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Asymptotic Behaviour of the Solutions of the Falkner-Skan Equations Governing the Swirling Flow

J. Singh

Department of Civil Engineering, Institute of Technology, Banaras Hindu University, Varanasi – 221005 (U.P.), India

B. B. Singh and I. M. Chandarki

Department of Mathematics, Dr. Babasaheb Ambedkar Technological University
Lonere – 402103, Dist. Raigad (M.S.), India
brijbhansingh@yahoo.com, imran_2204@yahoo.co.in

Abstract

Here the asymptotic nature, as the independent similarity variable tends to infinity, of the solutions of the equations governing the swirling flow in a laminar compressible boundary layer over an axi-symmetric surface with variable cross-section has been studied; the results being based on the asymptotic integrations of second order linear differential equations.

Keywords: Asymptotic behaviour, principal and linearly independent Solutions, injection rates, pressure gradient, swirling flow, longitudinal acceleration, enthalpy difference ratio

1 Introduction

Mass addition is an interesting phenomenon of boundary layer blow off where large amounts of fluid are added into the boundary layer filling the region near the wall and causing significant alterations in the profiles of variables. On account of this fact, for large rates of injection the boundary layers are characterized by (i) an inner layer close to the surface where viscous forces are negligible and (ii) a relatively thin outer viscous layer in which transition from the inner to the inviscid external flow takes place.

Due to smaller pressure gradients and greater fields of integration, the usual shooting methods of handling these problems fail for large blowing parameters. This is due to the poor convergence and grater instability of the methods used.

Since the prediction of massive blowing on slender aerodynamic bodies is of technological significance, hence problems of such type have been studied by a number of investigators [4-12] and [15-17]. These workers have used the methods of matched asymptotic expansions, inverse methods and numerical methods obtained by combining the finite difference technique with quasi-linearization.

The objective of the present paper lies in the study of the asymptotic behaviours, as the independent similarity variable tends to infinity, of the solutions of the Falkner – Skan equations which govern the above problem for large rates of injection. The results are based on a significant method, namely the asymptotic integrations of second order linear differential equations, of finding solutions of second order linear differential equations for very large (approaching infinity) values of the independent variables. In the present paper, we have considered the acceleration parameter lying in the range $1 \le \bar{\beta} \le 1$ and the case $\bar{\beta} < 0$ has been disposed of. It is due to the fact that the latter case is not of practical significance as regards the present problem where very high velocities of flow are required to be attained with the help of large injection rates. Though the case $\bar{\beta} < 0$ is of mathematical significance, but the study of existence and uniqueness of the solutions of the Falkner-Skan equations governing the present problem for this case is full of complications and various assumptions are required to be imposed on $\bar{\alpha}$ and $\bar{\beta}$. There are even certain assumptions which are less likely to occur in actual practice. On account of this fact, it is by far the best to consider the case $\bar{\beta} > 0$.

2 Analysis

The similarity equations governing the low-speed swirling laminar compressible boundary layer flow of a perfect gas with density ρ, constant specific heat C_p, viscosity μ proportional to temperature T and Prandtl number unity caused by free vortex on the longitudinal flow over an axi-symmetric surface of radius r with large injection at the surface are [7]

$$f''' + ff'' \bar{\beta}[G(1-g_w) + g_w - f'^2] + \bar{\alpha}[G(1-g_w) + g_w - G^2] = 0 \qquad (1)$$

$$G'' + fG' = 0 \qquad (2)$$

under the boundary conditions

$$f(0) = f_w(\leq -1), \quad f'(0) = 0, \quad f'(\infty) = 1 \tag{3}$$

$$G(0) = 0, \quad G(\infty) = 1 \tag{4}$$

Here f is a dimensionless stream function defined in such a way that $f' = u/u_e$ where u and u_e are the longitudinal velocity components in ε-direction inside and outside the boundary layer respectively. G stands for both the normalized swirl velocity component and enthalpy difference ratio and is given by

$$G = v/v_e = (g - g_w)/(1 - g_w) \tag{5}$$

where v is the swirl velocity in the η- direction,

$$g = H/H_e \quad , \quad H = c_p T + (u^2 + v^2)/2 \tag{6}$$

and suffixes w and e denote values at the wall and the edge of the boundary layer respectively. Also $\bar{\alpha}$, $\bar{\beta}$, g_w and f_w are the swirl, longitudinal acceleration, wall temperature and mass transfer parameter respectively and are given by

$$\bar{\alpha} = \frac{2X}{r}\left(-\frac{dr}{dX}\right)\left(\frac{v_e}{u_e}\right)^2 \frac{H_e}{h_e} \quad , \quad \bar{\beta} = \frac{2X}{u_e}\left(-\frac{du_e}{dX}\right)\left(\frac{H_e}{h_e}\right),$$

$$g_w = \frac{H_w}{H_e}, \qquad f_w = \frac{-\rho_w W_w (2X)^{1/2}}{r\rho_e \mu_e u_e} \tag{7}$$

$$X = \int_0^{\xi} \rho_e \mu_e u_e r^2 d\xi, \qquad h = c_p T$$

Here w is the velocity component normal to the surface in ξ-direction. Primes denote differentiation w.r.t. the independent similarity variable Z defined by

$$Z = \left(r\rho_e u_e/(2X)^{1/2}\right) \cdot \int_0^{\xi} (\rho/\rho_e) d\xi \tag{8}$$

The equations (1), (3) can as well be written as

$$f''' + (f + f_w) f'' + \bar{\beta}\left(1 - f'^2\right) + \bar{\alpha}\left(1 - G^2\right) = 0 \tag{9}$$

under the boundary conditions

$$f(0) = f'(0) = 0, \quad f'(\infty) = 1 \tag{10}$$

In the above equation, we have taken $g_w = 1$ to avoid complication in the discussion of the problem. Moreover, the above form of the equation (1) shall help us to study the effect of large injection rates on the flow field.

If $f(Z)$ be the solution of the equation (9), let us put

$$h = 1 - f' \tag{11}$$

in it to obtain

$$h'' + (f + f_w) h' - \bar{\beta}(1 + f')\left[1 + \left(\bar{\alpha}/\bar{\beta}\right)\left(1 - G^2/1 - f'^2\right)\right] h = 0 \tag{12}$$

To remove the middle term in (12), let us substitute

$$h = k\exp\left(-1/2\right)\int_0^Z (f + f_w)ds \tag{13}$$

in it to obtain

$$k'' - Q(Z)k = 0 \tag{14}$$

where $Q(Z) = \frac{1}{2}f' + \frac{1}{4}(f + f_w)^2 + \bar{\beta}((1 + f')\left[1 + \bar{\alpha}\left(\frac{1-G^2}{1-f'^2}\right)\right]$

Since $Q(Z)$ is a continuous complex valued function for $Z \geq 0$ satisfying $\int^\infty Z^{2l-1}|Q(Z)|^l\, dZ < \infty$ for some l on the range $1 \leq l \leq 2$, hence following ([14], p. 384) the equation (14) shall have a solution satisfying

$$k \sim \exp\left(-\int^Z sQ(s)ds\right), \quad k'/k = o(1/Z) \tag{16}$$

and a solution satisfying

$$k \sim Z\exp\int^Z sQ(s)ds \quad \text{and} \quad k'/k \sim 1/Z, \qquad \text{as} \quad Z \to \infty \tag{17}$$

Using (15) in (16) and (17), we have in view of (13) that there exists a solution satisfying

$$h \sim \exp\left(-\int^Z\left[\frac{1}{2}(f + f_w) + I(s)\right]ds\right), \tag{18a}$$

$$h' \sim \left[o\left(\frac{1}{Z}\right) - \frac{1}{2}(f + f_w)\right]\exp\left(-\int^Z\left[\frac{1}{2}(f + f_w) + I(sa)\right]ds\right) \tag{18b}$$

and a solution satisfying

$$h \sim Z\exp\left(\int^Z\left\{-\frac{1}{2}(f + f_w) + I(s)\right\}ds\right), \tag{19a}$$

$$h' \sim \left[1 - \frac{Z}{2}(f + f_w)\right]\exp\left(\int^Z\left\{-\frac{1}{2}(f + f_w) + I(s)\right\}ds\right), \quad \text{as} \quad Z \to \infty \tag{19b}$$

where $I(Z) = Z\left[\frac{1}{2}f' + \frac{1}{4}(f + f_w)^2 + \bar{\beta}(1 + f')\left\{1 + \frac{\bar{\alpha}}{\bar{\beta}}\left(\frac{1-G^2}{1-f'^2}\right)\right\}\right]$

Since $f' \sim 1$ as $Z \to \infty$, hence $f \sim Z + Constant$,or that $f \sim Z$ as $Z \to \infty$.Substituting this into (18a) and (18b), we have

$$1 - f' \sim \exp\left(-\left[\frac{(Z + f_w)^2}{4} + J(Z)\right]\right) \tag{20a}$$

$$f'' \sim \left[\frac{1}{2}(Z + f_w) - O(1)\right]\exp\left(-\left[\frac{(Z + f_w)^2}{4} + J(Z)\right]\right) \tag{20b}$$

and into (19a) and (19b),we have

$$1 - f' \sim Z\exp\left(-\left[\frac{(Z + f_w)^2}{4} + J(Z)\right]\right) \tag{21a}$$

$$f'' \sim \left[\frac{Z}{2}(Z + f_w) - 1\right]\exp\left(-\left[\frac{(Z + f_w)^2}{4} + J(Z)\right]\right) \quad \text{as} \quad Z \to \infty \tag{21b}$$

where $J(Z) = \frac{Z^4}{16} + \frac{f_w Z^3}{6} + \left(\frac{1}{4} + \bar{\beta} + \frac{f_w^2}{8}\right)Z^2$

We now substitute

$$\tau = 1 - G \tag{22}$$

in (2) to obtain

$$\tau'' + f\tau' = 0 \tag{23}$$

under the boundary conditions

$$\tau(0) = 1, \ \tau(\infty) = 0 \tag{24}$$

The asymptotic behaviours of (23) and (24) have earlier been discussed by the authors [15], hence they do not need any thorough investigation here. The results obtained in the above paper are only required here to study the asymptotic nature of the solutions in the present set-up.

The authors [2] investigated that (2), (4) have principal solutions satisfying, as $Z \to \infty$,

$$1 - G \sim c_0 Z^{-1}\exp\left(-\frac{Z^2}{2} - c_1 Z\right), \qquad G' \sim Z(G - 1) \tag{25}$$

and linearly independent solutions satisfying, as $Z \to \infty$,

$$1 - G \sim c_0 Z^0, \qquad G' \approx 0$$

where $c_0 > 0$ and c_1 are the constants.

3 Results and Discussions

In calculus, the nature of the solutions of the differential or differential – difference equations is studied as the independent variable tends to infinity. This property is completely fulfilled by the study of the asymptotic nature of the solutions. A solution which tends to zero or to an infinitesimal limit as the independent variable tends to infinity is said to exhibit asymptotic nature, or is said to be asymptotically stable. It is also of much more practical significance in boundary value problems and hence has been studied by a number of workers [1-3] and [13-14]. On the contrary, those solutions which do not behave in the above fashion are said to be asymptotically unstable.

The criteria over which our entire discussion is based are $\lim_{Z\to\infty}(1-f')=0$, $\lim_{Z\to\infty} f''=0$, $\lim_{Z\to\infty}(1-G)=0$ and $\lim_{Z\to\infty} G'=0$. These criteria require a little clarification. The second criterion can be derived from the following theorem:

Theorem: Given that $\bar{\alpha}>0$, $0<\bar{\beta}\le 1$ there exists a unique solution $f(Z)$ of (9), (10) such that $f''>0$ on $(0,\infty)$ and $\lim_{Z\to\infty} f''(Z)=0$.

The proof of the above theorem follows from [13] and from ([14],p.521). Only stylistic changes are required to be carried out.

The third criterion is obvious, for $\lim_{Z\to\infty} G=1$. The last criterion follows from the fact that $G'=e^{-\int_0^Z f(s)ds}\Big/\int_0^\infty e^{\int_0^Z f(s)ds}ds$

Since $f\sim Z$ as $Z\to\infty$, hence $G'\sim e^{-Z^2}\big/2\sqrt{\pi/2}$ as $\int_0^\infty\left(e^{-Z^2}\big/2\right)dz=\sqrt{\pi/2}$ for large Z. Therefore, $G'\to 0$ *as* $Z\to\infty$.

We impose these criteria on the LHS' of the solutions and observe whether RHS' of the solutions are also tending to the same limits or not. If the answer is affirmative, they will show asymptotic nature as $Z\to\infty$, and on the contrary not.

Since large injection rates have overwriting influence in aeronautical engineering where very high velocities of the flight are attained in modern times, hence our major concern is to study the effect of $f_w(\le -1)$ on velocity profiles and enthalpy difference ratio.

Here we see that the LHS' of the asymptotic relations (20a) and (20b) tend to zero as $Z\to\infty$ and the RHS' are also approaching the same limit (i.e. zero). Hence these relations shall hold true as $Z\to\infty$. Contrary to it, the relations (21a) and (21b) do not exist as $Z\to\infty$, for their RHS' do not tend to the limit zero to which the LHS' tend.

Here on noticeable fact is that the swirl parameter $\bar{\alpha}$ and longitudinal acceleration $\bar{\beta}$ do not influence the asymptotic nature of the solutions, for they are absent in them. Second important thing is that even if we take $f_w=0$ or $f_w>0$ (suction), the asymptotic nature of the solutions is the same as in the case $f_w\le -1$ (large injection).

Finally, we see that (25) shall exist as Z→∞, whereas (26) shall not.

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