

Drag on a Fluid Sphere Embedded in a Porous Medium

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Abstract. The problem of the Stokes flow past a fluid sphere embedded in a porous medium is studied. The Brinkman equation for the flow outside the fluid sphere and the Stokes equation for inside the fluid sphere, in their stream function formulations are used. The drag force experienced by a fluid sphere embedded in a porous medium is evaluated. The dependence of the drag coefficient on permeability and viscosity ratio is presented graphically and discussed. Some previous known results are then also deduced from the present analysis.

Mathematics Subject Classification: 76D07

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1. Introduction:

The Stokes flow past a class of axi-symmetric bodies with uniform stream at infinity parallel to the axis of symmetry was studied by Payne and Pell [5] and obtained a general formula for the drag force experienced by axisymmetric bodies in forms of the stream function. The Stokes flow due to the translation of a spherical fluid particle in an unbounded fluid medium has been discussed in the classical book of Happel and Brenner [4]. Ramkissoon [7] studied the problem of Stokes flow past a slightly deformed fluid sphere and evaluated the drag force on a fluid oblate spheroid. Berman [2] discussed the flow of a viscous fluid past an impervious sphere embedded in a porous medium. He found that the viscous sublayer increases with the increase of the permeability of the porous medium.

The problem of symmetrical flow of a classical fluid past a Reiner-Rivlin liquid sphere was studied by Ramkissoon [8]. He found that the drag experienced

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by a Reiner-Rivlin liquid sphere is greater than that on a purely Newtonian fluid sphere. However for the first order expansion of Newtonian parameters, both are identical. Flow past a sphere in a porous medium based on the Brinkman model studied by Pop and Ingham [6] and they presented a closed form, exact solution for the forced flow past a sphere which is embedded in a porous medium using the Brinkman model. Flow past an axisymmetric body embedded in a saturated porous medium using Brinkman's extension investigated by Srinivasacharya and Murthy [9]. Stokes flow past a fluid prolate spheroid was studied by Deo and Datta [3] and evaluated the drag force experienced by it.

This paper concerns the solution of the problem of Stokes flow past a fluid sphere embedded in a porous medium. The Brinkman equation for the flow outside the fluid sphere and the Stokes equation inside the fluid sphere, in their stream function formulations are used. The drag force experienced by a fluid sphere embedded in a porous medium is evaluated. The dependence of drag coefficient on permeability is presented graphically and discussed. Some previous known results are then also deduced.

2. Mathematical formulation of the problem

Let us consider the flow of an incompressible viscous fluid with a uniform velocity U directed in the positive z -direction in a porous medium of permeability k in which a fluid sphere of radius a is situated. The inside and outside regions of the fluid sphere are fully saturated with the viscous fluid. We shall denote $i = 1$ in an entity for inside and $i = 2$ for outside regions of the fluid sphere, respectively. For the inside region (1) within the fluid sphere we assume the Stokes equation (Happel and Brenner [4]) as

$$\mu_1 \nabla^2 \mathbf{v}^{(1)} = \nabla p^{(1)}. \quad (1)$$

The governing Brinkman equation for the outside porous region (2) can be expressed as

$$\nabla^2 \mathbf{v}^{(2)} - \left(\frac{\sigma}{a}\right)^2 \mathbf{v}^{(2)} = \frac{1}{\mu_e} \nabla p^{(2)}. \quad (2)$$

Here, $\sigma^2 = \frac{a^2}{k}$, μ_1 and μ_2 are the viscosities of fluids for inside and outside of the fluid sphere, respectively and k being the permeability of the porous medium. Since, σ are dimensionless quantity related inversely with the permeability, therefore, we named σ as the dimensionless permeability parameter. In addition, the equations of continuity for incompressible fluids must be satisfied in both regions:

$$\text{div } \mathbf{v}^{(i)} = 0, \quad i = 1, 2. \quad (3)$$

These equations of continuity for axisymmetric, incompressible viscous fluid in spherical polar coordinates (r^*, θ, φ) for both regions can also be written as

$$\frac{\partial}{\partial r^*}(r^{*2} v_{r^*}^{(i)}) + \frac{r^*}{\sin \theta} \frac{\partial}{\partial \theta}(v_{\theta}^{(i)} \sin \theta) = 0, \quad (4)$$

where, $v_{r^*}^{(i)}$ and $v_{\theta}^{(i)}$, are components of velocity in the direction of r^* and θ , respectively. The Stokes stream functions $\psi^{(i)}(r^*, \theta)$ in both regions satisfying equations of continuity (4) can be defined as

$$v_{r^*}^{(i)} = -\frac{1}{r^{*2} \sin \theta} \frac{\partial \psi^{(i)}}{\partial \theta} ; v_{\theta}^{(i)} = \frac{1}{r^* \sin \theta} \frac{\partial \psi^{(i)}}{\partial r^*}. \quad (5)$$

In order to make non-dimensional the physical quantities, we shall use the following non-dimensional variables

$$\psi^{(i)} = Ua^2 \tilde{\psi}^{(i)}, \quad p^{(i)} = \frac{\mu U}{a} \tilde{p}^{(i)}, \quad \mathbf{v}^{(i)} = U \tilde{\mathbf{v}}^{(i)}, \quad r^* = ar, \quad i=1,2. \quad (6)$$

Eliminating the pressures from equations (1) and (2) and dropping the tildes, we obtain the following non-dimensional equations, respectively as

$$E^2 (E^2 \psi^{(1)}) = 0, \quad (7)$$

$$E^2 (E^2 - \sigma^2) \psi^{(2)} = 0, \quad (8)$$

where the dimensionless operator

$$E^2 = \frac{\partial^2}{\partial r^2} + \frac{(1-\zeta^2)}{r^2} \frac{\partial^2}{\partial \zeta^2}, \quad \text{and } \zeta = \cos \theta. \quad (9)$$

Further, the expressions for tangential stress $T_{r\theta}^{(i)}, i=1,2$ for both regions can be expressed as

$$T_{r\theta}^{(i)}(r, \zeta) = \frac{\mu_i}{r\sqrt{1-\zeta^2}} \left[\frac{\partial^2 \psi^{(i)}}{\partial r^2} - \frac{2}{r} \frac{\partial \psi^{(i)}}{\partial r} - \frac{(1-\zeta^2)}{r^2} \frac{\partial^2 \psi^{(i)}}{\partial \zeta^2} \right]. \quad (10)$$

In the case of axisymmetric incompressible creeping flow, the general solution of the Stokes equation (7) in spherical polar coordinates comes out as (Happel and Brenner [4])

$$\begin{aligned} \psi^{(1)}(r, \zeta) = & \sum_{n=0}^{\infty} [A_n r^{-n+1} + B_n r^n + C_n r^{-n+3} + D_n r^{n+2}] G_n(\zeta) \\ & + \sum_{n=0}^{\infty} [A'_n r^{-n+1} + B'_n r^n + C'_n r^{-n+3} + D'_n r^{n+2}] H_n(\zeta), \end{aligned} \quad (11)$$

where $G_n(\zeta)$ and $H_n(\zeta)$ are Gegenbauer functions of first and second kinds, respectively[1]. If we retain the terms which are multiplied by $G_0(\zeta)$ and $G_1(\zeta)$, then velocity will become irregular on the symmetry axis z . Also, $H_n(\zeta)$ are irregular on the z -axis for all values of integers n . Therefore, for the regular solution, we have ignored the terms which are multiplied by $G_0(\zeta)$, $G_1(\zeta)$ and $H_n(\zeta)$ for all values of n . Thus, the complete regular solution of Stokes equation takes the form

$$\psi^{(1)}(r, \zeta) = \sum_{n=2}^{\infty} [A_n r^{-n+1} + B_n r^n + C_n r^{-n+3} + D_n r^{n+2}] G_n(\zeta). \quad (12)$$

Therefore, a particular solution which satisfies the regularity condition at origin and for spherical case we can take the above expansion (12) for $n = 2$ only, i.e.

$$\psi^{(1)}(r, \zeta) = [B_2 r^2 + D_2 r^4] G_2(\zeta). \quad (13)$$

The general regular solution on the symmetry axis z of the Brinkman's equation (8) comes out as (Zlatanovski [10])

$$\psi^{(2)}(r, \zeta) = \sum_{n=0}^{\infty} [A_n^* r^{-n+1} + B_n^* r^n + C_n^* \sqrt{r} K_\nu(\sigma r) + D_n^* \sqrt{r} I_\nu(\sigma r)] G_n(\zeta). \quad (14)$$

Here, $I_\nu(\sigma r)$ and $K_\nu(\sigma r)$ are the modified Bessel functions of first and second kinds and of non-integer index $\nu = n - \frac{1}{2}$, respectively as defined in Abramowitz and Stegun [1].

Therefore, the above solution which satisfies the uniform condition at infinity and requirement for spherical case reduces to

$$\psi^{(2)}(r, \zeta) = [A_2^* r^{-1} + B_2^* r^2 + C_2^* \sqrt{r} K_{3/2}(\sigma r)] G_2(\zeta). \quad (15)$$

3. Boundary conditions:

The boundary conditions those are physically realistic and mathematically consistent for this proposed problem can be taken as given below:

The kinematic condition of mutual impenetrability at the surface requires that we take

$$\psi^{(1)} = 0 \quad \text{on } r = 1, \quad (16)$$

$$\psi^{(2)} = 0 \quad \text{on } r = 1. \quad (17)$$

We assume that the tangential velocity is continuous across the surface, hence we have

$$\frac{\partial \psi^{(1)}}{\partial r} = \frac{\partial \psi^{(2)}}{\partial r} \quad \text{on } r = 1. \quad (18)$$

Now from the theory of interfacial tension, the presence of interfacial tension only produces a discontinuity in the normal stress T_{rr} and does not in anyway affect the tangential stress $T_{r\theta}$. Therefore, the latter is continuous across the surface and so that we may take

$$T_{r\theta}^{(1)} = T_{r\theta}^{(2)} \quad \text{on } r = 1. \quad (19)$$

Far away from the fluid sphere, the flow is uniform so that the condition at infinity implies that

$$\psi_\infty^{(2)}(r, \zeta) = \frac{1}{2} r^2 \sin^2 \theta = r^2 G_2(\zeta) \quad \text{as } r \rightarrow \infty. \quad (20)$$

Applying these above boundary conditions (16)-(20), we get the following equations, respectively as

$$B_2 + D_2 = 0 \quad (21)$$

$$A_2^* + B_2^* + C_2^* K_{3/2}(\sigma) = 0 \quad (22)$$

$$-A_2^* + 2B_2^* - C_2^* (\sigma K_{1/2}(\sigma) + K_{3/2}(\sigma)) - 2B_2 - 4D_2 = 0 \quad (23)$$

$$6\gamma^2 A_2^* + \gamma^2 C_2^* \{(\sigma^2 + 6)K_{3/2}(\sigma) + 2\sigma K_{1/2}(\sigma)\} - 6D_2 = 0 \quad (24)$$

$$B_2^* = 1, \quad (25)$$

where $\gamma^2 = \mu_2 / \mu_1$ being the viscosity ratio.

Solving these above equations (21) - (25), we get following values of unknown constants as

$$B_2 = -\frac{[3\gamma^2(1+\sigma)]}{6+2\gamma^2(3+\sigma)}, \quad (26)$$

$$D_2 = \frac{3\gamma^2(1+\sigma)}{6+2\gamma^2(3+\sigma)}, \quad (27)$$

$$A_2^* = -\frac{[3(3+3\sigma+\sigma^2)+\gamma^2(6+6\sigma+3\sigma^2+\sigma^3)]}{\sigma^2\{3+\gamma^2(3+\sigma)\}}, \quad (28)$$

$$B_2^* = 1, \quad (29)$$

$$C_2^* = \sqrt{\frac{2}{\pi\sigma} \left(\frac{3e^\sigma(3+2\gamma^2)}{\{3+\gamma^2(3+\sigma)\}} \right)}. \quad (30)$$

Thus the entire coefficients have been determined and hence, we get explicit expressions for the stream functions in both regions given by equations (13) and (15), respectively.

4. Evaluation of the drag force:

The drag force experienced by an axisymmetric body in a porous medium can be calculated by using the simple formula given by Srinivasacharya and Murthy [9]

$$F = 4\pi \mu_2 U a \sigma^2 \lim_{r \rightarrow \infty} \left[\frac{r^3 \psi^{(2)}}{\varpi^2} \right], \quad (31)$$

which is analogous to the result of Payne and Pell [5].

If the fluid is not rest at infinity the above formula is not applicable, so if ψ_∞ denotes the stream function corresponding to the fluid motion at infinity, then the stream function $\psi - \psi_\infty$ gives a state of rest at infinity. Therefore, the above formula (31) in our case becomes

$$F = 4\pi \mu_2 U a \sigma^2 \lim_{r \rightarrow \infty} \left[\frac{r^3 (\psi^{(2)} - \psi_\infty^{(2)})}{\varpi^2} \right], \quad (32)$$

where $\varpi = r \sin \theta$.

Thus for the present case, the above formula provides

$$F = 2\pi \mu_2 U a \sigma^2 A_2^*$$

$$= -2\pi \mu_2 U a \left[\frac{3(3+3\sigma+\sigma^2) + \gamma^2(6+6\sigma+3\sigma^2+\sigma^3)}{\{3+\gamma^2(3+\sigma)\}} \right]. \quad (33)$$

Also, the drag coefficient can be defined as

$$\begin{aligned} C_D &= \frac{-F}{(1/2)\rho U^2 \pi a^2} \\ &= \frac{8}{\text{Re}} \left[\frac{3(3+3\sigma+\sigma^2) + \gamma^2(6+6\sigma+3\sigma^2+\sigma^3)}{\{3+\gamma^2(3+\sigma)\}} \right], \end{aligned} \quad (34)$$

where $\text{Re} = \frac{2aU}{\nu}$ is the Reynolds number and $\nu = \mu_2 / \rho$ being the kinematic viscosity of fluid, respectively.

The following special cases can be deduced immediately as follows:

Case I: If $k \rightarrow \infty$, then $\sigma = a/\sqrt{k} \rightarrow 0$ i.e., the porous region will be a clear fluid. Therefore, the drag force experienced by a fluid sphere comes out as

$$F = -6\pi \mu_2 U a \frac{(1 + \frac{2}{3}\gamma^2)}{(1 + \gamma^2)}, \quad (35)$$

which agree with the result reported earlier in the book by Happel and Brenner[4] for the drag force experienced by a fluid sphere in a fluid medium.

Case II: If $\gamma^2 = 0$, then fluid sphere behaves like a solid sphere, thus the drag force from the equation (35) comes out as

$$F = -6\pi \mu_2 U a, \quad (36)$$

which is the well-known Stokes result for flow past a rigid sphere in an unbounded medium.

Case III: If $\mu_2 \gg \mu_1$ i.e. $\gamma^2 \rightarrow \infty$, then fluid sphere behaves like a gaseous spherical bubble, so in this case drag force from the equation (35) provides

$$F = -4\pi \mu_2 U a. \quad (37)$$

This result is identical to that previously given for a sphere at whose surface perfect slip occurs (Happel and Brenner [4]).

Conclusions:

The variation of the drag coefficient C_D with permeability parameter σ for various values of viscosity ratio γ^2 is shown in figure-1. It is evident from the figure that the variation of C_D increases with increasing permeability parameter σ , i.e. decreases with permeability, for various values of $\gamma^2 = 1, 3, 9$

and decreases very slowly with increasing viscosity ratio γ^2 . The variation of C_D with γ^2 is depicted in figure-2 for various values of σ which shows that after a slight decrease in C_D , it becomes almost constant with γ^2 and increases with increasing σ .

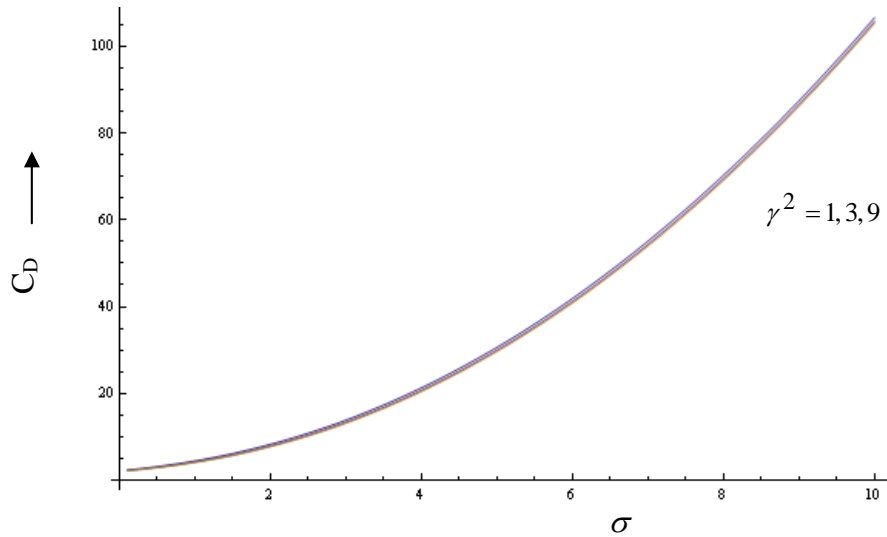


Figure-1: Variation of drag coefficient C_D for a fluid sphere versus permeability parameter σ for various values of viscosity ratio γ^2 .

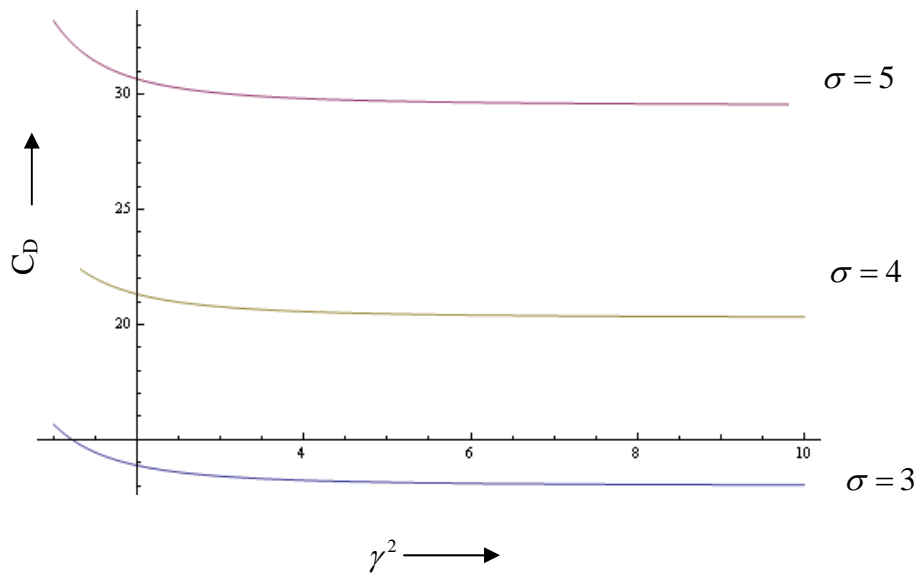


Figure-2: Variation of drag coefficient C_D for a fluid sphere versus viscosity ratio γ^2 for various values of permeability parameter σ .

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