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# A Further Generalization of the Bernoulli Polynomials and on the 2D-Bernoulli Polynomials $B_n^2(x,y)$

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**Abstract**

In this work we give some recurrence relations of the new generalized Bernoulli polynomials and numbers. Furthermore a relation is given between $2D$-Bernoulli polynomials and 2-variable Hermite Kampé de Feriet polynomials.

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## 1 Introduction

The ordinary Bernoulli numbers and ordinary Bernoulli polynomials are defined by the following generating functions respectively,

$$\begin{aligned} G(x,t) &= \frac{te^{xt}}{e^t-1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}, \quad |t|<2\pi, \qquad (1) \\ G(t) &= \frac{t}{e^t-1} = \sum_{n=0}^{\infty} B_n\frac{t^n}{n!}, \quad |t|<2\pi. \end{aligned}$$

First few of the rational numbers $B_n$ are,

$$B_0=1,\ B_1=-\frac{1}{2},\ B_2=\frac{1}{6},\ B_3=0,\ B_4=-\frac{1}{30},\cdots,B_{2k+1}=0 \text{ for } k=1,2,3,\cdots.$$

Some of the Bernoulli polynomials are

$$\begin{aligned} B_0(x) &= 1,\ B_1(x) = x - \frac{1}{2},\ B_2(x) = x^2 - x + \frac{1}{6},\ B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x, \\ B_4(x) &= x^4 - 2x^3 + x^2 - \frac{1}{30},\ B_5(x) = x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{1}{6}x, \cdots. \end{aligned}$$

**Proposition 1.1** *The Bernoulli polynomials satisfy the following relations*

$$B_n(x+1) = \sum_{k=0}^{n} \binom{n}{k} B_k(x),$$

$$\begin{aligned} B_n(x+y) &= \sum_{k=0}^{n} \binom{n}{k} B_k(x) y^{n-k}, \\ &= \sum_{k=0}^{n} \binom{n}{k} B_k(y) x^{n-k}, \end{aligned}$$

$$B_n(x+1) - B_n(x) = nx^{n-1},\ n \geqslant 1,$$

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k},\ n \geqslant 0,$$

$$B_n^{'} = nB_{n-1}(x),\ n \geqslant 1.$$

**Definition 1.2** *A new class of generalized Bernoulli polynomials $B_n^{[m-1]}(x), m \geq 1$ are defined by means of the generating function defined in a suitable neigbourhood of $t = 0$,*

$$G^{[m-1]}(x,t) = \frac{t^m e^{xt}}{e^t - \sum\limits_{h=0}^{m-1} \frac{t^h}{h!}} = \sum_{n=0}^{\infty} B_n^{[m-1]}(x) \frac{t^n}{n!} \tag{2}$$

*([1],[2] and [6]).*

For $m = 1$, we obtain from (2) the generating function,

$$G^{[0]}(x,t) = \frac{te^{xt}}{e^t - 1}$$

of classical Bernoulli polynomials $B_n^0(x)$.

In (2), we take $x = 0$, a generalized Bernoulli numbers are

$$\frac{t^m}{e^t - \sum_{h=0}^{m-1} \frac{t^h}{h!}} = \sum_{n=0}^{\infty} B_n^{[m-1]} \frac{t^n}{n!}. \tag{3}$$

Since $G^{[m-1]}(x,t) = A(t)e^{xt}$, the generalized Bernoulli polynomials belong to the class of Appell polynomials. From (2), we have

$$\begin{aligned} B^{[m-1]} &= B^{[m-1]}(0), \\ e^{xt} &= \sum_{h=m}^{\infty} \frac{t^{h-m}}{h!} \sum_{n=0}^{\infty} B_n^{[m-1]}(x) \frac{t^n}{n!}. \end{aligned} \tag{4}$$

Since $e^{xt} = \sum_{n=0}^{\infty} x^n \frac{t^n}{n!}$, (4) becomes

$$\sum_{n=0}^{\infty} x^n \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{h=0}^{n} \binom{n}{h} \frac{h!}{(h+m)!} B_{n-h}^{[m-1]}(x) \frac{t^n}{n!}.$$

By comparing the coefficients of the last equation, we obtain

$$x^n = \sum_{h=0}^{n} \binom{n}{h} \frac{h!}{(h+m)!} B_{n-h}^{[m-1]}(x). \tag{5}$$

From (5), we obtain the first generalized Bernoulli polynomials as follows respectively,

$$B_0^{[m-1]} = m!,$$

$$B_1^{[m-1]} = -\frac{m!}{m+1},$$

$$B_2^{[m-1]} = \frac{2m!}{(m+1)^2(m+2)}, \cdots$$

and

$$B_0^{[m-1]}(x) = m!,$$

$$B_1^{[m-1]}(x) = m!\left(x - \frac{1}{m+1}\right),$$

$$B_2^{[m-1]}(x) = m!\left(x^2 - \frac{2}{m+1}x + \frac{2}{(m+1)^2(m+2)}\right), \cdots .$$

Costabile et al gave the different classical definitions and relations. Also, Costabile et al have defined Bernoulli polynomials as a determinant relation in ([3]). Natalini et al in ([6]) studied on some differential equations which are satisfied the Bernoulli numbers $B_n^{[m-1]}(x)$. In ([1]), Bretti et al defined multidimensional extensions of the Bernoulli and Appell polynomials. They defined the $2D$-Bernoulli polynomials $B_n^{[j]}(x,y)$ in ([4]). Also, they gave an explicit forms of the polynomials $B_n^{[j]}(x,y)$ in terms of the Hermite-Kampé de Feriet polynomials $H_n^j(x,y)$ in ([1],[4]). Subuhi et al in ([7]) gave the Hermite-Bernoulli polynomials ${}_HB_n(x,y)$, Hermite-Euler polynomials ${}_HE_n(x,y)$. They proved some recurrence relations between these polynomials. In this work, we made a generalization on a new class of Bernoulli polynomials. We gave some relations in this subject. Also, we gave two equations of the $2D$-Bernoulli polynomials.

# 2 Main Theorems

**Proposition 2.1** *There is a relation between a generalized Bernoulli numbers and a generalized Bernoulli polynomial*

$$B_n^{[m-1]}(x) = \sum_{k=0}^{n} \binom{n}{k} B_k^{[m-1]} x^{n-k}. \tag{6}$$

**Proof.** From (2) and (3), we get (6). ■

**Theorem 2.2** *Generalized Bernoulli polynomials satisfy the following relations,*

$$\begin{aligned} B_n^{[m-1]}(x+y) &= \sum_{k=0}^{n} \binom{n}{k} B_k^{[m-1]}(x) y^{n-k} \qquad (7) \\ &= \sum_{k=0}^{n} \binom{n}{k} B_k^{[m-1]}(y) x^{n-k}. \end{aligned}$$

$$B_n^{[m-1]}(ax) = \sum_{k=0}^{n} \binom{n}{k} B_k^{[m-1]}(x)(a-1)^{n-k} x^{n-k}, a \in \mathbb{Z}^+. \tag{8}$$

**Proof.** From (2),

$$\sum_{n=0}^{\infty} B_n^{[m-1]}(x+y)\frac{t^n}{n!} = \frac{t^m e^{(x+y)t}}{e^t - \sum\limits_{h=0}^{m-1} \frac{t^h}{h!}} = \frac{t^m e^{xt}}{e^t - \sum\limits_{h=0}^{m-1} \frac{t^h}{h!}} e^{yt} = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k} B_k^{[m-1]}(x) y^{n-k}\right)\frac{t^n}{n!}.$$

By comparing the coefficients $\frac{t^n}{n!}$ in the both sides of the above equation, we obtain (7). Proof of (8) is similar to that of (7), so omit it. ■

**Definition 2.3** *We consider the generalized Bernoulli number* $B_n^{[m-1,\alpha]}$ *order* $\alpha$ *as respectively;*

$$\sum_{n=0}^{\infty} B_n^{[m-1,\alpha]}\frac{t^n}{n!} = \left(\frac{t^m}{e^t - \sum\limits_{h=0}^{m-1} \frac{t^h}{h!}}\right)^{\alpha}, \tag{9}$$

$$\sum_{n=0}^{\infty} B_n^{[m-1,\alpha]}(x)\frac{t^n}{n!} = \left(\frac{t^m}{e^t - \sum\limits_{h=0}^{m-1} \frac{t^h}{h!}}\right)^{\alpha} e^{xt}.$$

*For* $\alpha = 1$ *and* $m = 1$ *it reduces to classical Bernoulli number and classical Bernoulli polynomials.*

From (1), we have

$$B_n^{[m-1,\alpha]}(x) = \sum_{k=0}^{n}\binom{n}{k} B_k^{[m-1,\alpha]} x^{n-k}.$$

**Theorem 2.4** *The generalized Bernoulli polynomials* $B_n^{[m-1,\alpha]}(x)$ *order* $\alpha$ *satisfy the following relations;*

$$B_n^{[m-1,\alpha]}(x+y) = \sum_{k=0}^{n}\binom{n}{k} B_k^{[m-1,\alpha]}(x) y^{n-k}, \tag{10}$$

$$B_n^{[m-1,\alpha+\beta]}(x+y) = \sum_{k=0}^{n}\binom{n}{k} B_k^{[m-1,\alpha]}(x) B_{n-k}^{[m-1,\beta]}(y). \tag{11}$$

**Proof.** From (9),

$$\begin{aligned}
\sum_{n=0}^{\infty} B_n^{[m-1,\alpha+\beta]}(x+y)\frac{t^n}{n!} &= \left(\frac{t^m}{e^t-\sum\limits_{h=0}^{m-1}\frac{t^h}{h!}}\right)^{(\alpha+\beta)} e^{(x+y)t} \\
&= \left(\frac{t^m}{e^t-\sum\limits_{h=0}^{m-1}\frac{t^h}{h!}}\right)^{\alpha} e^{xt}\left(\frac{t^m}{e^t-\sum\limits_{h=0}^{m-1}\frac{t^h}{h!}}\right)^{\beta} e^{yt} \\
&= \left(\sum_{n=0}^{\infty} B_n^{[m-1,\alpha]}(x)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty} B_n^{[m-1,\beta]}(y)\frac{t^n}{n!}\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k} B_k^{[m-1,\alpha]}(x)B_{n-k}^{[m-1,\beta]}(y)\right)\frac{t^n}{n!}.
\end{aligned}$$

By comparing the coefficients $\frac{t^n}{n!}$ in the both sides of the above equation, we arrive at (11). Proof of (10) is similar to that of (11). ■

**Corollary 2.5** *Generalized Bernoulli numbers* $B_n^{[m-1,\alpha]}$ *order* $\alpha$ *satisfy the following relation*

$$B_n^{[m-1,\alpha]} = \sum_{k=0}^{n}\binom{n}{k} B_k^{[m-1,l]} B_{n-k}^{[m-1,\alpha-l]}, \tag{12}$$

*where* $k, l \in \mathbb{Z}^+$.

**Proof.** From (9),

$$\begin{aligned}
\sum_{n=0}^{\infty} B_n^{[m-1,\alpha]}\frac{t^n}{n!} &= \left(\frac{t^m}{e^t-\sum\limits_{h=0}^{m-1}\frac{t^h}{h!}}\right)^{\alpha} = \left(\frac{t^m}{e^t-\sum\limits_{h=0}^{m-1}\frac{t^h}{h!}}\right)^{l}\left(\frac{t^m}{e^t-\sum\limits_{h=0}^{m-1}\frac{t^h}{h!}}\right)^{\alpha-l} \\
&= \left(\sum_{n=0}^{\infty} B_n^{[m-1,l]}\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty} B_n^{[m-1,\alpha-l]}\frac{t^n}{n!}\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k} B_k^{[m-1,l]}B_{n-k}^{[m-1,\alpha-l]}\right)\frac{t^n}{n!}.
\end{aligned}$$

By comparing the coefficients $\frac{t^n}{n!}$ in the both sides of the above equation, we easily arrive at (12). ■

In this section, we will prove two relation of a class of $2D$-Bernoulli polynomials denoted by $B_n^2(x,y)$ which are a generalization the classical Bernoulli polynomials.

The $2-$variable Hermite Kampé de Feriet polynomials are defined by

$$\sum_{n=0}^{\infty} H_n^2(x,y)\frac{t^n}{n!} = e^{xt+yt^2} \tag{13}$$

or

$$H_n^2(x,y) = n!\sum_{s=0}^{\left[\frac{n}{2}\right]} \frac{x^{n-2s}y^s}{(n-2s)!(s)!}$$

([1], [4], [7]).

In [1] Bretti et al defined $2D-$Bernoulli polynomials by

$$\sum_{n=0}^{\infty} B_n^2(x,y)\frac{t^n}{n!} = \frac{t}{e^t-1}e^{xt+yt^2}. \tag{14}$$

They proved the following relations between the polynomials $H_n^2(x,y)$ and $B_n^2(x,y)$.

$$\begin{aligned} B_n^2(x,y) &= \sum_{h=0}^{n}\binom{n}{h}B_{n-h}H_h^2\left(x,y\right) \\ &= n!\sum_{h=0}^{n}\frac{B_{n-h}}{(n-h)!}\sum_{s=0}^{\left[\frac{h}{2}\right]}\frac{x^{h-2s}y^s}{(h-2s)!(s)!}, \end{aligned}$$

and

$$H_n^2\left(x,y\right) = \sum_{h=0}^{n}\binom{n}{h}\frac{1}{n-h+1}B_n^2(x,y).$$

**Theorem 2.6** *$2D-$Bernoulli polynomial satisfy the recurrence relation*

$$B_n^2(x+1,y) - B_n^2(x,y) = nH_{n-1}^2\left(x,y\right). \tag{15}$$

**Proof.** From (14),

$$\begin{aligned} \sum_{n=0}^{\infty} B_n^2(x+1,y)\frac{t^n}{n!} - \sum_{n=0}^{\infty} B_n^2(x,y)\frac{t^n}{n!} &= te^{xt+yt^2} \\ &= t\sum_{n=0}^{\infty} H_n^2(x,y)\frac{t^n}{n!}. \end{aligned}$$

Since $B_0^2(x,y)=1$, we have

$$\sum_{n=1}^{\infty}\left(B_n^2(x+1,y)-B_n^2(x,y)\right)\frac{t^n}{n!}=\sum_{n=1}^{\infty}nH_{n-1}^2\left(x,y\right)\frac{t^n}{n!}.$$

By comparing the coefficient $\frac{t^n}{n!}$ in the both sides of the above equation, we easily arrive at (15). ■

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