)(1 - r_1)E(S_{1B})$,

Prob [the server is busy with SSS(A)]$= P_{2A}(1) = \lambda c_1 E(X) r_2 E(S_{2A})$,

Prob [the server is busy with SSS(B)]$= P_{2B}(1) = \lambda c_1 E(X)(1 - r_2)E(S_{2B})$, and

Prob [the server is on vacation]$= Q(1) = p\lambda c_2 E(X)E(V)$.

Next, let denote the *PGF* of the queue size distribution at a random epoch can be found by adding equations (3.30)- (3.35). Thus we have

$$P_Q(z) = R_0 + P_{1A}(z) + P_{1B}(z) + P_{2A}(z) + P_{2B}(z) + zQ(z), \tag{3.36}$$

4   Particular Cases

Case 1   All Arriving Batches Are Allowed to Join the System (No RA Policy)

The main results for this case are obtained by letting $c_1 = c_2 = 1$ in equations (3.30) to (3.35), where now S_{1A}^*, S_{1B}^*, S_{2A}^*, S_{2B}^* and V^* in these equations represent $S_{1A}^*\big(\lambda(1 - X(Z))\big)$, $S_{1B}^*\big(\lambda(1 - X(Z))\big)$, $S_{2A}^*\big(\lambda(1 - X(Z))\big)$, $S_{2B}^*\big(\lambda(1 - X(Z))\big)$ and $V^*\big(\lambda(1 - X(Z))\big)$ respectively.

Case 2   The System Has Two First Stage Services (FSSA and FSSB) and Only One Second Stage Service (SSSA)

The main results of this case are obtained by letting $r_2 = 1$ in equations (3.30) to (3.35).

Thus we obtain

$$P_{1A}(z) = \frac{\left\langle \begin{array}{c} r_1\big[z - (1-r_1)[(1-p)+pV^*]S_{2A}^*S_{1B}^* + pV^*]S_{1B}^*S_{2B}^*\big] \\ -(1-r_1)\big[1 - r_1[(1-p)+pV^*]S_{2A}^*S_{1B}^* + pV^*]S_{1A}^*S_{2B}^*\big] \end{array} \right\rangle (S_{1A}^* - 1)R_0}{\left\langle 1 - \big[(1-p)+pV^*\big]\big[r_1 S_{1A}^* S_{2A}^* + (1-r_1)S_{2A}^* S_{1B}^*\big]\right\rangle} \tag{4.1}$$

$$P_{1B}(z) = \frac{\left\langle \begin{array}{c} r_1\big[-(1-r_1)[(1-p)+pV^*]S_{1A}^*S_{2A}^*\big] \\ +(1-r_1)\big[z + r_1[(1-p)+pV^*]S_{1A}^*S_{2A}^* + pV^*]S_{1A}^*S_{2B}^*\big] \end{array} \right\rangle (S_{1B}^* - 1)R_0}{\left\langle 1 - \big[(1-p)+pV^*\big]\big[r_1 S_{1A}^* S_{2A}^* + (1-r_1)S_{2A}^* S_{1B}^*\big]\right\rangle}$$

(4.2)

$$P_{2A}(z) = = \frac{\big[r_1 S_{1A}^* + (1-r_1)S_{1B}^*\big](S_{2A}^* - 1)R_0}{\left\langle 1 - \big[(1-p)+pV^*\big]\big[r_1 S_{1A}^* S_{2A}^* + (1-r_1)S_{2A}^* S_{1B}^*\big]\right\rangle} \tag{4.3}$$

$$P_{2B}(z) = 0 \tag{4.4}$$

$$Q(z) = \frac{p\left\langle \big[r_1 S_{1A}^* + (1-r_1) S_{1B}^*\big]S_{2A}^*\right\rangle \left\langle 1 - V^*\big(\lambda c_2(1 - X(z))\big)\right\rangle R_0}{z\left\langle 1 - \big[(1-p)+pV^*\big]\big[r_1 S_{1A}^* S_{2A}^* + (1-r_1) S_{2A}^* S_{1B}^*\big]\right\rangle} \tag{4.5}$$

$$R_0 = (1-\rho), \tag{4.6}$$

where

$$\rho = (1-p)[\lambda E(X)c_1\{r_1 E(S_{1A}) + (1-r_1)E(S_{1B}) + E(S_{2A})\}] $$
$$+ p[\lambda E(X)c_1\{r_1 E(S_{1A}) + (1-r_1)E(S_{1B}) + E(S_2)\} + c_2 \lambda E(X)E(V)]$$

$$= \lambda E(X)\big[c_1\big(r_1 E(S_1) + (1-r_1)E(S_{1B}) + E(S_{2A}) + c_2 \, pE(V)\big]\ < 1$$

is the utilization factor of this system.

Case 3   The System Has Only One First Stage Service (FSSA) and Two Second Stage Services (SSSA and SSSB)

The main results of this case are obtained by letting $r_1 = 1$ in equations (3.30) to (3.35). Thus we obtain

$$P_{1A}(z) = \frac{z\big(S_{1A}^* - 1\big)R_0}{\Big\langle 1 - \big[(1-p) + pV^*\big]\big[r_2 S_{1A}^* S_{2A}^* + (1-r_2)S_{1A}^* S_{2B}^*\big]\Big\rangle} \tag{4.7}$$

$$P_{1B}(z) = 0 \tag{4.8}$$

$$P_{2A}(z) = \frac{r_2 \, S_{1A}^* \big(S_{2A}^* - 1\big)R_0}{\Big\langle 1 - \big[(1-p) + pV^*\big]\big[r_2 S_{1A}^* S_{2A}^* + (1-r_2)S_{1A}^* S_{2B}^*\big]\Big\rangle} \tag{4.9}$$

$$P_{2B}(z) = \frac{\big[(1-r_2)S_{1B}^*\big]\big[S_{2B}^* - 1\big]R_0}{\Big\langle 1 - \big[(1-p) + pV^*\big]\big[r_2 S_{1A}^* S_{2A}^* + (1-r_2)S_{1A}^* S_{2B}^*\big]\Big\rangle} \tag{4.10}$$

$$Q(z) = \frac{p\big\langle\big[r_2 S_{1A}^*\big]S_{2B}^*\big\rangle \big\langle 1 - V^*(\lambda c_2(1 - X(z)))\big\rangle R_0}{z\Big\langle 1 - \big[(1-p) + pV^*\big]\big[r_2 S_{1A}^* S_{2A}^* + (1-r_2)S_{1A}^* S_{2B}^*\big]\Big\rangle} \tag{4.11}$$

$$R_0 = (1 - \rho), \tag{4.12}$$

where

$$\rho = (1-p)[\lambda E(X)c_1\{E(S_{1A}) + r_2 E(S_{2A}) + (1-r_2)E(S_{2B})\}]$$
$$+ p[\lambda E(X)c_1\{E(S_{1A}) + r_2 E(S_2) + (1-r_2)E(S_{2B})\} + c_2 \lambda E(X)E(V)]$$

$$= \lambda E(X)\left[c_1\left(r_1 E(S_{1A}) + (1-r_1)E(S_{1B}) + r_2 E(S_{2A}) + (1-r_2)E(S_{2B})\right) + c_2 pE(V)\right] < 1$$

is the utilization factor of this system.

Case 4   The System Has only One First Stage Service (FSSA) and Only One Second Stage Service (SSSA)

The main results of this case are obtained by letting $r_1 = 1$ and $r_2 = 1$ in equations (3.30) to (3.35). Thus we obtain

$$P_{1A}(z) = \frac{z\left(S_{1A}^* - 1\right)R_0}{\left\langle 1 - \left[(1-p) + pV^*\right]\left[S_{1A}^* S_{2A}^*\right]\right\rangle} \tag{4.13}$$

$$P_{1B}(z) = 0 \tag{4.14}$$

$$P_{2A}(z) = \frac{S_{1A}^*\left(S_{2A}^* - 1\right)R_0}{\left\langle 1 - \left[(1-p) + pV^*\right]\left[S_{1A}^* S_{2A}^*\right]\right\rangle} \tag{4.15}$$

$$P_{2B}(z) = 0 \tag{4.16}$$

$$Q(z) = \frac{p\,S_{1A}^*\left\langle 1 - V^*\left(\lambda c_2(1 - X(z))\right)\right\rangle R_0}{z\left\langle 1 - \left[(1-p) + pV^*\right]\left[S_{1A}^* S_{2A}^*\right]\right\rangle} \tag{4.17}$$

$$R_0 = (1-\rho), \tag{4.18}$$

where

$$\rho = (1-p)[\lambda E(X)c_1\{E(S_{1A}) + E(S_{2A})\}]$$
$$+ p[\lambda E(X)c_1\{E(S_{1A})) + E(S_2)\} + c_2\lambda E(X)E(V)]$$

$$= \lambda E(X)\big[c_1\big(E(S_1) + E(S_{2A})\big) + c_2 pE(V)\big] \quad < 1 \quad \text{is the utilization factor of this system.}$$

The results of case 4 agree with Madan and Choudhury [16]. The reader is referred to this paper for all further particular cases of this case.

5 Mean Queue Size and Mean Waiting Time in the Queue

In this section we derive the mean queue size of this

$$M^X / \left(\begin{pmatrix} G_{1A} \\ G_{1B} \end{pmatrix}, \begin{pmatrix} G_{2A} \\ G_{2B} \end{pmatrix} \right) / 1 / BS / SV / RA \quad \text{queue. Let } L_Q \text{ be the mean queue size at a}$$

random epoch, then

$$L_Q = \frac{dP_Q(z)}{dz}\bigg]_{z=1}$$

$$= \rho + \frac{\lambda^2 (E(X))^2 \big[c_1^2\big(r_1 E(S_{1A}^2) + (1-r_1)E(S_{1B}^2) + r_2 E(S_{2A}^2) + (1-r_2)E(S_{2B}^2 + 2r_1 r_2 E(S_1)E(S_2))\big]}{2(1-\rho)}$$

$$+ \frac{\lambda^2 (E(X))^2 \big[pc_2^2 E(V^2) + 2pc_1 c_2 E(V)\big(r_1 E(S_{1A}) + (1-r_1)E(S_{1B}) + r_2 E(S_{2A}) + (1-r_2)E(S_{2B})\big]}{2(1-\rho)}$$

$$+ \frac{\rho E(X_R)}{(1-\rho)}, \tag{5.1}$$

where $\rho = \lambda E(X)\big[c_1\big(r_1 E(S_{1A}) + (1-r_1)E(S_{1B}) + r_2 E(S_{2A}) + (1-r_2)E(S_{2B})\big) + c_2 pE(V)\big]$

and $E(X_R) = \dfrac{E(X(X-1))}{2E(X)}$ is the mean residual batch size.

If the above system has no FSSB service and no SSSB service, then we have $E(S_{1B}) = 0$, $E(S_{2B}) = 0$, $r_1 = 1$ and $r_2 = 1$. With these substitutions, the main result (5.1) reduces to

$$L_Q = \rho + \frac{\lambda^2 (E(X))^2 \left[c_1^2 \left(E(S_{1A}^2) + E(S_{2A}^2) + 2E(S_1)E(S_2) \right) \right]}{2(1-\rho)}$$

$$+ \frac{\lambda^2 (E(X))^2 \left[pc_2^2 E(V^2) + 2pc_1c_2 E(V) \left(E(S_{1A}) + E(S_{2A}) \right) \right]}{2(1-\rho)} + \frac{\rho E(X_R)}{(1-\rho)}, \qquad (5.2)$$

where $\rho = \lambda E(X) \left[c_1 \left(E(S_{1A}) + E(S_{2A}) \right) + c_2 pE(V) \right]$

Note that (5.2) is the result for the $M^X /(G_1, G_2)/1/BS/SV/RA$ studied by Madan and Choudhury [16].

Next, If we have $c_1 = c_2 = 1$ (no *RA* policy), then from (5.1) we obtain

$$L_Q = \rho + \frac{\lambda^2 (E(X))^2 \left[\left(r_1 E(S_{1A}^2) + (1-r_1)E(S_{1B}^2) + r_2 E(S_{2A}^2) + (1-r_2)E(S_{2B}^2 + 2r_1r_2 E(S_1)E(S_2) \right) \right]}{2(1-\rho)}$$

$$+ \frac{\lambda^2 (E(X))^2 \left[pE(V^2) + 2pE(V) \left(r_1 E(S_{1A}) + (1-r_1)E(S_{1B}) + r_2 E(S_{2A}) + (1-r_2)E(S_{2B}) \right) \right]}{2(1-\rho)}$$

$$+ \frac{\rho E(X_R)}{(1-\rho)}, \qquad (5.3)$$

where $\rho = \lambda E(X) \left[\left(r_1 E(S_{1A}) + (1-r_1)E(S_{1B}) + r_2 E(S_{2A}) + (1-r_2)E(S_{2B}) \right) + pE(V) \right]$.

Note that (5.3) gives the mean queue size at a random epoch for the

$$M^X / \left(\binom{G_{1A}}{G_{1B}}, \binom{G_{2A}}{G_{2B}} \right) /1/BS/SV \text{ queue without } RA \text{ policy.}$$

Further, with $E(S_{1B}) = 0$ (no FSSB service), $E(S_{2B}) = 0$ (no SSSB) , $E(V) = \dfrac{1}{\nu}$,

$E(V^2) = \dfrac{2}{\nu^2}$ and $E(X_R) = 0$, the result in (5.3) will reduce to the result obtained by Madan [12] for the case of single arrivals and exponential server vacations.

Further, in addition to $c_1 = c_2 = 1$ (no *RA* policy), if we have $E(S_{2A}) = 0 = E(S_{2A}^2)$, $E(S_{2B}) = 0 = E(S_{2B}^2)$ (no second stage service) and $p = 0$ (no server vacations), then (5.3) will reduce to

$$L_Q = \rho + \frac{\lambda^2 (E(X))^2 \left[\left(r_1 E(S_{1A}^2) + (1-r_1)E(S_{1B}^2) \right) \right]}{2(1-\rho)} + \frac{\rho E(X_R)}{(1-\rho)}, \qquad (5.4)$$

where $\rho = \lambda E(X)\big[\big(r_1 E(S_{1A}) + (1 - r_1) E(S_{1B})\big)\big]$.

Note that (5.4) gives the mean queue size for $M^X / \begin{pmatrix} G_1 \\ G_2 \end{pmatrix} / 1 / BS / SV$ queue with two first stage services (FSSA and FSSB) and no second stage service, without vacation policy

Next, let W_Q be the mean waiting time of an arbitrary customer for our general model $M^X / \left(\begin{pmatrix} G_{1A} \\ G_{1B} \end{pmatrix}, \begin{pmatrix} G_{2A} \\ G_{2B} \end{pmatrix} \right) / 1 / BS / SV / RA$ queue. Then utilizing Little's formula in equation (5.1), we may write

$$W_Q = \frac{L_Q}{\lambda_a}, \tag{5.5}$$

where, following the admissibility assumptions of our model, λ_a, the actual arrival rate of batches is given by

$$\lambda_a = \lambda \, c_1 \text{ (proportion of non-vacation time) } + \lambda \, c_2 \text{(proportion of vacation time)} \tag{5.6}$$

Note that in section 3, we found

The proportion of vacation time$= p\lambda c_2 E(X)E(V)$. $\tag{5.7}$

Consequently,

the proportion of non-vacation time including the first and second service times and the idle time$= 1 - p\lambda c_2 E(X)E(V)$. $\tag{5.8}$

Then utilizing (5.7) and (5.8) into equation (5.6), we obtain the actual arrival rate as

$$\lambda_a = \lambda \, c_1 \, \big[1 - p\lambda c_2 E(X)E(V) \big] + \lambda \, c_2 \big[p\lambda c_2 E(X)E(V) \big] \tag{5.9}$$

Remark: One may verify that when $p = 0$ (no vacations), then (5.9) reduces to $\lambda_a = \lambda c_1$, which is (as it should be) the actual arrival rate in case of no vacations, and further when $c_2 = c_1$ (same rate of restriction at all times), then also $\lambda_a = \lambda c_1$. Further

note that in both cases, we get $\lambda_a = \lambda$ for the case when $c_1 = 1$ (no restricted admissibility).

6 Mean Busy Period

We define the busy period as the length of time interval during which the server remains busy and this continues till the instant when the server becomes free again. This busy period is equivalent to the ordinary busy period generated by the units which arrive during the vacation period plus an idle period, which we may call as generalized idle period. We now define the following events:

$T_0 =$ length of the generalized idle period

$T_b =$ length of the busy period.

Now since T_0 and T_b generate an alternating renewal process, therefore we may write

$$\frac{E(T_b)}{E(T_0)} = \frac{\Pr ob[T_b]}{1 - \Pr ob[T_b]}. \tag{6.1}$$

Now, from section 3, we have

$$\Pr ob[[T_b] = P_{1A}(1) + P_{1B}(1) + P_{2A}(A) + P_{2B}(1) = \lambda c_1 \lambda E(X) E(B), \tag{6.2}$$

where $E(B) = r_1 E(S_{1A}) + (1 - r_1) E(S_{1B}) + r_2 E(S_{2A}) + (1 - r_2) E(S_{2B})$.

Again due to the well-known property of the Poisson input queueing system, we have

$$E(T_0) = \frac{1}{\lambda_a} + pE(V). \tag{6.3}$$

Next, utilizing (6.2) and (6.3) into (6.1), we get on simplifying

$$E(T_b) = \frac{c_1 E(B)}{\lambda_a \left[1 - c_1 \lambda E(X) E(B)\right]} + \frac{pc_1 \lambda E(X) E(V) E(B)}{\left[1 - c_1 \lambda E(X) E(B)\right]}, \tag{6.4}$$

where λ_a is given by (5.9).

Now if we take p $=0$ (i.e. no server vacations), then equation (5.8) gives $\lambda_a = \lambda c_1$ and consequently (6.4) will now reduce to

$$E(T_b) = \frac{E(B)}{1 - \lambda c_1 E(X) E(B)}, \tag{6.5}$$

which is the mean busy period for an ordinary $M^X / G / 1$ queueing system with *RA* pollicy. Note that this result agrees with the result obtained by Chaudhry [4] for $c_1 = 1$ (i.e no *RA* during the busy period)

Further, it is clear that the fraction of time the server remains in the generalized idle state T_0 (i.e. idle plus on vacation) is equivalent to $\dfrac{E(T_0)}{E(T_0) + E(T_b)}$.

(6.6)

Now, using $E(T_0)$ and $E(T_b)$ from (6.3) and (6.4) in the expression (7.6) and simplifying it , one may verify that

$$\frac{E(T_0)}{E(T_0) + E(T_b)} = (1-\rho) + c_2 \; p\lambda E(X)E(V)$$

$$= \text{Prob [the server is idle] + Prob [server is on vacation]}$$

$$= \text{Prob [} T_0 \text{],} \tag{6.7}$$

as it should be.

If our general system $M^X / \left(\begin{pmatrix} G_{1A} \\ G_{1B} \end{pmatrix} , \begin{pmatrix} G_{2A} \\ G_{2B} \end{pmatrix} \right) / 1 / BS / SV / RA$ has no FSSB service and

no SSSB service, then we have $E(S_{1B}) = 0$, $E(S_{2B}) = 0$, $r_1 = 1$ and $r_2 = 1$. With these substitutions, E(B) in (6.2) will become $E(B) = E(S_{1A}) + E(S_{2A})$ and consequently the main results (6.1) – (6.4) found above would agree with Madan and Choudhury [16].

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