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Applications of Operational Calculus: Trigonometric Interpolating Equation for the Nine-Point Prism and Robust Center Point Formulas

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Abstract

The derivation of a trigonometric equation for the nine-point, rectangular prismatic array is illustrated. In many cases, the equation can be used to reproduce a ninth datum at an arbitrary point near the center of the array. New estimators of central tendency are more resistant to the distorting effects of an outlier than the arithmetic mean. Examples illustrate this property. All of the methods derive from the shifting operator.

Mathematics Subject Classification: 65D07, 65D17

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1. Introduction

A method for interpolating the eight-point cube by means of the circular or the hyperbolic trigonometric functions was recently described [2]. This paper illustrates a new method for interpolating nine data in cubical array by means of those functions. Operational, polynomial- and exponential-type interpolating equations for prismatic arrays are invariant under data translation and rotation but the trigonometric equations do not have the advantage of translational invariance [3,4].

2. The nine-point rectangular prism

The first trigonometric equation for interpolating numbers arranged at the vertices of a cube applied the identities in Eqs. (1a) and (2a) [2]. Their operational interpretations are Eqs. (1b) and (2b), respectively. Eqs. (1b) and (2b) yield Eq. (3). The three finite-difference equations (1b), (2b), (3) apply to the 5-point rectangle ACEGI in Fig. 1. The vertices of that rectangle can be reinterpreted as vertices A,B,E,H,I, and F,G,E,C,D in the rectangular prism illustrated in Fig. 2.

$$(2)\sin(x)\cos(x)\cos(x+y) - (2)\sin(x+y)\cos(x)^2 + \sin(x+y) = \sin(x-y) \quad (1a)$$
$$(F-D)(F+D)(I+A) - (I-A)(F+D)^2 + 2E^2(I-A) = 2E^2(C-G) \quad (1b)$$

$$(2)\sin(x)\cos(x)\cos(y-x) + (2)\cos(x)^2\sin(y-x) - \sin(y-x) = \sin(x+y) \quad (2a)$$
$$(F-D)(F+D)(G+C) + (F+D)^2(G-C) = 2E^2(G-C+I-A) \quad (2b)$$

$$E^2 - [FD(I+G-A-C)-F^2(A-G)-D^2(C-I)] / [2(I-A-C+G)] = 0 \quad (3)$$

Eq. (3) can be substituted with a new expression for the midpoint D of one side of the rectangle in Fig. 1. That new expression is $D=(F^2-IC+AG)^{(1/2)}$ [5]. Choose the positive root. It contains F^2. Let the four-letter sequence BDIG represent the midpoint of the corresponding face of the prism in Fig. 2. Substitute BDIG for the letter F in substituted Eq. (3). In three dimensions, two five-point rectangles are ABEHI and FGECD. By a change of notation, both rectangles can be represented by substituted Eq. (3). Multiply the two expressions, simplify the product for convenience, and solve it for BDIG. The tedious result is presented in terms of its numerator and denominator, Eqs. (4) and (5), respectively. It is the three-dimensional, nine-point analog of the Eq. (6) that appears in Ref. [2]. Equation (6) is restricted to positive numbers as data because the positive root for D was selected. A single letter represents a number at a vertex in Fig. 2. Double-letter combinations represent the product of two single letters. If A .. I are 1 .. 9, respectively, then BDIG=11/2.

numerator of BDIG =

$(2E^2(H+I-A-B) + BI(I-B) + AH(B-I))^2(2E^2(C+D-F-G) + DG(D-G) + FC(G-D))^2$ (4)

denominator of BDIG =
$(I+H-A-B)(4(I+H-B-A)E^2 - (BI-AH)(H+B-I-A))(F+G-C-D)(4(G+F-C-D)E^2 - (D-G+F-C)(DG-FC))$ (5)

BDIG = $([\text{Eq. (4)}] / [\text{Eq. (5)}])^{(1/4)}$ (6)

The right hand side of Eq. (6) above replaces the right hand side of Eq. (7) in Ref. [2]. The remaining expressions for the midpoints of sides ACHF, ABGF, CDIH, ABDC, and FGIH can be obtained by rotating the cube and reapplying Eq. (6). The new algorithm then proceeds starting at Eq. (16) in Ref. [2]. The factors W_1 .. W_6 are found as Eqs. (19)-(24) in Ref. [2]. Eq. (25) in Ref. [2] thus becomes Eq. (7) and similarly for the remaining side point expressions that appear as Eqs. (25)-(30) in Ref. [2].

BDIG = $(W_1)([\text{Eq. (4)}] / [\text{Eq. (5)}])^{(1/4)}$ (7)

The letter E in Eqs. (4) and (5) represents the datum at the center of the prism in Fig. 2. Its presence means that new algorithm generates an interpolating equation for the nine-point prism. However, the new interpolating equation will not necessarily reproduce the center point datum unless the value of the exponent NN is properly adjusted [2]. In certain cases, no adjustment of NN is necessary. Those cases include data that are generated by 2^x where x is 1 .. 9 at vertices A .. I, respectively in Fig. 2. Other cases include data that represent the circular sines or cosines of 10^o .. 90^o at vertices A .. I, respectively. In other words, some of the properties of the eight-point trigonometric interpolating equation described in Ref. [2] are maintained by the nine-point trigonometric equation described in this paper. Neither the eight- nor the nine-point trigonometric equations apply to trilinear numbers in prismatic array [2].

For example, let the numbers 1^3, 2^3, 3^3, 4^3 5^3, 6^3, 7^3, 8^3, 9^3 be positioned at vertices A .. I in Fig. 2, respectively. The equation interpolating them, as developed according to the preceding text, and assigning NN=0, renders R=124.958 at (x,y,z)=(0,0,0). If NN is changed to (–0.00285), the center point prediction is much closer to the true center point datum, R=125.0. That nine-point interpolating equation, with rounded coefficients, appears as Eq. (8) below.

$$\begin{aligned} R = {} & (125.0)\cosh(0.1556x)\cosh(0.3622y)\cosh(1.166z) + \\ & (135.9)\cosh(0.1556x)\cosh(0.3622y)\sinh(1.166z) + \\ & (144.8)\cosh(0.1556x)\sinh(0.3622y)\cosh(1.166z) + \\ & (138.2)\cosh(0.1556x)\sinh(0.3622y)\sinh(1.166z) + \\ & (165.4)\sinh(0.1556x)\cosh(0.3622y)\cosh(1.166z) + \\ & (155.4)\sinh(0.1556x)\cosh(0.3622y)\sinh(1.166z) + \\ & (147.3)\sinh(0.1556x)\sinh(0.3622y)\cosh(1.166z) + \\ & (89.51)\sinh(0.1556x)\sinh(0.3622y)\sinh(1.166z) \end{aligned} \quad (8)$$

Table 1 illustrates the sums of squares of deviations of three equations that interpolate the nine-point cube. The quadratic equation appears as Eq. (1) in Ref. [3]. It uses the quadratic-term expressions given as Eqs. (14)-(19) therein. The cubic equation is described as Eq. (D) in Ref. [6]. The trigonometric equation cited in Table 1 is the nine-point method illustrated in this paper. Let its exponent NN be taken as zero. The trial data are generated by the monotonic functions applied to the integers 1 .. 9 as A .. I, respectively, in Fig. 2. The eight- and nine-point trigonometric equations apply only to positive numbers as data [2]. They do not apply to every configuration of positive numbers.

3. Operational measures of central tendency

One of the interesting applications of the shifting operator is the development of two series of new formulas for the estimation of central tendency. The members of each series are tedious expressions but modern computers make them useful for the cited purpose. The members of both series require sorted data, and both series require a minimum of four measurements. The first series is denoted PXP where the X represents the number of data to which a particular formula applies and the trailing P represents a polynomial-type estimator. The second series of formulas are denoted PXE where the trailing E represents an exponential-type estimator.

The polynomial-type estimators are insensitive to data translation so they compete with the arithmetic mean. They have the advantage of exactness on linear numbers and their squares whereas the mean does not have the latter property. As illustrated by the examples in Refs. [7-9], the formulas offer the promise of greater accuracy and greater robustness than the mean. The PXE formulas also promise these advantages but they are sensitive to translation of the data. The first three members of each series have been described in the Refs. [7-9].

Below are the numerators and the denominators of the fourth member of the polynomial-type estimators. They are separated for the sake of clarity. The numerators and denominators of the fourth member of the PXE series follow them. Both P10P and P10E estimate the centers of ten sorted numbers. Patterns in the numerators and denominators of P10P and P10E are apparent. The patterns assist the generation of successive members of each series. Both series appear to be indefinitely extensible.

Numerator of P10P:

$$
\begin{aligned}
&4(z10-z1)^2(z9-z2)^2(z8-z3)^2(z7-z4)^2(z6+z5)\\
&-(z10-z1)^2(z9-z2)^2(z8-z3)^2(z7+z4)(z6-z5)^2\\
&-(z10-z1)^2(z9-z2)^2(z8+z3)(z7-z4)^2(z6-z5)^2\\
&-(z10-z1)^2(z9+z2)(z8-z3)^2(z7-z4)^2(z6-z5)^2\\
&-(z10+z1)(z9-z2)^2(z8-z3)^2(z7-z4)^2(z6-z5)^2 \qquad (9)
\end{aligned}
$$

Denominator of P10P:

$$8(z10-z1)^2(z9-z2)^2(z8-z3)^2(z7-z4)^2 - 2(z10-z1)^2(z9-z2)^2(z8-z3)^2(z6-z5)^2$$
$$- 2(z10-z1)^2(z9-z2)^2(z7-z4)^2(z6-z5)^2 - 2(z10-z1)^2(z8-z3)^2(z7-z4)^2(z6-z5)^2$$
$$- 2(z9-z2)^2(z8-z3)^2(z7-z4)^2(z6-z5)^2 \quad (10)$$

Numerator of $(P10E)^2$:

$$4(z10-z1)^2(z9-z2)^2(z8-z3)^2(z7-z4)^2(z6+z5)^2$$
$$- (z10-z1)^2(z9-z2)^2(z8-z3)^2(z7+z4)^2(z6-z5)^2$$
$$- (z10-z1)^2(z9-z2)^2(z8+z3)^2(z7-z4)^2(z6-z5)^2$$
$$- (z10-z1)^2(z9+z2)^2(z8-z3)^2(z7-z4)^2(z6-z5)^2$$
$$- (z10+z1)^2(z9-z2)^2(z8-z3)^2(z7-z4)^2(z6-z5)^2 \quad (11)$$

Denominator of $(P10E)^2$:

$$16(z10-z1)^2(z9-z2)^2(z8-z3)^2(z7-z4)^2 - 4(z10-z1)^2(z9-z2)^2(z8-z3)^2(z6-z5)^2$$
$$- 4(z10-z1)^2(z9-z2)^2(z7-z4)^2(z6-z5)^2 - 4(z10-z1)^2(z8-z3)^2(z7-z4)^2(z6-z5)^2$$
$$- 4(z9-z2)^2(z8-z3)^2(z7-z4)^2(z6-z5)^2 \quad (12)$$

An outlier is a datum that gives the impression of not being a member of a series of repeated measurements. The arithmetic mean has been criticized for its sensitivity to the distorting effects of outliers. In contrast to this undesirable property of the mean, center point estimates rendered by the PXP and the PXE formulas are resistant to an outlier. This property is illustrated in Table 2 for the four polynomial-type estimators P4P, P6P, P8P, P10P as well as for the four exponential-type estimators $P4E^2$, $P6E^2$, $P8E^2$ and $P10E^2$. Note that it is the squares of the center point estimates that are rendered by the PXE formulas. The sorted numbers used for the following illustrations are taken from Rousseuw [1]. The first datum in the series of Rousseeuw's ten numbers is incremented by an added error so that it gradually assumes the role of an outlier. A formula such as P4P or $P4E^2$ uses only the first four members of the series taken from left to right. Formulas P6P and $P6E^2$ use the first six numbers from left to right.

95+error, 93, 92, 90, 88, 86, 83, 80, 75, 40 (13)

The columns in Table 2 headed by the abbreviation AVG represent the arithmetic mean of four, six, eight, or ten numbers taken from left to right in the preceding series. The upper left, upper right, lower left, and lower right quadrants in Table 2 use 4, 6, 8, and 10 data, respectively. The entries for (AVG) illustrate that as the error increases, the mean also increases. The mean changes so much that an error of 5000 makes it an unrecognizable approximation of its true value when the error was zero. The operational estimates do not show this adverse effect. The operational estimates change somewhat as the error increases but they are essentially the same no matter what number for the error is assumed. That is, the operational center points of Rousseeuw's numbers are nearly the same whether the error is zero or 5000.

The operational measures of central tendency combine two desirable properties: enhanced accuracy in known cases [7-9] and enhanced resistance to an outlier in many other cases. See Table 2. There is no reason to think that these desirable properties will be nullified on application of the formulas to laboratory data. The rationale for the operational formulas has been summarized in Ref. [9] but the formulas therein have awkward formulations. See Refs. [7,8] for commentary on Ref. [9].

4. Discussion

There seems to be no popular nine-point analog of the trilinear equation for the cube in Fig. 2. Even so, nine positive numbers A .. I in Fig. 2 can be represented in many ways and some of them are exact on trilinear numbers. The preference for linear relationships is partly a tradition: the straight line will always be with us. However, the tradition does not exclude alternatives that contain curvature coefficients. Among them

are the operational equations that contain second-order coefficients and others that contain second- and third-order coefficients. Polynomial, trigonometric, and exponential laws often fit natural phenomena better than straight lines.

When the exponent NN is zero in both trigonometric methods, the nine-point method described in this paper typically renders lower sums of squares of deviations from simple trial surfaces that the eight-point method [2]. This remark applies primarily to data that are monotonic-increasing or -decreasing. Compare the entries in Table 1 to the like entries in Table 1 of Ref. [2]. The author can supply a Maple® worksheet for the nine-point trigonometric interpolating equation described above [10].

Let the coefficients of the arguments of the sine and cosine functions be determined as described in Section 2 and let a term be added to the expression for the prismatic array. It now contains nine coefficients external to the trigonometric functions including the added term. There are also nine data including the center point datum. Alternative external coefficients can usually be obtained by forming nine simultaneous equations and solving them for the external coefficients. This approach is easier than adjusting the value of the exponent NN. It can also be applied with various assignments for exponent NN. A selection of trigonometric interpolating equations for the nine-point array can therefore be generated at pleasure.

Operational estimators of central tendency are so tedious that computer assistance is essential. Their numerators and denominators adhere to patterns that can be discerned on examination. The formulas can apparently be extended indefinitely to accommodate more data. Table 2 provides evidence that the operational formulas resist the distorting influence of an outlying datum better than the mean. The operational formulas are often more accurate than the mean, as well [7,8]. The entries in the table were obtained by 10-digit precision to minimize the effects of error propagation. Copies of formulas for four to sixteen ordered numbers can be supplied by the author.

A recent paper describes the generation of exponential equations for eight- and nine-point prismatic arrays [4]. It is therein stated that the ranges searched for the numerical values of J, K, and L should not include unity. This statement seems to be too strong. When convergence is possible, modern software usually finds the appropriate values of J, K, L even if the searched ranges include unity. If the x-, y-, and z-coefficients in the trilinear equation for the eight-point cube are greater or less than zero, the corresponding values of J, K, and L are often greater or less than unity, respectively. Although it is not an infallible guide, this observation may assist the assignment of the ranges to be searched for the numerical values of J, K, and L in the exponential method [4].

Table 1. Approximate sums of squares of deviations of three interpolating equations from typical trial surfaces. The equations are based on different approaches to the nine-point cube. The data are generated by applying the listed functions to the integers 1 .. 9 at vertices A .. I, respectively, as in Fig. 2. The coefficient NN in the trigonometric equation is taken zero. The quadratic equation is formed from Eqs. (14)-(19) in Ref. [3]. Eq. (D) is taken from Ref. [6].

Function*	Quadratic equation	Cubic equation (D)	Trigonometric equation (text)
M^2	0	0	15.2
M^3	1201	0	1852
2^M	10063	1463	0
sinh(M/2)	33.2	2.87	0
tan(9M^o)	1.53	0.393	0.443
cosh(M/2)	32.5	2.92	0
cosh(M/2) + M	32.5	2.92	1.24
(M)cosh(M/2)	4737	540	16.4
(5)sin(10M^o) + cos(10M^o)	0.00563	0.000294	0

*M = (5+x/2+y+5z/2)

Table 2. Center point estimates as obtained by the arithmetic average (AVG) and two series of operational center point estimators. Estimates are obtained after adding an error to the largest of four, six, eight, and ten sorted numbers. The original numbers appear in the line for Eq. (13). Estimates as rendered by the 4-, 6-, 8- and 10-point polynomial-type and exponential-type formulas are denoted by the suffixes P and E, respectively. The entries have been rounded.

Error	**P4P**	**P4E**	**AVG**		**P6P**	**P6E**	**AVG**
0	92.50	92.50	92.50		91.06	91.06	90.67
1	92.49	92.49	92.75		91.04	91.04	90.83
5	92.47	92.47	93.75		91.02	91.02	91.50
10	92.48	92.48	95.00		91.02	91.02	92.33
50	92.49	92.49	105.0		91.03	91.03	99.00
100	92.50	92.49	117.5		91.03	91.03	107.3
500	92.50	92.50	217.5		91.04	91.04	174.0
1000	92.50	92.50	342.5		91.04	91.04	257.3
5000	92.50	92.50	1343		91.04	91.04	924.0
Error	**P8P**	**P8E**	**AVG**		**P10P**	**P10E**	**AVG**
0	89.02	89.02	88.38		87.03	87.03	82.20
1	89.02	89.02	88.50		87.03	87.03	82.30
5	89.01	89.01	89.00		87.03	87.03	82.70
10	89.01	89.01	89.63		87.03	87.03	83.20
50	89.01	89.01	94.63		87.03	87.03	87.20
100	89.01	89.01	100.9		87.03	87.03	92.20
500	89.01	89.01	150.9		87.03	87.03	132.2
1000	89.01	89.01	213.4		87.03	87.03	182.2
5000	89.01	89.01	713.4		87.03	87.03	582.2

G H I
D E F
A B C

Fig. 1. The nine-point rectangle.

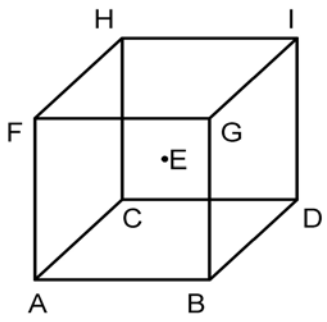

Fig. 2. The nine-point rectangular prism. It is abbreviated by the word cube in the text.

References

1. P. J. Rousseeuw and A. M. Leroy, Robust regression and outlier detection. Wiley-Interscience, John Wiley & Sons, Inc., New Jersey, 2003, p. 169.

2. G. L. Silver, Trigonometric interpolating equation for the eight-point cube. Applied Math. Sci. 4 (2010) 1057-1064.

3. G. L. Silver, Analysis of three-dimensional grids: five- and nine-point cubes. Appl. Math. Comput. 160 (2005) 133-140.

4. G. L. Silver, General Method for exponential-type equations for eight- and nine-point prismatic arrays. Appl. Math. Sci. 3 (2009) 2143-2156.

5. G. L. Silver, Operational method of data treatment. J. Comp. Chem. 6 (1985) 229-236.

6. G. L. Silver, Analysis of three-dimensional grids: cubic equations for nine-point prismatic arrays. Appl. Math. Comput. 170 (2005) 752-760.

7. G. L. Silver, Numerical relations within cubes and the estimation of central tendency. Appl. Math. Sci. 3 (2009) 119-128.

8. G. L. Silver, Operational equations for the five-point rectangle, the geometric mean, and data in prismatic array. Appl. Math. Sci. 3 (2009) 1461-1471.

9. G. L. Silver, Operational measures of central tendency. Appl. Math. Comput. 186 (2007) 1379-1384. The third and fourth columns of Table 5 should read Eq. (6) and Eq. (7), respectively, and the title of Table 6 should read Eqs. (8) and (10), respectively.

10. Waterloo Maple, Inc. http://www.maplesoft.com

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