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Common Fixed Point Theorems

with Integral Inequality

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Abstract

The aim of this paper is to obtain fixed point theorems for hybrid pairs of single valued and multivalued mappings satisfying a contractive condition of integral type in general settings. Several well known recent results are also obtained as special cases.

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1. INTRODUCTION

After the celebrated Banach contraction principle (which ensures the existence of unique fixed point of a contraction map on a complete metric space) in 1922, there have been numerous results in the literature, dealing with mappings satisfying the contractive conditions of various types including even nonlinear expressions. The study on fixed point theorems involving four single-valued maps started with the assumption of commutativity of all the maps. Jungck [5] obtained common fixed point theorems for such type of mappings. Sessa [12] weakened this condition of commutativity to weakly commuting mappings. Further, Jungck [6] introduced a more general concept than that of weak commutativity, called compatibility and it is generalized to weak compatibility by Jungck and Rhoades [7]. Recently, Aamri and El Moutawakil [1] defined a property (*E.A*) for self maps which further extended by Kamran [8] for hybrid maps. Branciari [3] obtained a fixed point theorem for single valued maps satisfying an analogue of Banach contraction principle for integral type inequality. This result was further generalized by many authors, see for instance ([2], [3], [11] and references thereof). More recently, Liu et al. [9] have obtained common fixed point theorems under hybrid contractive condition for the maps satisfying a new property, more general than that of (*E.A*) property. Motivated by their result we obtain common fixed point theorems for a hybrid pair of single and multivalued maps satisfying an integral type contractive condition in the settings of *b*-metric spaces.

2. PRELIMINARIES

In this section we have introduced some notations and definitions required for our results. Throughout this paper we have considered (X, d) to be any *b*-metric space and for any $x \in X$ and $A \subset X, d(x, A) = \inf\{d(x, y), y \in A\}$. Let $CB(X)$ be the class of all nonempty bounded closed subsets of X. Then Hausdorff metric H with respect to d is defined as

$$H(A, B) = \max\left\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\right\}, \text{ for every } A, B \in CB(X).$$

Definition 2.1 [4]. Let X be a set and $s \geq 1$ be a given real number. A function $d : X \times X \to R_+$ is said to be a *b*-metric iff for all $x, y, z \in X$ the following conditions are satisfied:

(i) $d(x, y) = 0$ *iff* $x = y$,
(ii) $d(x, y) = d(y, x)$,
(iii) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

A pair (X, d) is called a *b*-metric space.

The class of *b*-metric spaces is effectively larger than that of metric spaces, since a *b*-metric space is a metric space when $s = 1$ in the above condition (iii). The following example shows that a *b*-metric on *X* need not be a metric on *X*.

Example 2.1 [10, 13]. Let $X=\{x_1, x_2, x_3, x_4\}$ and $d(x_1, x_2)=k\geq 2$, $d(x_1, x_3)=d(x_1, x_4)=d(x_2, x_3)=d(x_2, x_4)=d(x_3, x_4)=1,\ d(x_i, x_j)=d(x_j, x_i)$ for all $i,\ j=1, 2, 3, 4$ and $d(x_i, x_i)=0,\ i=1, 2, 3, 4.$

Then $d(x_i, x_j)\leq \frac{k}{2}\left[d(x_i, x_n)+d(x_n, x_j)\right]$ for $n, i, j=1, 2, 3, 4$ and if $k>2,$ the ordinary triangle inequality does not hold.

Definition 2.2 [7]. Maps $f: X\to X$ and $F: X\to CB(X)$ are weakly compatible if they commute at their coincidence points, that is, if $fFx=Ffx$ whenever $fx\in Fx$.

Definition 2.3 [1]. Maps $f, g: X\to X$ are said to satisfy the property (*E.A*) if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n=\lim_{n\to\infty} gx_n=t\in X$.

Definition 2.4 [8]. Maps $f: X\to X$ and $F: X\to CB(X)$ are said to satisfy the property (*E.A*) if there exist a sequence $\{x_n\}$ in X, some $t\in X$ and $A\in CB(X)$ such that $\lim_{n\to\infty} fx_n=t\in A=\lim_{n\to\infty} Fx_n$.

Definition 2.5 [8]. Let $F: X\to CB(X)$. The map $f: X\to X$ is said to be *F*-weakly commuting at $x\in X$ if $ffx\in Ffx$.

Definition 2.6 [9]. Let $f, g: X\to X$ and $F, G: X\to CB(X)$. The pairs (f, F) and (g, G) are said to satisfy the common property (*E.A*) if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X , some $t\in X$ and $A, B\in CB(X)$ such that

$$\lim_{n\to\infty} Fx_n=A, \lim_{n\to\infty} Gy_n=B, \lim_{n\to\infty} fx_n=\lim_{n\to\infty} gy_n=t\in A\cap B.$$

3. MAIN RESULTS

Theorem 3.1. Let (X, d) be a complete *b*-metric space and $f, g: X\to X$ and $F, G: X\to CB(X)$ such that

(i) $FX\subseteq gX$, $GX\subseteq fX$;

(ii) The pairs (F, f) and (G, g) satisfy the common property (*E.A*);

(iii) for all $x, y \in X$,

$$\int_0^{H(Fx,\, Gy)} \phi(t)\,dt \le q\left(\int_0^{M(x,\, y)} \phi(t)\,dt\right) \tag{3.1}$$

Where $\phi : \Re^+ \to \Re^+$ is a Lebesgue integrable mapping which is summable, non-negative and such that

$$\int_0^{\varepsilon} \phi(t)\,dt > 0, \text{ for each } \varepsilon > 0 \tag{3.2}$$

and

$$M(x, y) = \max\{d(fx,\, gy), d(Fx,\, fx), d(Gy,\, gy), [d(Fx,\, gy) + d(Gy,\, fx)]/2\} \tag{3.3}$$

with $qs < 1$, $\lambda s < 1$, where $\lambda = \max\{q, \dfrac{qs}{2 - qs}\}$.

If fX and gX are closed subspace of X, then

(1) f and F have a coincidence point;

(2) g and G have a coincidence point;

(3) f and F have a common fixed point provided that f is F-weakly commuting at u and $ffu = fu$ for $u \in C(f, F)$.

(4) g and G have a common fixed point provided that g is G-weakly commuting at v and $ggv = gv$ for $v \in C(g, G)$.

(5) f, g, F and G have a common fixed point provided (3) and (4) are true.

Proof. Let $x_0 \in X$. From (i) we can construct a sequence $\{y_n\}$ in X such that

$$y_{2n+1} = fx_{2n+1} \in Gx_{2n}$$
$$y_{2n+2} = gx_{2n+2} \in Fx_{2n+1} \text{ for all } n \ge 0.$$

It follows from equation (3.1) that

$$\int_0^{H(y_{2n+2},\, y_{2n+3})} \phi(t)\,dt = \int_0^{H(Gx_{2n+1},\, Fx_{2n+2})} \phi(t)\,dt \le q\left(\int_0^{M(x_{2n+1},\, x_{2n+2})} \phi(t)\,dt\right)$$

where,

$$\begin{aligned} M(x_{2n+1,}\, x_{2n+2}) &= \max\{d(fx_{2n+1},\, gx_{2n+2}), d(Fx_{2n+1},\, fx_{2n+1}), d(Gx_{2n+2},\, gx_{2n+2}), \\ &\qquad [d(Fx_{2n+1},\, gx_{2n+2}) + d(Gx_{2n+2},\, fx_{2n+1})]/2\} \\ &= \max\{d(y_{2n+1},\, y_{2n+2}), d(y_{2n+2},\, y_{2n+3}), d(y_{2n+1},\, y_{2n+3})/2\} \end{aligned}$$

Thus

$$\int_0^{H(y_{2n+2},\, y_{2n+3})} \phi(t)\, dt \le q\left(\int_0^{\max\{d(y_{2n+1},\, y_{2n+2}),\, d(y_{2n+2},\, y_{2n+3}),\, d(y_{2n+1},\, y_{2n+3})/2\}} \phi(t)\, dt \right)$$

$$\le \lambda \left(\int_0^{H(y_{2n+1},\, y_{2n+2})} \phi(t)\, dt \right).$$

Similarly,

$$\int_0^{H(y_{2n+1},\, y_{2n+2})} \phi(t)\, dt \le \lambda \left(\int_0^{H(y_{2n},\, y_{2n+1})} \phi(t)\, dt \right), \qquad \text{where } \lambda = \max\{q,\, \frac{qs}{2-qs}\}.$$

Thus, we have proved that for all $n \ge 0$

$$\int_0^{H(y_{n+1},\, y_{n+2})} \phi(t)\, dt \le \lambda \left(\int_0^{H(y_n,\, y_{n+1})} \phi(t)\, dt \right) \le \lambda^n \left(\int_0^{H(y_0,\, y_1)} \phi(t)\, dt \right).$$

Hence for all $m \ge n \ge 0,$ noting $\lambda = \max\{q,\, \frac{qs}{2-qs}\}$, a constant

$$\int_0^{H(y_m,\, y_n)} \phi(t)\, dt \le \sum_{i=n}^{m-1} \int_0^{H(y_i,\, y_{i+1})} \phi(t)\, dt \le \sum_{i=n}^{m-1} \lambda^i \left(\int_0^{H(y_0,\, y_1)} \phi(t)\, dt \right) \le \frac{\lambda^n}{1-\lambda} \left(\int_0^{H(y_0,\, y_1)} \phi(t)\, dt \right).$$

Then

$$\lim_{m,\, n \to \infty} \int_0^{H(y_m,\, y_n)} \phi(t)\, dt = 0,$$ i.e., $\{y_n\}$ is a Cauchy sequence.

Since $\{y_n\}$ is a Cauchy sequence, there exist a z satisfying

$$\lim_{n\to\infty} y_n = z = \lim_{n\to\infty} fx_{2n+1} = \lim_{n\to\infty} gx_{2n+2}.$$

Since fX and gX are closed, there exist u and v such that $fu = z = gv$. A similar argument proves that

$$\lim_{n\to\infty} Fx_{2n+1} = \lim_{n\to\infty} Gx_{2n+2} \quad \text{and} \quad z \in \lim_{n\to\infty} Fx_{2n+1} = \lim_{n\to\infty} Gx_{2n+2}.$$

If $\lim_{n\to\infty} Fx_{2n+1} = A$ and $\lim_{n\to\infty} Gx_{2n+2} = B$, then $z \in A \cap B$. Thus $(F,\, f)$ and $(g,\, G)$ satisfy common property $(E.A)$

We claim that $fu \in Fu$. To prove it, taking $x = u$ and $y = x_{2n+1}$ in (3.1),

$$\int_0^{H(Fu,\, Gx_{2n+1})} \phi(t)\, dt \le q\left(\int_0^{M(u,\, x_{2n+1})} \phi(t)\, dt \right),$$

where

$$M(u,\ x_{2n+1}) = \max\{d(\ fu,\ gx_{2n+1}), d(Fu,\ fu), d(Gx_{2n+1},\ gx_{2n+1}),$$
$$[d(Fu,\ gx_{2n+1}) + d(Gx_{2n+1},\ fu)]/2\} \quad .$$

Since $fx_{2n+1} \in Gx_{2n}$, so,

$$d(Gx_{2n+1},\ gx_{2n+1}) \le d(\ fx_{2n+2},\ gx_{2n+1}),$$
$$d(Gx_{2n+1},\ fu) \le d(\ fx_{2n+2},\ fu) \quad H(Fu,\ Gx_{2n+1}) \le H(Fu,\ fx_{2n+2})$$

and

$$M(u, x_{2n+1}) = \max\{d(\ fu,\ gx_{2n+1}), d(Fu,\ fu), d(\ fx_{2n+2},\ gx_{2n+1}),$$
$$[d(Fu,\ gx_{2n+1}) + d(\ fx_{2n+2},\ fu)]/2\}.$$

Taking the limit as $n \to \infty$, we obtain

$$\max\{d(Fu, z), d(Fu, z)/2\} = d(Fu, z).$$

So, we have $H(Fu, z) \le d(Fu, z)$. Since $fu = gv \in A$, it follows from the definition of Hausdorff metric that

$$d(Fu, z) \le H(Fu, z) \le d(Fu, z), \text{ where } fu = z.$$

We may conclude that, $\displaystyle\int_0^r \phi(t)\,dt \le q\left(\int_0^r \phi(t)\,dt\right).$

This is a contradiction. Hence from (3.2), $fu \in Fu$ and result (1) is proved.

Similarly, we claim that $gv \in Gv$. Putting $x = x_{n+1}$ and $y = v$ in (3.1) and taking the limit, it can be easily verified that $gv \in Gv$, i.e. result (2) is true.

Thus f and F have a coincidence point u, g and G have a coincidence point v. This ends the proofs of part (1) and (2).

Furthermore, by virtue of condition (3), we obtain $ffu = fu$ and $ffu \in Ffu$. Thus $u = fu \in Fu$. This proves (3). A similar argument proves (4). Then (5) holds immediately.

Corollary 3.1. [9]. Let $(X,\ d)$ be a complete metric space and $f,\ g : X \to X$ and $F,\ G : X \to CB(X)$ such that

(i) $FX \subseteq gX$, $GX \subseteq fX$;

(ii) The pairs $(F,\ f)$ and $(G,\ g)$ satisfy the common property ($E.A$)

Let $\lambda \in (0, 1)$ be a constant, such that for all $x \ne y$ in X,

$H(Fx, Gy) \le q \max\{d(fx, gy), d(Fx, fx), d(Gy, gy), [d(Fx, gy) + d(Gy, fx)]/2\}$

If fX and gX are closed subspace of X, then

(1) f and F have a coincidence point;

(2) g and G have a coincidence point;

(3) f and F have a common fixed point provided that f is F-weakly commuting at u and $ffu = fu$ for $u \in C(f, F)$.

(4) g and G have a common fixed point provided that g is G-weakly commuting at v and $ggv = gv$ for $v \in C(g, G)$.

(5) f, g, F and G have a common fixed point provided (3) and (4) are true.

Proof. The proof follows by putting $\phi(t) = 1$ and $s = 1$ in theorem 3.1.

Theorem 3.2. Let (X, d) be a complete b-metric space and $f, g : X \to X$ and $F, G : X \to CB(X)$ such that

(i) $FX \subseteq gX$, $GX \subseteq fX$;

(ii) The pairs (F, f) and (G, g) satisfy the common property (E.A.);

(iii) for all $x, y \in X$,

$\int_0^{H(Fx, Gy)} \phi(t)\,dt \le q\left(\int_0^{M(x, y)} \phi(t)\,dt\right)$, where $\phi : \Re^+ \to \Re^+$ is a Lebesgue integrable mapping which is summable, non-negative and such that $\int_0^{\varepsilon} \phi(t)\,dt > 0$, for each $\varepsilon > 0$ and

$$M(x, y) = \alpha\, d(fx, gy) + \beta \max\{d(Fx, fx), d(Gy, gy)\} + \gamma \max\{d(Fx, gy) + d(Gy, fx), d(Fx, fx) + d(Gy, gy)\} \tag{3.4}$$

with $\alpha + \beta + 2\gamma < 1$, $qs < 1$, $\lambda s < 1$, where $\lambda = \max\{q, \frac{qs}{2 - qs}\}$.

If fX and gX are closed subspace of X, then

(1) f and F have a coincidence point;

(2) g and G have a coincidence point;

(3) f and F have a common fixed point provided that f is F-weakly commuting at u and $ffu = fu$ for $u \in C(f, F)$.

(4) g and G have a common fixed point provided that g is G-weakly commuting at v and $ggv = gv$ for $v \in C(g, G)$.

(5) f, g, F and G have a common fixed point provided (3) and (4) are true.

Proof. Let $q = \alpha + \beta + 2\gamma < 1$. Following (3.4) and
$\max\{d(Fx, fx), d(Gy, gy)\} \geq (d(Fx, fx) + d(Gy, gy))/2$, it is easy to see that
$H(Fx, Gy) \leq q \max\{d(fx, gy), d(Fx, fx), d(Gy, gy), [d(Fx, gy) + d(Gy, fx)]/2\}$
Thus by theorem 3.1, we arrive to the conclusion in theorem 3.2.

Corollary 3.2. [9] Let (X, d) be a complete metric space and $f, g : X \to X$ and $F, G : X \to CB(X)$ such that

(i) $FX \subseteq gX$, $GX \subseteq fX$;

(ii) The pairs (F, f) and (G, g) satisfy the common property $(E.A)$

Let $\lambda \in (0, 1)$ be a constant, such that for all $x \neq y$ in X,

$$H(Fx, Gy) \leq \alpha\, d(fx, gy) + \beta \max\{d(Fx, fx), d(Gy, gy)\} + \gamma \max\{d(Fx, gy) + d(Gy, fx), d(Fx, fx) + d(Gy, gy)\}\}$$

and $\alpha + \beta + 2\gamma < 1$. If fX and gX are closed subspace of X, then

(1) f and F have a coincidence point;

(2) g and G have a coincidence point;

(3) f and F have a common fixed point provided that f is F-weakly commuting at u and $ffu = fu$ for $u \in C(f, F)$.

(4) g and G have a common fixed point provided that g is G-weakly commuting at v and $ggv = gv$ for $v \in C(g, G)$.

(5) f, g, F and G have a common fixed point provided (3) and (4) are true.

Proof. Letting $\phi(t) = 1$ and $s = 1$ in theorem 3.2, we get the result.

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