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A Note on Proper Affine Symmetry in Bianchi Types VI_0 and VII_0 Space-Times

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Abstract

A study proper affine symmetry in the most general form of the Bianchi types VI_0 and VII_0 space-times is given by using holonomy and decomposability, the rank of the 6×6 Riemann matrix and direct integration techniques. It is shown that the very special classes of the above space-times admit proper affine symmetry.

Keywords: Affine vector fields, holonomy and decomposability, direct integration technique.

1 INTRODUCTION

In this paper we will explore all the possiblities when the Bianchi types VI_0 and VII_0 space-times admit proper affine symmetry. We use holonomy and decomposability, the rank of the 6×6 Rieman matrix and direct integration techinques to study proper affine symmetry in the above space-times. Throughout

M represents a four dimensional, connected, Hausdorff space-time manifold with Lorentz metric g of signature (-, +, +, +). The curvature tensor associated with g_{ab}, through the Levi-Civita connection, is denoted in component form by $R^a{}_{bcd}$. The usual covariant, partial and Lie derivatives are denoted by a semicolon, a comma and the symbol L, respectively. Round and square brackets denote the usual symmetrization and skew-symmetrization, respectively. Here, M is assumed non-flat in the sense that the curvature tensor does not vanish over any non-empty open subset of M.

A vector field X on M is called an affine vector field if it satisfies

$$X_{a;bc} = R_{abcd}\, X^d, \qquad (1)$$

where $R_{abcd} = g_{af} R^f{}_{bcd} = g_{af}(\Gamma^f_{bd,c} - \Gamma^f_{bc,d} + \Gamma^f_{ce}\Gamma^e_{bd} - \Gamma^f_{ed}\Gamma^e_{bc})$. If one decomposes $X_{a;b}$ on M into its symmetric and skew-symmetric parts

$$X_{a;b} = \frac{1}{2}H_{ab} + G_{ab}, \ (H_{ab}(\equiv X_{a;b} + X_{b;a}) = H_{ba}, \ G_{ab} = -G_{ba}) \qquad (2)$$

then equation (1) is equivalent to

$$(i)\ H_{ab;c} = 0 \quad (ii)\ G_{ab;c} = R_{abcd}\, X^d \quad (iii)\ G_{ab;c} X^c = 0. \qquad (3)$$

The proof of the above equation (1) implies (3) or equations (3) implies (1) can be found in [2, 3]. If $H_{ab} = 2cg_{ab}, c \in R,$ then the vector field X is called homothetic (and *Killing* if $c = 0$). The vector field X is said to be proper affine if it is not homothetic vector field and also X is said to be proper homothetic vector field if it is not Killing vector field on M [2]. Define the subspace S_p of the tangent space T_pM to M at p as those $k \in T_pM$ satisfying

$$R_{abcd} k^d = 0. \qquad (4)$$

2 Affine Vector Fields

Suppose that M is a simple connected space-time. Then the holonomy group of M is a connected Lie subgroup of the idenity component of the Lorentz group and is thus characterized by its subalgebra in the Lorentz algebra. These have been labeled into fifteen types $R_1 - R_{15}$ [1]. It follows from [2] that the only such space-times which could admit proper affine vector fields are those which admit nowhere zero covariantly constant second order symmetric tensor field H_{ab}. This forces the holonomy type to be either R_2, R_3, R_4, R_6, R_7, R_8, R_{10}, R_{11} or R_{13} [2]. A study of the affine vector fields for the above holonomy type can be found in [2]. It follows from [4] that the rank of the 6×6 Riemann matrix of the above space-times which have holonomy type R_2, R_3, R_4, R_6, R_7,

R_8, R_{10}, R_{11} or R_{13} is at most three. Hence for studying affine vector fields we are interested in those cases when the rank of the 6×6 Riemann matrix is less than or equal to three.

3 Main Results

Consider Bianchi types VI_0 and VII_0 space-times in usual coordinate system (t,x,y,z) (labeled by (x^0,x^1,x^2,x^3), respectively) with line element [5]

$$ds^2 = -dt^2 + \left(A\, f^2(z) + B\, h^2(z)\right)dx^2 + \left(A\, h^2(z) + B\, f^2(z)\right)dy^2 + 2(A+B)f(z)h(z)dx\,dy + C(t)dz^2, \tag{5}$$

where A, B and C are nowhere zero functions of t only. For $f(z)=\cosh z$, $h(z)=\sinh z$ or $f(z)=\cos z$, $h(z)=\sin z$ the above space-time (5) becomes Bianchi type VI_0 or VII_0, respectively. The above space-time admits three linearly independent Killing vector fields which are

$$\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, -y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + \frac{\partial}{\partial z}. \tag{6}$$

The non-zero components of the Riemann tensor are

$$R_{0101} = -\frac{1}{4}\left[\left(\frac{2A\ddot{A}-\dot{A}^2}{A}\right)f^2(z) + \left(\frac{2B\ddot{B}-\dot{B}^2}{B}\right)h^2(z)\right] = \alpha_1,$$

$$R_{0102} = -\frac{1}{4}\left[\left(\frac{2A\ddot{A}-\dot{A}^2}{A}\right) + \left(\frac{2B\ddot{B}-\dot{B}^2}{B}\right)\right] f(z)h(z) = \alpha_7,$$

$$R_{0113} = R_{0223} = -\frac{1}{4ABC}\left[A^2\dot{B}C + \dot{A}B^2C + 2AB\dot{C}(A+B) - 3ABC\left(\dot{A}+\dot{B}\right)\right]f(z)h(z) = \alpha_8,$$

$$R_{0123} = -\frac{1}{4}\left[\left\{(A+B)\left(\frac{\dot{A}}{A}+\frac{\dot{C}}{C}\right) - 2\left(\dot{A}+\dot{B}\right)\right\}h^2(z) + \left\{(A+B)\left(\frac{\dot{B}}{B}+\frac{\dot{C}}{C}\right) - 2\left(\dot{A}+\dot{B}\right)\right\}f^2(z)\right]$$

$$= \alpha_9,\ R_{0202} = -\frac{1}{4}\left[\left(\frac{2A\ddot{A}-\dot{A}^2}{A}\right)h^2(z) + \left(\frac{2B\ddot{B}-\dot{B}^2}{B}\right)f^2(z)\right] = \alpha_2,$$

$$R_{0213} = -\frac{1}{4}\left[\left\{(A+B)\left(\frac{\dot{A}}{A}+\frac{\dot{C}}{C}\right) - 2\left(\dot{A}+\dot{B}\right)\right\}f^2(z) + \left\{(A+B)\left(\frac{\dot{B}}{B}+\frac{\dot{C}}{C}\right) - 2\left(\dot{A}+\dot{B}\right)\right\}h^2(z)\right]$$

$$= \alpha_{10},\ R_{0303} = -\frac{1}{4}\left(\frac{2C\ddot{C}-\dot{C}^2}{C}\right) = \alpha_3,\ R_{0312} = -\frac{1}{4}(A+B)\left(\frac{\dot{A}B-A\dot{B}}{AB}\right) = \alpha_{11},$$

$$R_{1212} = \frac{1}{4ABC}\left[(A+B)^2 + \dot{A}\dot{B}C\right]\left[Af^2(z) + Bh^2(z)\right]\left[Ah^2(z) + Bf^2(z)\right] = \alpha_4,$$

$$R_{1313} = \frac{1}{4AB}\left[\left\{A^3 - AB\left(2A+3B-\dot{A}\dot{C}\right)\right\}f^2(z) - \left\{AB\left(3A+2B-\dot{B}\dot{C}\right) - B^3\right\}h^2(z)\right] = \alpha_5,$$

$$R_{1323} = \frac{1}{4AB}\left[A^2(A-5B) + B^2(B-5A) + AB\dot{C}\left(\dot{A}+\dot{B}\right)\right]f(z)h(z) = \alpha_{12},$$

$$R_{2323} = \frac{1}{4AB}\left[\left\{A^3 - AB\left(2A+3B-\dot{A}\dot{C}\right)\right\}h^2(z) - \left\{AB\left(3A+2B-\dot{B}\dot{C}\right) - B^3\right\}f^2(z)\right] = \alpha_6.$$

Writing the curvature tensor with components R_{abcd} at p as a 6×6 symmetric matrix [6]

$$R_{abcd} = \begin{pmatrix} \alpha_1 & \alpha_7 & 0 & 0 & \alpha_8 & \alpha_9 \\ \alpha_7 & \alpha_2 & 0 & 0 & \alpha_{10} & \alpha_8 \\ 0 & 0 & \alpha_3 & \alpha_{11} & 0 & 0 \\ 0 & 0 & \alpha_{11} & \alpha_4 & 0 & 0 \\ \alpha_8 & \alpha_{10} & 0 & 0 & \alpha_5 & \alpha_{12} \\ \alpha_9 & \alpha_8 & 0 & 0 & \alpha_{12} & \alpha_6 \end{pmatrix}. \quad (7)$$

As mentioned in section 2, the space-times which can admit proper affine vector fields have holonomy type R_2, R_3, R_4, R_6, R_7, R_8, R_{10}, R_{11} or R_{13} and the rank of the 6×6 Riemann matrix is at most three. Therefore we are only interested in those cases when the rank of the 6×6 Riemann matrix is less than or equal to three. Hence there exist following two possibilities when the rank of the 6×6 symmetric matrix is less or equal to three which are:

(A) Rank=3, $\alpha_1=\alpha_2=\alpha_3=\alpha_7=\alpha_8=\alpha_9=\alpha_{10}=\alpha_{11}=0,$ $\alpha_4\neq 0, \alpha_5\neq 0, \alpha_6\neq 0$ and $\alpha_{12}\neq 0.$

(B) Rank=1, $\alpha_4\neq 0,$ and

$\alpha_1=\alpha_2=\alpha_3=\alpha_5=\alpha_6=\alpha_7=\alpha_8=\alpha_9=\alpha_{10}=\alpha_{11}=\alpha_{12}=0.$

We will consider each case in turn.

Case (A):

In this case we have $\alpha_4\neq 0, \alpha_5\neq 0, \alpha_6\neq 0, \alpha_{12}\neq 0,$

$\alpha_1=\alpha_2=\alpha_3=\alpha_7=\alpha_8=\alpha_9=\alpha_{10}=\alpha_{11}=0,$ the rank of the 6×6 Riemann matrix is three and there exists a unique (up to a multiple) no where zero time like vector field $t_a=t_{,a}$ solution of equation (4) and $t_{a;b}\neq 0.$ From the above constraints we have $B(t)=d\,A(t),$ $C(t)=eA(t)$ and $A(t)=(at+b)^2,$ where $a,b,d,e\in R(d,e>0).$ The line element in this case takes the form

$$ds^2=-dt^2+(a\,t+b)^2\left[\begin{matrix}\left(f^2(z)+e\,h^2(z)\right)dx^2+\left(d\,h^2(z)+f^2(z)\right)dy^2+\\ 2(1+d)f(z)h(z)dx\,dy+e\,dz^2\end{matrix}\right]. \quad (8)$$

Substituting the above information into the affine equations one find that

$$X^0=c_4t+c_5,\; X^1=-c_1y+c_2,\; X^2=-c_1x+c_3,\; X^3=c_1, \quad (9)$$

where $c_1,c_2,c_3,c_4,c_5\in R.$ One can write the above equation (9) subtracting Killing vector fields as

$$X=(c_4t+c_5,0,0,0). \quad (10)$$

Clearly in this case one can easily see that the above space-times (8) admit proper affine symmetry. It is important to note that the constants a and b can not be zero simultaneously.

Case (B):
In this case we have
$\alpha_1 = \alpha_2 = \alpha_3 = \alpha_5 = \alpha_6 = \alpha_7 = \alpha_8 = \alpha_9 = \alpha_{10} = \alpha_{11} = \alpha_{12} = 0, \quad \alpha_4 \neq 0$ and the rank of the 6×6 Riemann matrix is one. Here, there exist two linear independent solutions $t_a = t_{,a}$ and $z_a = z_{,a}$ of equation (4). The vector field t_a is covariantly constant whereas z_a is not covariantly constant. From the above constraints we have $A(t) = B(t) = C(t) = (t+q)^2$, where $q \in R$. The line element takes the form

$$ds^2 = -dt^2 + (t+q)^2 \begin{bmatrix} \left(f^2(z) + h^2(z)\right)dx^2 + \left(h^2(z) + f^2(z)\right)dy^2 + \\ 4f(z)h(z)dx\,dy + dz^2 \end{bmatrix}. \quad (11)$$

Affine vector fields in this case

$$X^0 = c_4 t + c_5 z + c_6,\; X^1 = -c_1 y + c_2,\; X^2 = -c_1 x + c_3,\; X^3 = c_1, \quad (12)$$

where $c_1, c_2, c_3, c_4, c_5, c_6 \in R.$ One can write the above equation (12) subtracting Killing vector fields as

$$X = (c_4 t + c_5 z + c_6, 0, 0, 0). \quad (13)$$

In this case clearly the above space-times (11) admit proper affine symmetry.

SUMMARY

In this paper an attempt is made to explore all the possibilities when the Bianchi types VI_0 and VII_0 space-times admit proper affine vector fields. A different approach is adopted to study proper affine vector fields of the above space-times by using holonomy and decomposability, the rank of the 6×6 Riemann matrix and direct integration techniques. From the above study we obtain the following results:

(i) We obtain the space-time (8) that admits proper affine vector fields when the rank of the 6×6 Riemann matrix is three and there exists a unique nowhere zero independent timelike vector field, which is the solution of equation (4) and is not covariantly constant (for details see case A).

(ii) The space-time (11) is obtained, which admits proper affine vector fields (see case B) when the rank of the 6×6 Riemann matrix is one and there exist two independent solutions of equation (4) but only one independent covariantly constant vector field.

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