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M/M/1 Retrial Queueing System with Negative Arrival under Erlang-K Service by Matrix Geometric Method

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Abstract

Consider a single server retrial queueing system in which customers arrive in a Poisson process with arrival rate **λ** and **negative** customers arrive at a rate **ν** which also follows a Poisson process**.** Let K be the number of phases in the service station. The service time has Erlang-K distribution with service rate Kμ for each phase. We assume that the services in all phases are independent and identical and only one customer at a time is in the service mechanism. If the server is **free** at the time of a primary call arrival, the arriving call begins to be served in Phase 1 immediately by the server then progresses through the remaining phases and must complete the last phase and leave the system before the next customer enters the first phase. If the server is **busy**, then the arriving customer goes to orbit and becomes a source of **repeated calls**. This pool of sources of repeated calls may be viewed as a sort of queue. Every such source produces a Poisson process of repeated calls with intensity **σ**. If an incoming repeated call finds the server free, it is served in the same manner and leaves the system after service, while the source which produced this repeated call disappears.We assume that the access from orbit to the service facility is governed by the **classical retrial policy. This model is solved by using Matrix geometric Technique**. Numerical study have been done for Analysis of Mean number of customers in the orbit (MNCO),Truncation level (OCUT),Probability of server free and busy for various values of **λ , μ** , **ν, k** and **σ** in elaborate manne**r and also various particular cases of this model have been discussed.**

Mathematics Subject Classification: 60K25, 65K30

Keywords: Single Server – Stochastic nature – Erlang type service –K phases – negative arrival - Matrix Geometric Method – Orbit – classical retrial policy – stability

1. Introduction

Queueing systems in which arriving customers who find the server busy may retry for service after a period of time is called **Retrial queues [1,2,3,8,9,10]** Because of the complexity of the retrial queueing models, analytic results are generally difficult to obtain. There are a great number of numerical and approximations methods available, in this paper we will place more emphasis on the solutions by **Matrix geometric method [11, 12, 13].**

2. Description of the Queueing System

Consider a single server retrial queueing system in which customers arrive in a Poisson process with arrival rate **λ**. These customers are identified as primary calls**.** Further assume that **negative** customers arrive at a rate **ν** which follows a Poisson process**.** Let k be the number of phases in the service station.Assume that the service time has **Erlang-k distribution [7]** with service rate kμ for each phase. We assume that the services in all phases are independent and identical and only one customer at a time is in the service mechanism. If the server is **free** at the time of a primary call arrival, the arriving call begins to be served in Phase 1 immediately by the server then progresses through the remaining phases and must complete the last phase and leaves the system before the next customer enters the first phase. If the server is **busy**, then the arriving customer goes to orbit and becomes a source of repeated calls. This pool of sources of repeated calls may be viewed as a sort of queue. Every such source produces a Poisson process of repeated calls with intensity **σ**. If an incoming repeated call finds the server free, it is served in the same manner and leaves the system after service, while the source which produced this repeated call disappears. Otherwise, the system state does not change.

2.1 Negative Arrival

Gelenbe (1991) has introduced a new class of queueing processes in which customers are either **Positive or Negative. Positive** means a regular customer who is treated in the usual way by a server. **Negative customers [4, 5, 6, 14, 15]** have the effect of **deleting some customer in the queue.** In the simplest version, a negative arrival removes an ordinary positive customer or a batch of positive customers according to some strategy. It is noted that the existence of a flow of negative arrivals provides a control mechanism to control excessive congestion at the retrial group in tele communication and computer networks. The control mechanism is such that whenever server is busy, an exponential timer is activated. If the timer expires and the server is still busy then at random one of the customers who are stored at the retrial pool is automatically removed. A negative arrival has the effect of removing a random customer from the retrial group. We assume that the negative customers only act when the server is busy.

2.2 Retrial Policy

We assume that the access from the orbit to the service facility follows the exponential distribution with rate $\mathbf{n\sigma}$ which may depend on the current number n, ($n \geq 0$) the number of customers in the orbit. That is, the probability of repeated attempt during the interval (t, $t + \Delta t$), given that there are n customers in the orbit at time t is $\mathbf{n\sigma\ \Delta t}$. It is called the **classical retrial rate policy.** The input flow of primary calls, interval between repetitions and service time in phases are mutually independent.

3. Matrix Geometric Methods

Let N(t) be the random variable which represents the number of customers in orbit at time t and S(t) be the random variable which represents the phase in which customer is getting service at time t.

The random process is described as

$$\{\langle N(t), S(t)\rangle \,/\, N(t)=0,1,2,3\ldots; S(t)=0,1,2,3,\ldots,k\}$$

The value of S(t) = 0 for the server being idle and S(t) = i for server being busy with the customer in the i th phase (i=1,2,3…,k).The possible state space for single server retrial queueing with Erlang-k phases service are

$$\mathbf{\{ (i, j) \,/\, i = 0,1,2,3,\ldots \;;\; j = 0,1,2,3,\ldots,k \}}$$

The infinitesimal generator matrix **Q** for this model is given below

$$Q = \begin{pmatrix} A_{00} & A_0 & O & O & O & \cdots \\ A_{10} & A_{11} & A_0 & O & O & \cdots \\ O & A_{21} & A_{22} & A_0 & O & \cdots \\ O & O & A_{32} & A_{33} & A_0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

The matrices $\mathbf{A_{00}}$, $\mathbf{A_{n\,n-1}}$, $\mathbf{A_{n\,n}}$ and $\mathbf{A_{n\,n+1}}$ are square matrices of order k+1, where

$$\mathbf{A_{00}} = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & \cdots & 0 & 0 \\ 0 & -(\lambda+k\mu) & k\mu & 0 & \cdots & 0 & 0 \\ 0 & 0 & -(\lambda+k\mu) & k\mu & \cdots & 0 & 0 \\ 0 & 0 & 0 & -(\lambda+k\mu) & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & -(\lambda+k\mu) & k\mu \\ k\mu & 0 & 0 & 0 & \cdots & 0 & -(\lambda+k\mu) \end{pmatrix}$$

$$\mathbf{A_{nn-1} = (a_{ij})} \text{ where } \mathbf{a_{ij}} = \begin{cases} n\sigma & \text{if } i= 1, j=2 \\ \nu & \text{if } i = j,\ i=2, 3, 4,\ldots, k+1 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbf{A_{nn}} = \begin{pmatrix} -(\lambda+n\sigma) & \lambda & 0 & 0 & \cdots & 0 & 0 \\ 0 & -(\lambda+k\mu+\nu) & k\mu & 0 & \cdots & 0 & 0 \\ 0 & 0 & -(\lambda+k\mu+\nu) & k\mu & \cdots & 0 & 0 \\ 0 & 0 & 0 & -(\lambda+k\mu+\nu) & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & -(\lambda+k\mu+\nu) & k\mu \\ k\mu & 0 & 0 & 0 & \cdots & 0 & -(\lambda+k\mu+\nu) \end{pmatrix}$$

$$\mathbf{A_{nn+1} = A_0 = (a_{ij})} \text{ where } \mathbf{a_{ij}} = \begin{cases} \lambda & \text{if } i = j,\ i = 2,3,4,\ldots,k+1 \\ 0 & \text{otherwise} \end{cases}$$

If the capacity of the orbit is finite say M, then

$$A_{MM} = \begin{pmatrix} -(\lambda+M\sigma) & \lambda & 0 & 0 & \cdots & 0 & 0 \\ 0 & -(k\mu+\nu) & k\mu & 0 & \cdots & 0 & 0 \\ 0 & 0 & -(k\mu+\nu) & k\mu & \cdots & 0 & 0 \\ 0 & 0 & 0 & -(k\mu+\nu) & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & -(k\mu+\nu) & k\mu \\ k\mu & 0 & 0 & 0 & \cdots & 0 & -(k\mu+\nu) \end{pmatrix}$$

Let **X** be a steady-state probability vector of **Q** and partitioned as
X= (x(0),x(1),x(2), ….) and **X** satisfies

$$\mathbf{XQ = 0}, \ \mathbf{Xe = 1}, \text{where } x(i) = (P_{i0}, P_{i1}, P_{i2}, \ldots, P_{ik}) \qquad (1)$$

4. Direct truncation method

In this method one can truncate the system of equations in (1) for sufficiently large value of the number of customers in the orbit, say M. That is, the orbit size is restricted to M such that any arriving customer finding the orbit full is considered lost. The value of M can be chosen so that the loss probability is very small. Due to the intrinsic nature of the system in (1), the only choice available for studying M is through algorithmic methods. While a number of approaches are available for determining the cut-off point M , The one that seems to perform well (w.r.t approximating the system performance measures) is to increase M until the largest individual change in the elements of **X** for successive values is less than ϵ a predetermined infinitesimal value.

5. Stability condition

Theorem :

The inequality $\left(\frac{\lambda-\nu}{\mu}\right)<1$ is the necessary and sufficient condition for system to be stable.

Proof:

Let Q be an infinitesimal generator matrix for the queueing system (without retrial)

The stationary probability vector **X** satisfies

$$\mathbf{XQ = 0 \quad and \quad Xe=1} \qquad \mathbf{(2)}$$

Let R be the rate matrix and satisfing the equation

$$\mathbf{A_0+RA_1+R^2 A_2 = 0} \qquad \mathbf{(3)}$$

The system is stable if sp(R) <1

We know that the Matrix R satisfies sp(R) <1 if and only if

$$\mathbf{\Pi A_0 e < \Pi A_2 e} \quad (4)$$

where $\mathbf{\Pi} = (\pi_1,\ldots,\pi_k)$ and satisfies

$$\mathbf{\Pi A = 0 \text{ and } \Pi e = 1} \quad (5)$$

and

$$\mathbf{A = A_0 + A_1 + A_2} \quad (6)$$

Here $\mathbf{A_0}$, $\mathbf{A_1}$ **and** $\mathbf{A_2}$ are square matrices of order **k** and

$\mathbf{A_0} = \boldsymbol{\lambda}\,\mathbf{I}$, where **I** is the identity matrix of order k.

$$\mathbf{A_1} = \begin{pmatrix} -(\lambda+k\mu+\nu) & k\mu & 0 & \ldots & 0 & 0 \\ 0 & -(\lambda+k\mu+\nu) & k\mu & \ldots & 0 & 0 \\ 0 & 0 & -(\lambda+k\mu+\nu) & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & -(\lambda+k\mu+\nu) & k\mu \\ 0 & 0 & 0 & \ldots & 0 & -(\lambda+k\mu+\nu) \end{pmatrix}$$

$$\mathbf{A_2 = (a_{ij})} \text{ where } \mathbf{a_{ij}} = \begin{cases} k\mu & \text{for } i=1,\ j=k \\ \nu & \text{for } i=j \text{ and } i=1,2,3,\ldots,k \\ 0 & \text{otherwise} \end{cases}$$

By substituting $\mathbf{A_0}$, $\mathbf{A_1}$, $\mathbf{A_2}$ in equations (4) ,(5) and (6) ,we get

$$\left(\frac{\lambda-\nu}{\mu}\right)<1$$

The inequality $\left(\frac{\lambda-\nu}{\mu}\right)<1$ is also a sufficient condition for the retrial queueing system to be stable. Let Q_n be the number of customers in the orbit after the departure of n^{th} customer from the service station. We first prove the embedded Markov chain $\{\mathbf{Q_n}, \mathbf{n\geq 0}\}$ is ergodic if $\left(\frac{\lambda-\nu}{\mu}\right)<1$. $\{\mathbf{Q_n}, \mathbf{n\geq 0}\}$ is irreducible and aperiodic. It remains to be proved that $\{\mathbf{Q_n}, \mathbf{n\geq 0}\}$ is positive recurrent. The irreducible and aperiodic Markov chain $\{\mathbf{Q_n}, \mathbf{n\geq 0}\}$ is positive recurrent if $|\psi_m| < \infty$ for all m and $\lim_{m\to\infty} \sup \psi_m < 0$, where

$$\psi_m = E\left((Q_{n+1} - Q_n) / Q_n = m \right) \qquad (m=0,1,2,3,4,5,\ldots)$$

$$\psi_m = \left(\frac{\lambda-\nu}{\mu}\right) - \left(\frac{\lambda}{\lambda+m\sigma}\right)$$

If $\left(\frac{\lambda-\nu}{\mu}\right)<1$, then $|\psi_m|<\infty$ for all **m** and $\lim_{m\to\infty}\sup\ \psi_m<0$

Therefore the embedded Markov chain $\{Q_n, n\geq 0\}$ is ergodic.

6. Analysis of steady state probabilities

In this paper we are applying the **Direct Truncation Method** to find the Steady state probability vector **X**. Let M denote the cut-off point for this truncation method.The steady state probability vector $X^{(M)}$ is now partitioned as $X^{(M)} = (x(0), x(1), x(2), \ldots x(M))$ which satisfies $X^{(M)} Q = 0$, $X^{(M)} e = 1$, where $x(i) = (P_{i0}, P_{i1}, P_{i2}, \ldots, P_{ik})$ i =0, 1, 2,3,…,M .

The above system of equations is solved by exploiting the special structure of the co-efficient matrix. It is solved by GAUSS-JORDAN elementary transformation method. Since there is no clear cut choice for M, we may start the iterative process by taking, say M=1 and increase it until the individual elements of **x** do not change significantly. That is, if M^* denotes the truncation point then

$$\| x^{M^*}(i) - x^{M^*-1}(i) \|_\infty < \epsilon \quad \text{,where } \epsilon \text{ is an infinitesimal quantity.}$$

7. Special cases

1. As $\nu\to 0$, the above model reduces to Single server retrial queueing system with erlang-k type service.
2. If K=1 and $\nu\to 0$, this model becomes the Single server retrial queueing model and our numerical results coincide with the following closed form of Number of customers in the orbit in the steady state **[9]**

   Mean Number of Customers in the orbit = $\frac{\rho(\lambda+\rho\sigma)}{(1-\rho)\sigma}$
3. As $\sigma\to\infty$ and $\nu\to 0$,the closed form of number of customers in the orbit tends to length of the queue in standard queueing system with Erlang type service

   $$L_q = \left(\frac{k+1}{2k}\right)\left(\frac{\rho^2}{1-\rho}\right)$$

   For many values of λ, μ, K and very high values of σ (>10000), the above result coincides with our numerical results.

8. Systems performance measures

In this section some important performance measures along with formulas and their qualitative behaviour for various values of λ, μ, k , ν and σ are studied. Numerical study has been dealt in very large scale to study these measures. Defining

P(n , 0) = Probability that there are n customers in the orbit and server is free

P(n , i) = Probability that there are n customers in the orbit and server is busy with customer in the i^{th} phase

1. **The probability mass function of Server state**

Let S(t) be the random variable which represents the phase in which customer is getting service at time t .

S :	**0**	**1**	**2**	**3**	**…**	**k**
P :	$\sum_{i=0}^{\infty} p(i,0)$	$\sum_{i=0}^{\infty} p(i,1)$	$\sum_{i=0}^{\infty} p(i,2)$	$\sum_{i=0}^{\infty} p(i,3)$	…	$\sum_{i=0}^{\infty} p(i,k)$

2. **The Mean number of busy servers**

$$\text{MNBS} = \sum_{i=0}^{\infty}\sum_{j=1}^{k} p(i,j)$$

3. **The probability mass function number of customers in the orbit**

Let X(t) be the random variable representing the number of customers in the orbit.

$$\text{Prob (No customers in the orbit)} = \sum_{j=0}^{k} p(0,j)$$

$$\text{Prob (i customers in the orbit)} = \sum_{j=0}^{k} p(i,j)$$

4. **The Mean number of customers in the orbit**

$$\text{Mnco} = \sum_{i=0}^{\infty} i\left(\sum_{j=0}^{k} p(i,j)\right)$$

5. **The probability that the orbiting customer is blocked**

$$\text{Blocking Probability} = \sum_{i=1}^{\infty}\sum_{j=1}^{k} p(i,j)$$

6. **The probability that an arriving customer enter into service immediately**

$$\text{PSI} = \sum_{i=0}^{\infty} p(i,0)$$

9. Numerical Study

The Numerical study is done subject to the condition that the parameters **λ, μ, ν** satisfy the stability condition $\left(\frac{\lambda-\nu}{\mu}\right)<1$

From the following tables we conclude that

- Mean number of cutomers in the orbit decreases as **σ** increases.
- Mean number of cutomers in the orbit decreases as **ν** increases
- As the number of phases **K** increases, Mean number of customers in the orbit decreases

Table 1: Mean number of customers in the orbit for λ =6, μ=10, K= 5, ν = 2 and various values of σ

Sigma	O_cut	Mnco	P0	P1
10	16	0.768	0.4857	0.5143
20	15	0.5914	0.4701	0.5299
30	15	0.5236	0.4633	0.5367
40	15	0.4875	0.4594	0.5406
50	15	0.4649	0.457	0.543
60	15	0.4495	0.4553	0.5447
70	14	0.4383	0.454	0.546
80	14	0.4298	0.4531	0.5469
90	14	0.4231	0.4523	0.5477
100	14	0.4177	0.4517	0.5483
200	14	0.3929	0.4488	0.5512
300	14	0.3844	0.4478	0.5522
400	14	0.3801	0.4473	0.5527
500	14	0.3776	0.447	0.553
600	14	0.3758	0.4468	0.5532
700	14	0.3746	0.4466	0.5534
800	14	0.3737	0.4465	0.5535
900	14	0.373	0.4464	0.5536
1000	14	0.3724	0.4464	0.5536
2000	14	0.3698	0.446	0.554
3000	14	0.3689	0.4459	0.5541
4000	14	0.3685	0.4459	0.5541
5000	14	0.3682	0.4459	0.5541
6000	14	0.3681	0.4458	0.5542
7000	14	0.3679	0.4458	0.5542
8000	14	0.3678	0.4458	0.5542
9000	14	0.3678	0.4458	0.5542

Table 2: Mean number of customers in the orbit for λ =5, μ=10, ν = 2, σ=100 and various values of K.

K	σ	Ocut	Mnco	P_0	P_1
5	100	11	0.2529	0.5354	0.4646
6	100	11	0.2483	0.5351	0.4649
7	100	11	0.2449	0.5348	0.4652
8	100	11	0.2424	0.5346	0.4654
9	100	11	0.2405	0.5345	0.4655
10	100	11	0.2389	0.5344	0.4656
11	100	11	0.2376	0.5343	0.4657
12	100	11	0.2365	0.5342	0.4658
13	100	11	0.2356	0.5341	0.4659
14	100	11	0.2348	0.5341	0.4659
15	100	11	0.2342	0.534	0.466
16	100	11	0.2336	0.534	0.466
17	100	11	0.2331	0.5339	0.4661
18	100	11	0.2326	0.5339	0.4661
19	100	11	0.2322	0.5338	0.4662
20	100	11	0.2318	0.5338	0.4662
21	100	11	0.2314	0.5338	0.4662
22	100	11	0.2311	0.5338	0.4662
23	100	11	0.2309	0.5337	0.4663
24	100	11	0.2306	0.5337	0.4663
25	100	11	0.2304	0.5337	0.4663

Table 3: Mean number of customers in the orbit for λ =5, μ=10, σ=100, K=5 and various values of ν.

ν	σ	Ocut	Mnco	P_0	P_1
0.2	100	13	0.3370	0.5044	0.4956
0.4	100	13	0.3249	0.5085	0.4915
0.6	100	13	0.3137	0.5125	0.4875
0.8	100	13	0.3032	0.5162	0.4838
1.0	100	12	0.2935	0.5198	0.4802
1.2	100	12	0.2843	0.5232	0.4768
1.4	100	12	0.2758	0.5265	0.4735
1.6	100	12	0.2677	0.5296	0.4704
1.8	100	12	0.2601	0.5326	0.4674
2.0	100	11	0.2529	0.5354	0.4646
2.2	100	11	0.2462	0.5382	0.4618
2.4	100	11	0.2397	0.5408	0.4592
2.6	100	11	0.2337	0.5434	0.4566
2.8	100	11	0.2279	0.5458	0.4542
3.0	100	11	0.2224	0.5481	0.4519
3.2	100	11	0.2172	0.5504	0.4496
3.4	100	11	0.2122	0.5526	0.4474
3.6	100	10	0.2074	0.5547	0.4453
3.8	100	10	0.2029	0.5567	0.4433
4.0	100	10	0.1985	0.5587	0.4413

10. Graphical Study

Figure 1. Mean Number of customers in the orbit for λ = 6 μ =10 ν = 2 K=5 various values of σ.

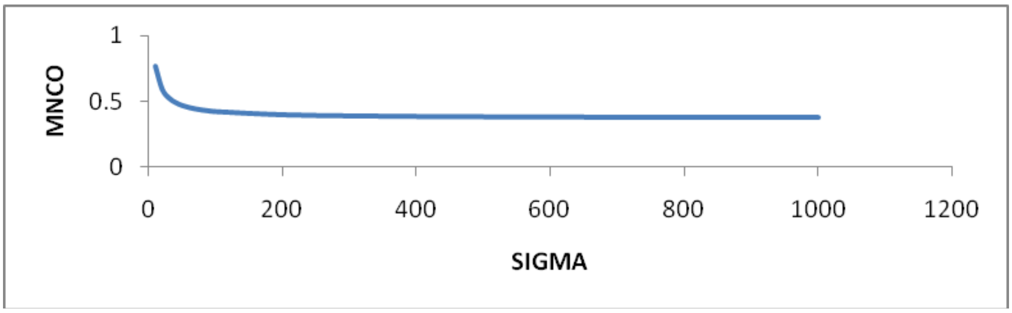


Figure 2. Mean Number of customers in the orbit for λ = 5 μ =10 ν = 2 σ =100 and various values of K.

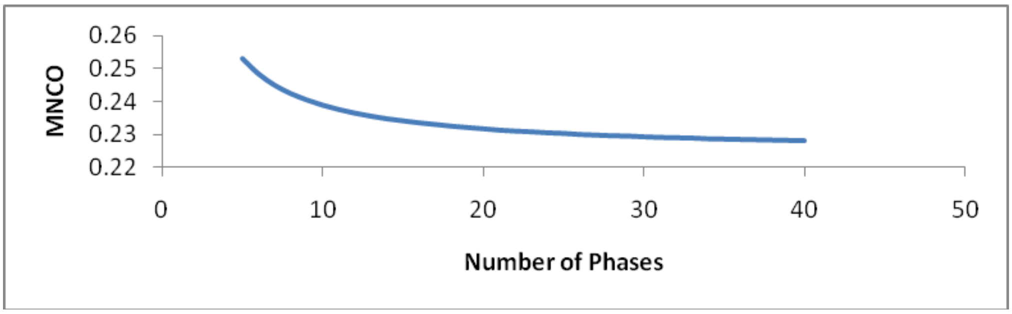
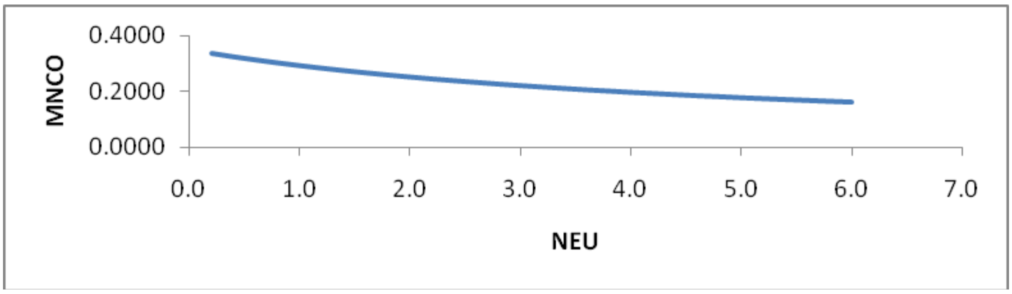


Figure 3. Mean Number of customers in the orbit for λ = 5 μ =10 K = 5 σ =100 and various values of ν.



11. Conclusion

It is observed from numerical and graphical studies that Mean number of customers in the orbit decreases as the retrial rate increases, the probabilities for the server being idle, busy are dependent over retrial rate. The various special cases discussed in section 7 are particular cases of this research work. This research work can further be extended by introducing various parameters like vacation policies, second optional services etc.,

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