

M/M/1 Retrial Queueing System with Loss and Feedback under Non-pre-emptive Priority Service by Matrix Geometric Method

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Abstract

Consider a single server retrial queueing system with loss and feedback under **Non-pre-emptive priority** service in which two types of customers arrive in a Poisson process with arrival rate λ_1 for low priority customers and λ_2 for high priority customers. These customers are identified as primary calls. The service times follow an exponential distribution with parameters μ_1 and μ_2 for both types of customers respectively. The retrial, loss and feedback are introduced for low priority customers only. Let k be the maximum number of waiting spaces for high priority customers in front of the service station. The high priorities customers will be governed by the **Non-Pre-emptive priority service**. The access from orbit to the service facility is governed by the **classical retrial policy**. This model is solved by using Matrix geometric Technique. Numerical study have been done for Analysis of Mean number of low priority customers in the orbit (MNCO), Mean number of high priority customers in the queue (MPQL), Truncation level (OCUT), Probability of server free and Probabilities of server busy with low, high

priority customers for various values of λ_1 , λ_2 , μ_1 , μ_2 , p , q , σ and k in elaborate manner and also various particular cases of this model have been discussed.

Mathematics Subject Classification: 60K25, 65K30

Keywords: Single Server – Stochastic nature – Low priority – high priority – Non-Pre-emptive priority service – loss-feedback - Matrix Geometric Method – Orbit – classical retrial policy

1. INTRODUCTION

Queueing systems in which arriving customers who find all servers and waiting positions (if any) occupied may retry for service after a period of time are called **Retrial queues** [1, 2, 7]. Because of the complexity of the retrial queueing models, analytic results are generally difficult to obtain. There are a great number of numerical and approximations methods are available, in this paper we will place more emphasis on the solutions by **Matrix geometric method** [9, 11, 13].

2. DESCRIPTION OF QUEUEING SYSTEM

Consider a single server retrial queueing system with loss and feedback under **Non-pre-emptive priority** [3, 5, 6] service in which two types of customers arrive in a Poisson process with arrival rate λ_1 for low priority customers and λ_2 for high priority customers. These customers are identified as primary calls. The service times follow an exponential distribution with parameters μ_1 and μ_2 for both types of customers. The retrial, loss and feedback are introduced for low priority customers only. This concept is recently (2009) discussed by K.Farahmand and T.Li [8] for single server retrial queueing by analytic method. Let k be the maximum number of waiting spaces for high priority customers in front of the service station.

2.1 Description of loss and feedback

The concepts loss and feedback are introduced for low priority customers in this paper. If the server is free at the time of the arrival of low priority customer, then the arriving call begins to be served immediately by the server. After completion of service, if the low priority customer dissatisfied then he may re-join the orbit with probability q and with probability $(1-q)$ he leaves the system. This is called **feedback** [4, 10, 12] in queueing theory. If the server is busy at the time of the arrival of low priority customer, then due to impatient this low priority customer may or may not join the orbit. This is called **loss** [8] in queueing theory. We assume that p is the probability that the low priority customer joins the orbit and $(1-p)$ is the probability that he leaves the system without getting service (due to impatient).

If the server is free at the time of the arrival of high priority customer, then the arriving call begins to be served immediately by the server and high priority customer leaves the system after service completion. If the server is **busy** then the low priority arriving customer goes to orbit with probability p and becomes a source of repeated calls. The pool of sources of repeated calls may be

viewed as a sort of queue. Every such source produces a Poisson process of repeated calls with intensity σ . If an incoming repeated call (low) finds the server **free**, it is served and leaves the system after service, while the source which produced this repeated call disappears.

If any one of the waiting spaces is occupied by the high priority customers then the low priority customers (as a primary call) can not enter into service station and goes to orbit with probability p . If the server is busy and there are some waiting spaces then the high priority customer can enter into the service station and waits for his service. If there are no waiting spaces then the high priority customers can not enter into the service station and will be **lost** for the system. Otherwise, the system state does not change. If the server is engaging with low priority customer and at that time the higher priority customer comes then the high priority customer will get service only after completion of the service of low priority customer who is in service. This type of priority service is called the **Non-pre-emptive priority service**. This kind of priority service is followed in this paper.

2.2 Retrial Policy

Most of the queueing system with repeated attempts assume that each customer in the retrial group seeks service independently of each other after a random time exponentially distributed with rate σ so that the probability of repeated attempt during the interval $(t, t + \Delta t)$ given that there were n customers in orbit at time t is $n\sigma \Delta t + O(\Delta t)$. This discipline for access for the server from the retrial group is called **classical retrial rate policy**. The input flow of primary calls (low and high), interval between repetitions, service times, interval between returns from vacation are mutually independent.

3. MATRIX GEOMETRIC METHODS

Let $N(t)$ be the random variable which represents the number of low priority customers in the orbit at time t and $P(t)$ be the random variable which represents the number of high priority customers in the queue (in front of the service station) at time t and $S(t)$ represents the server state at time t . The random process is described as $\{ \langle N(t), P(t), S(t) \rangle / N(t)=0,1,2,3,4,\dots; P(t)=0,1,2,3,\dots,k; S(t)=0,1,2 \}$. $S(t)$ takes the values 0, 1, or 2 depends on the server idle, busy with low priority customer, busy with high priority customer at time t respectively.

The possible state spaces are

$$\{ (u,v,w) / u = 0,1,2,3,\dots ; v = 0; w=0,1,2 \} \cup \{ (u,v,w) / u = 0,1,2,3,\dots ; v=1,2,3,\dots,k; w=1,2 \}$$

The infinitesimal generator matrix Q is given below

$$Q = \begin{pmatrix} A_{00} & A_0 & O & O & O & \dots \\ A_{10} & A_{11} & A_0 & O & O & \dots \\ O & A_{21} & A_{22} & A_0 & O & \dots \\ O & O & A_{32} & A_{33} & A_0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

$A_{00}, A_{01}, A_{n-1}, A_{nn}, A_{n+1}$ are square matrices of order $2k+3$

Notations:

$$\begin{aligned} \#_1 &= -(\lambda_1 + \lambda_2) & \#_2 &= -(p\lambda_1 + \lambda_2 + (1-q)\mu_1) & \#_3 &= -(p\lambda_1 + \lambda_2 + \mu_2) & \#_5 &= -(p\lambda_1 + \mu_1) \\ \#_6 &= -(p\lambda_1 + \mu_2) & \#_8 &= -(n\sigma + \lambda_1 + \lambda_2) & \#_9 &= -(M\sigma + \lambda_1 + \lambda_2) & \#_{10} &= -(\lambda_2 + \mu_1) \\ \#_{11} &= -(\lambda_2 + \mu_2) & \#_{12} &= -(p\lambda_1 + \lambda_2 + \mu_1) \end{aligned}$$

$$A_{00} = \begin{pmatrix} \#_1 & \lambda_1 & \lambda_2 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ (1-q)\mu_1 & \#_{12} & 0 & \lambda_2 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ \mu_2 & 0 & \#_3 & 0 & \lambda_2 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & (1-q)\mu_1 & \#_{12} & 0 & \lambda_2 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu_2 & 0 & \#_3 & 0 & \lambda_2 & \dots & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 & 0 & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & \#_{12} & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & \#_3 & 0 & \lambda_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & (1-q)\mu_1 & \#_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & \mu_2 & 0 & \#_6 \end{pmatrix}$$

$A_{nn-1} = (a_{ij})$ where $a_{ij} = 0$ for all i and j except $i = 1$ and $j = 2$
 $a_{12} = n\sigma$

$$A_{nn} = \begin{pmatrix} \#_8 & \lambda_1 & \lambda_2 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ (1-q)\mu_1 & \#_{12} & 0 & \lambda_2 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ \mu_2 & 0 & \#_3 & 0 & \lambda_2 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & (1-q)\mu_1 & \#_{12} & 0 & \lambda_2 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu_2 & 0 & \#_3 & 0 & \lambda_2 & \dots & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & \#_{12} & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & \#_3 & 0 & \lambda_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & (1-q)\mu_1 & \#_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & \mu_2 & 0 & \#_6 \end{pmatrix}$$

$$A_{nn+1} = A_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ q\mu_1 & p\lambda_1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & p\lambda_1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & q\mu_1 & p\lambda_1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & p\lambda_1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & p\lambda_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & p\lambda_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & q\mu_1 & p\lambda_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & p\lambda_1 \end{pmatrix}$$

If the capacity of the orbit is finite say M then

$$A_{MM} = \begin{pmatrix} \#_9 & \lambda_1 & \lambda_2 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ \mu_1 & \#_{10} & 0 & \lambda_2 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ \mu_2 & 0 & \#_{11} & 0 & \lambda_2 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu_1 & \#_{10} & 0 & \lambda_2 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu_2 & 0 & \#_{11} & 0 & \lambda_2 & \dots & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & \#_{10} & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & \#_{11} & 0 & \lambda_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & \mu_1 & -\mu_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & \mu_2 & 0 & -\mu_2 \end{pmatrix}$$

Let \mathbf{x} be a steady-state probability vector of Q and partitioned as $\mathbf{x} = (x(0), x(1), x(2), \dots)$ and \mathbf{x} satisfies

$$\mathbf{xQ} = \mathbf{0}, \mathbf{xe} = \mathbf{1}. \tag{1}$$

Where $x(i) = (P_{i00}, P_{i01}, P_{i02}, P_{i11}, P_{i12}, P_{i21}, P_{i22}, \dots, P_{ik1}, P_{ik2})$
 $i=0, 1, 2, 3, \dots$

4. Stability condition

Theorem:

The inequality $(F)(\lambda_1/\mu_1) < 1$ where $S_1 = (1+G)*(p/(1-q))$,
 $S_2 = ((1-x)(1 - \pi_{2k} - \pi_{2k+1}) + x\pi_{2k+1})(1 + \lambda_2 / (1-q)\mu_1)$ $F = S_1/S_2$, $x = \lambda_2/\mu_2$,
 $G = t + t^2 + t^3 + \dots + t^{k-1}$ is the necessary and sufficient condition for the system to be stable. If $K \rightarrow \infty$, the above stability condition becomes $(p\lambda_1 / (1-q) \mu_1 + \lambda_2/\mu_2) < 1$

Proof:

Let Q be an infinitesimal generator matrix for the queueing system

The stationary probability vector X satisfying

$$\mathbf{XQ} = \mathbf{0} \text{ and } \mathbf{Xe} = \mathbf{1} \tag{2}$$

Let R be the rate matrix and satisfying the equation

$$\mathbf{A}_0 + \mathbf{RA}_1 + \mathbf{R}^2\mathbf{A}_2 = \mathbf{0} \tag{3}$$

The system is stable if $sp(R) < 1$

We know that the Matrix R satisfies $sp(R) < 1$ if and only if

$$\mathbf{\Pi A}_0 \mathbf{e} < \mathbf{\Pi A}_2 \mathbf{e} \tag{4}$$

and $\mathbf{\Pi} = (\pi_0, \pi_1, \pi_2, \dots, \pi_{2k}, \pi_{2k+1})$ satisfies

$$\mathbf{\Pi A} = \mathbf{0} \tag{5}$$

$$\mathbf{\Pi e} = \mathbf{1} \tag{6}$$

where

$$\mathbf{A} = \mathbf{A}_0 + \mathbf{A}_1 + \mathbf{A}_2 \tag{7}$$

A_0, A_1 and A_2 are square matrices of order $2k+2$

$$A_0 = \begin{pmatrix} p\lambda_1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & p\lambda_1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & q\mu_1 & p\lambda_1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & p\lambda_1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q\mu_1 & p\lambda_1 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & p\lambda_1 & \dots & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & p\lambda_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & p\lambda_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & q\mu_1 & p\lambda_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & p\lambda_1 \end{pmatrix}$$

$$A_2 = (a_{ij}) \text{ where } \begin{matrix} a_{ij} = (1-q) \mu_1 & \text{if } i = 1 \text{ and } j = 1 \\ a_{ij} = \mu_2 & \text{if } i = 2 \text{ and } j = 1 \\ a_{ij} = 0 & \text{otherwise} \end{matrix}$$

$$A_1 = \begin{pmatrix} \#_2 & 0 & \lambda_2 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & \#_3 & 0 & \lambda_2 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & (1-q)\mu_1 & \#_{12} & 0 & \lambda_2 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & \mu_2 & 0 & \#_3 & 0 & \lambda_2 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (1-q)\mu_1 & \#_{12} & 0 & \dots & \lambda_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu_2 & 0 & \#_3 & \dots & 0 & \lambda_2 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & \mu_1 & \dots & \#_{12} & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu_2 & \dots & 0 & \#_3 & 0 & \lambda_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & (1-q)\mu_1 & \#_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & \mu_2 & 0 & \#_6 \end{pmatrix}$$

By substituting A_0, A_1, A_2 in equations (5) and (7), we get

$$\begin{aligned} \pi_1 &= x\pi_0 \\ \pi_{2i} &= t^i \pi_0 \quad i = 1, 2, 3, \dots, k-1 \\ \pi_i &= x(\pi_{i-2} + \pi_{i-1}) \quad i = 3, 5, 7, \dots, 2k-1 \\ \pi_{2k} &= (\lambda_2 / (1-q)\mu_1) t^{k-1} \pi_0 \\ \pi_{2k+1} &= x\pi_{2k-1} \end{aligned}$$

From (6), $\pi_0 + \pi_1 + \pi_2 + \pi_3 + \pi_4 + \pi_5 + \dots + \pi_{2k-1} + \pi_{2k} + \pi_{2k+1} = 1$

By substituting π_i values in the above equation we get

$$\pi_0(1+G) = [(1-x)(1 - \pi_{2k} - \pi_{2k+1}) + x\pi_{2k+1}]$$

From (4), $(p/(1-q))(\lambda_1/\mu_1) < \pi_0 (1 + \lambda_2/(1-q)\mu_1)$

By substituting π_0 we get

$$(1+G)(p/(1-q))(\lambda_1/\mu_1) < ((1-x)(1 - \pi_{2k} - \pi_{2k+1}) + x\pi_{2k+1})(1 + \lambda_2 / (1-q)\mu_1)$$

Therefore $(F)(\lambda_1/\mu_1) < 1$

The inequality $(F)(\lambda_1/\mu_1) < 1$ is also a sufficient condition for the retrial queueing system to be stable. Let Q_n be the number of customers in the orbit after departure n^{th} customer from the service station. We first prove the embedded Markov chain $\{Q_n, n \geq 0\}$ is ergodic if $(F)(\lambda_1/\mu_1) < 1$ is readily to see that $\{Q_n, n \geq 0\}$ is irreducible and aperiodic. It remains to be proved that $\{Q_n, n \geq 0\}$ is positive recurrent. The irreducible and aperiodic Markov chain $\{Q_n, n \geq 0\}$ is positive recurrent if $|\psi_i| < \infty$ for all i and $\lim_{i \rightarrow \infty} \sup \psi_i < 0$ where

$$\psi_i = E(Q_{n+1} - Q_n / Q_n = i) \quad (i=0, 1, 2, 3, 4, 5, \dots)$$

$$\psi_i = (F)(\lambda_1/\mu_1) - i\sigma / (p\lambda_1 + \lambda_2 + i\sigma)$$

If $(F)(\lambda_1/\mu_1) < 1$, then $|\psi_i| < \infty$ for all i and $\lim_{i \rightarrow \infty} \sup \psi_i < 0$

Therefore the embedded Markov chain $\{Q_n, n \geq 0\}$ is ergodic.

If $K \rightarrow \infty$ then $\pi_{2k} \rightarrow 0$ and $\pi_{2k+1} \rightarrow 0$ and $G \rightarrow \lambda_2/\mu_1$.

The above stability condition becomes $(p\lambda_1 / (1-q)\mu_1) + (\lambda_2/\mu_2) < 1$

5. Analysis of steady state probabilities

We are applying **Direct Truncation Method** to find Steady state probability vector x . Let M denote the cut-off point or Truncation level. The steady state probability vector $x^{(M)}$ is now partitioned as $x^{(M)} = (x(0), x(1), x(2), \dots, x(M))$

and $\mathbf{x}^{(M)}$ satisfies $\mathbf{x}^{(M)} Q = 0, \mathbf{x}^{(M)} \mathbf{e} = 1.$

Where $\mathbf{x}^{(i)} = (P_{i00}, P_{i01}, P_{i02}, P_{i11}, P_{i12}, P_{i21}, P_{i22}, \dots, P_{ik1}, P_{ik2})$
 $i=0,1,2,3 \dots M$

The above system of equations is solved exploiting the special structure of the coefficient matrix. It is solved by Numerical method such as GAUSS-JORDAN elementary transformation method. Since there is no clear cut choice for M, we may start the iterative process by taking, say M=1 and increase it until the individual elements of \mathbf{x} do not change significantly. That is, if M^* denotes the truncation point then

$$\| \mathbf{x}^{M^*}(\mathbf{i}) - \mathbf{x}^{M^*-1}(\mathbf{i}) \|_{\infty} < \mathbf{e} \text{ where } \mathbf{e} \text{ is an infinitesimal quantity.}$$

6. Special cases

- a) This model becomes Single Server Retrial queueing system with non-preemptive priority service if $q \rightarrow 0$ and $p \rightarrow 1$
- b) This model becomes Single Server Retrial queueing system and results coincide with analytic solutions given by Falin and Templeton for various values of $\lambda_1, (\lambda_2 \rightarrow 0), \mu_1, (\mu_2 \rightarrow \infty), q \rightarrow 0, p \rightarrow 1, \sigma$ and K
- c) This model becomes Single Server Standard Queueing System and coincide with standard results if $\lambda_2 \rightarrow 0, \mu_2 \rightarrow \infty, q \rightarrow 0, p \rightarrow 1$ and $\sigma \rightarrow \infty$

7. Systems performance measures

We can find various probabilities for various values of $\lambda_1, \lambda_2, \mu_1, \mu_2, p, q, \sigma$ and K and the following system measures can be easily study with these probabilities

a) The probability mass function of Server state

Let S(t) be the random variable which represents the server state at time t .

S :	0_{idle}	1_{low}	2_{high}
P :	$\sum_{i=0}^{\infty} p(i, 0, 0)$	$\sum_{i=0}^{\infty} \sum_{j=0}^k p(i, j, 1)$	$\sum_{i=0}^{\infty} \sum_{j=0}^k p(i, j, 2)$

b) The probability mass function of number of customers in the orbit

Let X(t) be the random variable which represents the number of low priority customers in the orbit.

No.of low priority customers (orbit)	Probability
i	$\sum_{j=0}^k \sum_{l=1}^2 p(i, j, l) + p(i, 0, 0)$

- c) **The Probability mass function of number of high priority customers in the queue.**

Let $P(t)$ be the random variable which represents number of high priority customers in the queue at time t .

No. of high priority customers (queue)	Probability
0	$\sum_{i=0}^{\infty} \sum_{l=0}^2 p(i, 0, l)$
j	$\sum_{i=0}^{\infty} \sum_{l=1}^2 p(i, j, l)$

- d) **The Mean number of high priority customers in the queue**

$$\text{MPQL} = \sum_{j=1}^k j \left(\sum_{i=0}^{\infty} \sum_{l=1}^2 p(i, j, l) \right)$$

- e) **The Mean number of low priority customers in the orbit**

$$\text{MNCO} = \left(\sum_{i=0}^{\infty} i \left(\sum_{j=0}^k \sum_{l=1}^2 p(i, j, l) + p(i, 0, 0) \right) \right)$$

- f) **The probability that the orbiting customer (low) is blocked**

$$\text{Blocking Probability} = \sum_{i=1}^{\infty} \sum_{j=0}^k \sum_{l=1}^2 p(i, j, l)$$

- g) **The probability that an arriving customer (low/high) enter into service station immediately**

$$\text{PSI} = \sum_{i=0}^{\infty} p(i, 0, 0)$$

9. NUMERICAL STUDY

The values for parameters $\lambda_1, \lambda_2, \mu_1, \mu_2, p, q$ will be chosen so that it satisfies the stability condition discussed section 5

From the following tables we conclude that

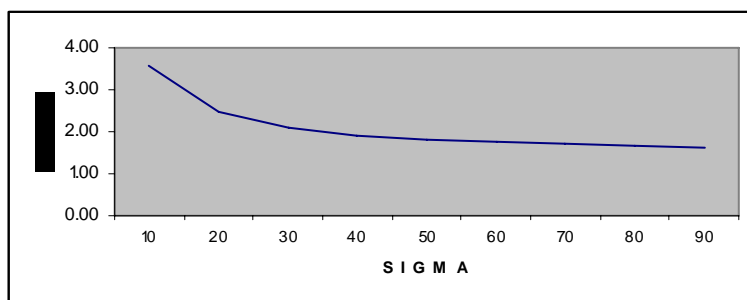
- Mean number of cutomers in the orbit decreases as σ increases.
- Probabilities P_0, P_1 and P_2 are independent of σ .

Table 1: System Measures for $\lambda_1=10$ $\lambda_2=5$ $\mu_1=20$ $\mu_2=25$ $p=0.8$ $q=0.2$ $K=6$

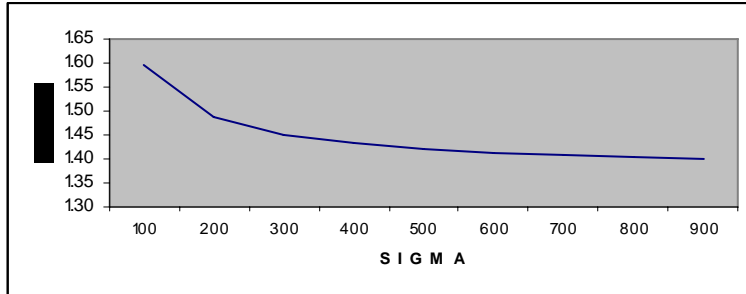
Sigma	Ocut	P_0	P_1	P_2	MNCO	MPQL
10	48	0.2667	0.5333	0.2000	3.5767	0.2580
20	44	0.2667	0.5333	0.2000	2.4769	0.2580
30	43	0.2667	0.5333	0.2000	2.1102	0.2580
40	42	0.2667	0.5333	0.2000	1.9269	0.2580
50	42	0.2667	0.5333	0.2000	1.8170	0.2580
60	42	0.2667	0.5333	0.2000	1.7436	0.2580
70	42	0.2667	0.5333	0.2000	1.6913	0.2580
80	42	0.2667	0.5333	0.2000	1.6520	0.2580
90	42	0.2667	0.5333	0.2000	1.6214	0.2580
100	42	0.2667	0.5333	0.2000	1.5970	0.2580
200	42	0.2667	0.5333	0.2000	1.4870	0.2580
300	42	0.2667	0.5333	0.2000	1.4503	0.2580
400	42	0.2667	0.5333	0.2000	1.4320	0.2580
500	42	0.2667	0.5333	0.2000	1.4210	0.2580
600	42	0.2667	0.5333	0.2000	1.4137	0.2580
700	42	0.2667	0.5333	0.2000	1.4084	0.2580
800	42	0.2667	0.5333	0.2000	1.4045	0.2580
900	42	0.2667	0.5333	0.2000	1.4015	0.2580
1000	42	0.2667	0.5333	0.2000	1.3990	0.2580
2000	42	0.2667	0.5333	0.2000	1.3880	0.2580
3000	42	0.2667	0.5333	0.2000	1.3843	0.2580
4000	42	0.2667	0.5333	0.2000	1.3825	0.2580
5000	42	0.2667	0.5333	0.2000	1.3814	0.2580
6000	42	0.2667	0.5333	0.2000	1.3807	0.2580
7000	42	0.2667	0.5333	0.2000	1.3802	0.2580
8000	42	0.2667	0.5333	0.2000	1.3798	0.2580
9000	42	0.2667	0.5333	0.2000	1.3795	0.2580

10. Graphical Study

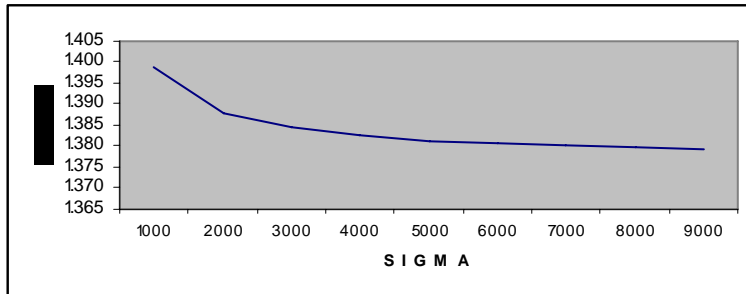
Fig 1. Mean No. of low priority customers in the orbit for $\lambda_1 = 10$ $\lambda_2=5$ $\mu_1=20$ $\mu_2=25$ $p = 0.8$ $q = 0.2$ $K = 6$ and σ varies from 10 to 90



**Fig 2. Mean No. of low priority customers in the orbit for $\lambda_1 = 10$ $\lambda_2 = 5$ $\mu_1 = 20$ $\mu_2 = 25$
 $p = 0.8$ $q = 0.2$ $K = 6$ and σ varies from 100 to 900**



**Fig 3. Mean No. of low priority customers in the orbit for $\lambda_1 = 10$ $\lambda_2 = 5$ $\mu_1 = 20$ $\mu_2 = 25$
 $p = 0.8$ $q = 0.2$ $K = 6$ and σ varies from 1000 to 9000**



11. Conclusions

It is observed from sections 9 and 10 that Mean number of low priority customers in the orbit decreases as the retrial rate increases, the Probabilities for the server being idle, busy with low priority customers independent of retrial rate. From this study, we further state that mean number of customers in the orbit increases as the probability q increases and it decreases as the probability p decreases. The various special cases which have been discussed in section 7 are particular cases of this research work. This research work can further be extended by introducing various vacation policies.

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