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A Variation Problem for Null Curves

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Abstract

H.K. Nickerson and G. S Manning [1] derived the intrinsic equations for a relaxed elastic line on an oriented surface in $\mathbb{E}^3$.

Considerable works were done on physical and geometric use of null curves in the literature ([2]-[9]). In this work, we give the analogue of the Frenet -Serret formulas on the lightlike surface and we derive the formula square curvature for a non-unit velocity null curve in $\mathbb{E}^3_1$. We obtain the intrinsic equations for a relaxed elastic line on a lightlike surface in the 3-dimensional semi-Euclidean space

Keywords: Variational problem, null curve, lightlike surface

1 Preliminaries

In this section, we will give some new definitions and propositions except fundamental definitions in literature.

Definition 1 *Let* $\mathbb{E}^3 = \{(x^1, x^2, x^3, x^4)|x^1, x^2, x^3 \in R\}$ *be a 3-dimensional Euclidean space. For any vectors* $x = (x^1, x^2, x^3)$, $y = (y^1, y^2, y^3)$ *in* $\mathbb{E}^3$*, the pseudo scalar product of* x *and* y *is defined to be* $\langle x, y\rangle = -x^1y^1 + x^2y^2 + x^3y^3$ $\mathbb{E}^3_1$ *is called 3-dimensional semi-Euclidean space* $\mathbb{E}^3_1$ [10].

Definition 2 *Let* $\mathbb{E}^3_1$ *be a semi-Euclidean space furnished with a metric tensor* $\langle , \rangle$ *.A vector* v *to* $\mathbb{E}^3_1$ *is called if* $\langle v, v\rangle = 0$ *and* $v \neq 0, v$ *is null* [10]

Proposition 3 *Let* $C : [a, b] \to \mathbb{E}^3_1$ *, a positively oriented set* $\{C'(p), C''(p), C'''(p)\}$ *, there exist a local frame* $F = \{\xi = C', N, W\}$*, called Cartan frame satisfying* $\langle \xi, \xi\rangle = \langle N, N\rangle = 0, \langle \xi, N\rangle = 1, \langle W, \xi\rangle = \langle W, N\rangle = 0, \langle W, W\rangle = 1$*. The Cartan equations* $\xi' = \kappa W$*,* $N' = \tau W$*,* $W' = \tau\xi - \kappa N$ *where* κ *and* τ*, the curvature and torsion of* C ([2] − [9]).

Definition 4 *Let ξ denote an arc on a connected oriented lightlike surface L parametrized by arc length s, $0 \leq s \leq l$. Let $\kappa(s)$ be the curvature of $\xi(s)$. The total square curvature C of ξ is defined by $C = \int_0^l \kappa^2 ds$.*

Definition 5 *The curve null ξ is called a relaxed elastic line if it is an extremal for the variational problem of the minimizing value of L within the family of all curves of length l on L lightlike surface having the same initial point and initial direction with ξ.*

Definition 6 *Let ξ be a null curve on the lightlike surface L. Apart from the Frenet frame, there also exist a second frame at every point of the null curve ξ. At the point $\xi(s)$ of ξ, let $T(s) = \xi'(s)$ denote the unit tangent vector to ξ, let $N(s)$ denote the unit normal to L lightlike surface. We take $T \times N = -T, T \times R = -N$, $N \times R = -R$. Then, $\{T, R, N\}$ gives an orthonormal basis for all vectors at $\xi(s)$.*

Proposition 7 *Let L be the lightlike surface and ξ denote an null curve on L. We give the analogue of the Frenet -Serret formulas on a lightlike surface in $\mathbb{E}_1^3$.*

$$\begin{bmatrix} T' \\ R' \\ N' \end{bmatrix} = \begin{bmatrix} k_n & k_g & 0 \\ \tau_g & 0 & -k_g \\ 0 & -\tau_g & -k_n \end{bmatrix} \begin{bmatrix} T \\ R \\ N \end{bmatrix} \tag{1}$$

k_g is the geodesic curvature, τ_g is the geodesic torsion, k_n is the normal curvature. We can write

$$\begin{aligned} T' &= a_{11}T + a_{12}R + a_{13}N, \\ R' &= a_{21}T + a_{22}R + a_{23}N, \\ N' &= a_{31}T + a_{32}R + a_{33}N \end{aligned} \tag{2}$$

$\langle T, T\rangle = \langle N, N\rangle = 0, \langle T, N\rangle = 1$, $\langle R, T\rangle = \langle R, N\rangle = 0, \langle R, R\rangle = 1$ and $< T', T >= 0 \Rightarrow a_{13} = 0, < R', R >= 0 \Rightarrow a_{22} = 0$ and $< N', N >= 0 \Rightarrow a_{31} = 0$ $< T', R >= k_g \Rightarrow a_{12} = k_g$ and $< R', T >= a_{23} = -k_g, < T', N >= k_n \Rightarrow a_{11} = k_n; < R', N >= \tau_g \Rightarrow a_{21} = \tau_g; < N', R >= -\tau_g \Rightarrow a_{32} = -\tau_g$.

With substituting $a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}$, a_{33} in (2), we obtain (1).

Proposition 8 *Let $\gamma : I \to \mathbb{E}_1^3$ denote a non-unit velocity null curve. The square curvature κ^2 along the line is given $\kappa^2 = \left\langle \frac{\partial\gamma}{\partial\partial}, \frac{\partial^3\gamma}{\partial\partial^3} \right\rangle \left\langle \frac{\partial^2\gamma}{\partial\partial^2}, \frac{\partial^2\gamma}{\partial\partial^2} \right\rangle^{-1}$.*

Proof. With the first and third differentiation of γ , respectively, we obtain

$$\frac{\partial\gamma}{\partial\sigma}=\frac{d\overset{*}{\gamma}}{ds}\frac{ds}{d\sigma}=T\left\langle\frac{\partial^2\gamma}{\partial\partial^2},\frac{\partial^2\gamma}{\partial\partial^2}\right\rangle^{1/4} \tag{3}$$

$$\frac{\partial^3\gamma}{\partial\sigma^3}=[\frac{\partial^2}{\partial\partial^2}(\left\langle\frac{\partial^2\gamma}{\partial\partial^2},\frac{\partial^2\gamma}{\partial\partial^2}\right\rangle^{1/4})-\kappa\tau(\left\langle\frac{\partial^2\gamma}{\partial\partial^2},\frac{\partial^2\gamma}{\partial\partial^2}\right\rangle^{1/4})^3T+[\kappa\left\langle\frac{\partial^2\gamma}{\partial\partial^2},\frac{\partial^2\gamma}{\partial\partial^2}\right\rangle^{1/4}\frac{d(\left\langle\frac{\partial^2\gamma}{\partial\partial^2},\frac{\partial^2\gamma}{\partial\partial^2}\right\rangle^{1/4})}{d\partial}$$
$$+2\kappa\left\langle\frac{\partial^2\gamma}{\partial\partial^2},\frac{\partial^2\gamma}{\partial\partial^2}\right\rangle^{1/4}\frac{d(\left\langle\frac{\partial^2\gamma}{\partial\partial^2},\frac{\partial^2\gamma}{\partial\partial^2}\right\rangle^{1/4})}{d\partial}]W+\kappa^2(\left\langle\frac{\partial^2\gamma}{\partial\partial^2},\frac{\partial^2\gamma}{\partial\partial^2}\right\rangle^{1/4})^3N. \tag{4}$$

■

From $\left\langle\frac{\partial\gamma}{\partial\partial},\frac{\partial^3\gamma}{\partial\partial^3}\right\rangle=\kappa^2\left\langle\frac{\partial^2\gamma}{\partial\partial^2},\frac{\partial^2\gamma}{\partial\partial^2}\right\rangle$,proof is trivial.

2 The solution of $\mathbf{C}'(0)=0$

Let ξ null curve be lies in a coordinate patch $(u,v)\to r(u,v)$ of lightlike surface L. Let be $r_u=\dfrac{\partial r}{\partial u}$, $r_v=\dfrac{\partial r}{\partial v}$. Then, ξ is expressed as $\xi(s)=r\left(u(s),v(s)\right),\ 0\le s\le l$ with $T(s)=\xi'(s)=\frac{\partial r}{\partial u}\frac{du}{ds}+\frac{\partial r}{\partial v}\frac{dv}{ds}$ and $R(s)=v(s)r_u+y(s)r_v$ for scalar functions $v(s)$ and $y(s)$.

We extend ξ to an arc ξ^* defined for $0\le s\le l^*$, with $l^*>l$. l^* close to l . $\mu(s)$, $0\le s\le l^*$, is not vanishing identically. Define $\eta(s)=\mu(s)v^*(s)$, $\rho(s)=\mu(s)y^*(s)$ Along ξ

$$\mu(0)=0,\mu'(0)=0 \tag{5}$$

Define $\eta(s)r_u+\rho(s)r_v=\mu(s)R(s)$ and

$$\eta(\sigma;t)=ru(\sigma)+t\rho(\sigma),v(\sigma)+t\rho(\sigma))\qquad,\ 0\le\sigma\le l^* \tag{6}$$

For $|t|<\varepsilon$ (where $\varepsilon>0$ depends upon the choice of ξ^* and of μ), the point $\eta(\sigma;t)$ lies in the coordinate patch. For fixed t, $\eta(\sigma;t)$ gives an arc with the same initial point and initial direction as ξ. σ is not arc length for $t\neq 0$. For fixed t, $|t|<\varepsilon$, let $J^*(t)$ denote the length of the arc $\eta(\sigma;t),0\le\sigma\le l^*$. Then $J^*(t)=\int\limits_0^{l^*}\left\langle\frac{\partial^2\eta}{\partial\sigma^2}(\sigma;t),\frac{\partial^2\eta}{\partial\sigma^2}(\sigma;t)\right\rangle^{1/4}d\sigma$. $J^*(0)=l^*>l$. For $0<\varepsilon_1\le\varepsilon$, $J^*(t)>l$ ($|t|<\varepsilon_1$). We can restrict $\eta(\sigma;t),0\le|t|<\varepsilon_1$, to an arc of length l by restricting the parameter σ to an interval $0\le\sigma\le\lambda(t)\le l^*$.

$$\int_0^{\lambda(t)} \left\langle \frac{\partial^2 \eta}{\partial \sigma^2}(\sigma;t), \frac{\partial^2 \eta}{\partial \sigma^2}(\sigma;t) \right\rangle^{1/4} d\sigma = l. \tag{7}$$

For $t = 0$, If $\lambda(t)$ differentiate with respect to t, we obtain

$$\left.\frac{d\lambda}{dt}\right|_{t=0} \sqrt[4]{\left\langle \frac{\partial^2 \eta}{\partial \sigma^2}, \frac{\partial^2 \eta}{\partial \sigma^2} \right\rangle_{\sigma=\lambda(0)}} + \frac{1}{2}\int_0^l \left\langle \left.\frac{\partial^2 \eta}{\partial \sigma^2}\right|_{t=0}, \left.\frac{\partial^2 \eta}{\partial \sigma^2}\right|_{t=0} \right\rangle^{-3/4} \left\langle \left.\frac{\partial^3 \eta}{\partial t \partial \sigma^2}\right|_{t=0}, \left.\frac{\partial^2 \eta}{\partial \sigma^2}\right|_{t=0} \right\rangle d\sigma = 0$$

$$\left.\frac{d\lambda}{dt}\right|_{t=0} = -\frac{1}{2}k_g^{-1/2}(l)\int_0^l (\mu'' + 2\mu\tau_g k_g + \mu k_n^2 - \mu k_g' k_n k_g^{-1} - 2\mu' k_n) k_g^{-1} ds \tag{8}$$

$$\left.\frac{\partial \eta}{\partial \sigma}\right|_{t=0} = T, \qquad 0 \le \sigma \le l \ , \tag{9}$$

$$\left.\frac{\partial^2 \eta}{\partial \sigma^2}\right|_{t=0} = T' = k_n T + k_g R, \tag{10}$$

$$\left.\frac{\partial^3 \eta}{\partial \sigma^3}\right|_{t=0} = T' = (k_n' + k_n^2 + k_g\tau_g)T + (k_n k_g + k_g')R - k_g^2 N. \tag{11}$$

From (6),

$$\left.\frac{\partial \eta}{\partial t}\right|_{t=0} = \mu R. \tag{12}$$

We obtain respectively(13), (14),(15), with second ,third and fourth differentiation of (12)

$$\frac{\partial^2 \eta}{\partial t \partial \sigma} = \mu' R + \mu\tau_g T - \mu k_g N, \tag{13}$$

$$\left.\frac{\partial^3 \eta}{\partial t \partial \sigma^2}\right|_{t=0} = \left(2\mu'\tau_g + \mu\tau_g' + \mu\tau_g k_n\right) T + \left(\mu'' + 2\mu\tau_g k_n\right) R + \left(\mu k_g k_n - \mu k_g' - 2\mu' k_g\right) N, \tag{14}$$

$$\begin{aligned}\left.\frac{\partial^4\eta}{\partial t\partial\sigma^3}\right|_{t=0} &= \left(3\mu''\tau_g + 3\mu'\tau_g' + \mu'\tau_g'' + 3\mu'\tau_g k_n + 2\mu\tau_g' k_n + \mu\tau_g k_n' + \mu\tau_g k_n^2 + 2\mu\tau_g^2 k_g\right) T \\ &+ \left(6\mu'\tau_g k_g + 3\mu\tau_g' k_g + \mu''' + 3\mu k_g'\tau_g\right) R \\ &+ \left(3\mu' k_g k_n - 3\mu'' k_g + 2\mu k_g' k_n - 2\mu\tau_g k_g^2 - 3\mu' k_g' - \mu k_g'' + \mu k_g k_n' - \mu k_g k_n^2\right) N.\end{aligned} \tag{15}$$

From Theorem 1.3 , the total square curvature of the curve $\eta(\sigma;t)$, $0 \le \sigma \le \lambda(t)$, $t \neq 0$. $C(t) = \int\limits_0^{\lambda(t)} \left\langle \frac{\partial\eta}{\partial\sigma}, \frac{\partial^3\eta}{\partial\sigma^3}\right\rangle \left\langle \frac{\partial^2\eta}{\partial\sigma^2}, \frac{\partial^2\eta}{\partial\sigma^2}\right\rangle^{-3/4} d\sigma$. If ξ extremal. $C'(0)$ is equal zero. In calculating of $C'(t)$,

$$\begin{aligned} C'(t) &= \frac{d\lambda}{dt}\left\{\left\langle \frac{\partial\eta}{\partial\sigma}, \frac{\partial^3\eta}{\partial\sigma^3}\right\rangle^{-3/2} \left\langle \frac{\partial^2\eta}{\partial\sigma^2}, \frac{\partial^2\eta}{\partial\sigma^2}\right\rangle^{-3/4}\right\}_{\sigma=\lambda(t)} \\ &+ \int\limits_0^{\lambda(t)} [\left\langle \frac{\partial^2\eta}{\partial t\partial\sigma}, \frac{\partial^3\eta}{\partial\sigma^3}\right\rangle + \left\langle \left.\frac{\partial^4\eta}{\partial t\partial\sigma^3}\right|_{t=0}, \left.\frac{\partial\eta}{\partial\sigma}\right|_{t=0}\right\rangle] \left\langle \frac{\partial^2\eta}{\partial\sigma^2}, \frac{\partial^2\eta}{\partial\sigma^2}\right\rangle^{-3/4} d\sigma \\ &- \frac{3}{2}\int\limits_0^{\lambda(t)} \left\langle \frac{\partial\eta}{\partial\sigma}, \frac{\partial^3\eta}{\partial\sigma^3}\right\rangle \left\langle \frac{\partial^2\eta}{\partial\sigma^2}, \frac{\partial^2\eta}{\partial\sigma^2}\right\rangle^{-7/4} \left\langle \frac{\partial^3\eta}{\partial t\partial\sigma^2}, \frac{\partial^2\eta}{\partial\sigma^2}\right\rangle d\sigma + \ldots \end{aligned} \tag{16}$$

With using (8),(10), (11) , (13), (14), (15) in , (16), we find

$$\begin{aligned} C'(0) &= \int\limits_0^l \mu'' k_g^{-1/2} ds - \int\limits_0^l \mu'(2k_n + 2k_g' k_g^{-1}) k_g^{-1} ds \\ &+ \int\limits_0^l \mu\left(\tau_g - k_g'' k_g^{-2} + k_n' k_g^{-1} + \frac{1}{2}k_n^2 k_g^{-1} - \frac{3}{2}k_g' k_n k_g^{-2}\right) ds, \end{aligned} \tag{17}$$

With integration by parts and (5),

$$\int\limits_0^l \mu'' k_g^{-1/2} ds = \mu'(l) k_g^{-1/2}(l) + \frac{1}{2}\mu(l) k_g'(l) k_g^{-3/2}(l) - \frac{1}{2}\int\limits_0^l \mu(k_g'' k_g^{-3/2} - \frac{3}{2}k_g^{-5/2}(k_g')^2) ds \tag{18}$$

$$\int_0^l \mu'(2k_n + 2k_g' k_g^{-1})k_g^{-1} ds = 2\mu(l)(k_n(l)k_g^{-1}(l) + k_g'(l)k_g^{-2}(l)) \qquad (19)$$
$$-2\int_0^l \mu(k_n' k_g^{-1} - k_n k_g^{-2} k_g' + k_g'' k_g^{-2} - 2(k_g')^2 k_g^{-3}) ds$$

Substituting (18) in (17), we find

$$C'(0) = \int_0^l \mu \begin{pmatrix} \tau_g - k_g'' k_g^{-2} + k_n' k_g^{-1} + \frac{1}{2}k_n^2 k_g^{-1} - \frac{3}{2}k_g' k_n k_g^{-2} - \frac{1}{2}k_g'' k_g^{-3/2} + \\ \frac{3}{2}k_g^{-5/2}(k_g')^2 + 2k_n' k_g^{-1} - k_n k_g^{-2} k_g' + k_g'' k_g^{-2} - 2(k_g')^2 k_g^{-3} \end{pmatrix} ds$$
$$-2\mu(l)(k_n(l)k_g^{-1}(l) + k_g'(l)k_g^{-2}(l) - \frac{1}{4}k_g'(l)k_g^{-3/2}(l)) + \mu'(l)k_g^{-1/2}(l)$$

$C'(0) = 0$ for all choices of the function $\mu(s)$ satisfying (5), with arbitrary values of $\mu(l)$ and $\mu'(l)$, the given null arc ξ must satisfy two boundary conditions and differential equation

$$(1) = k_g^{-1/2}(l) = 0$$
$$(2) = k_g'(l) = -k_n(l)k_g(l)$$
$$(3) \quad \begin{matrix} \tau_g - k_g'' k_g^{-2} + k_n' k_g^{-1} + \frac{1}{2}k_n^2 k_g^{-1} - \frac{3}{2}k_g' k_n k_g^{-2} - \frac{1}{2}k_g'' k_g^{-3/2} + \\ \frac{3}{2}k_g^{-5/2}(k_g')^2 + 2k_n' k_g^{-1} - k_n k_g^{-2} k_g' + k_g'' k_g^{-2} - 2(k_g')^2 k_g^{-3} = 0 \end{matrix}.$$

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