Optimal Control of Reverse Logistics Model with Logistic Demand and Return Rates

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Abstract

Pontryagin minimum principle (PMP) is applied in this paper to find the optimal control of the reverse logistics model of a production inventory system with deteriorating items. The total cost of the system consists of the sum of holding costs of inventory levels in two different stores, the manufacturing and remanufacturing costs, and the disposal cost. The general solution of the controlled system is obtained. Finally, numerical example are presented to explain the analytical solution.

1   Introduction

Reverse logistic is a term for logistic environments with reuse of products and materials. In these systems, products are manufactured and the returned units from market are remanufactured and are considered to be new. The demand is to be satisfied with new manufactured and remanufactured products. So, there is no difference between manufactured and remanufactured items. The operation of such a system is described by the return process of the used products and the disposal activity. In order not to pollute the environment with waste, it is wised to recycle used items if it is economical. Reuse products and materials is not new. Metal scrap brokers, waste paper recycling and deposit systems for soft drink bottles are all examples that have been around for a long time. In these instances, recovery of the used products is economically more attractive than disposal.

Many authors have discussed the reverse logistics models such Al-Gohary and others [1] dealt with HMMS-model with constant, linear and periodic demand and return rates. Dobos [4] who has investigated optimal inventory policies in a reverse logistic system with special structure. He assumed that demand is known as a continuous function in a given planning horizon, and return rate

of used items is a given function. His model is considered as a generalization of the model of Holt et al. [7]. Dobos and Kistner [3] have presented optimal production-inventory strategies for a reverse logistics system where the costs consist of linear holding costs for the two stores and the convex non-decreasing manufacturing and remanufacturing costs, and there is no delay between using process and return process. The result of their study is that the optimal manufacturing-remanufacturing strategy is extreme, which means that the optimal trajectory contains always constraints either on the inventory levels or on the control variables. A similar result was shown by Kleber et al. [8] for a multiple reuse model with linear cost structure. They have examined a reverse logistic model without disposal and have generalized the convex model with disposal activity. Their model can be interpreted as a multi-product generalization of the Arrow-Karlin model (1958) [2]. After having characterized the optimal path, a forward algorithm was given by Kistner and Dobos [3] to construct the optimal manufacturing-inventory control system. Fleischmann et al. [6] has presented the recently emerged field of reverse logistics. Dobos [5] has analyzed the effects of tradable permits (emission trading or environmental license) on production and inventory.

In this paper, we will be concerned with optimal control of reverse logistics for a production inventory system with deteriorating items. The results of this study complement and extend recent work by Al-Gohary and others [1].

2   The problem

The production inventory system consists of two stores and demand is satisfied from the first store, where the manufactured items are stored. The returned products are collected in the second store and then remanufactured or disposed off. The costs of this system consist of the holding costs for these two stores and the manufacturing cost, the remanufacturing cost and the disposal cost. The model is represented as an optimal control problem with two state variables which are: the inventory levels in the first store and the second store and with three control variables which are: the manufacturing, remanufacturing and disposal rates. The objective is to minimize the sum of the quadratic deviation from described inventory levels in two stores and from described manufacturing, remanufacturing and disposal rates. The numerical example include the logistic demand and return rates as a functions of time.

In this model, We assume that the deterioration rates in two stores are constant ratios of the on-hand inventory levels. Also, the inventory goal levels in two stores, the goal manufacturing rate, the goal remanufacturing rate and the goal disposal rate are constants. There is a constant delay between the using and return process.

The following parameters are used in the mathematical formulation of the

model:

T	:	Length of the planning period,
$S(t)$	:	Rate of demand at time t,
τ	:	Delay of return, non-negative ,
r	:	Proportion of return in store(2), $0 \leq r \leq 1$,
$R(t)$	:	Rate of return at time t,
$\bar{x}_1$	:	Inventory goal level in store(1),
$\bar{x}_2$	:	Inventory goal level in store(2),
$\bar{P}_m$	:	Manufacturing goal rate,
$\bar{P}_r$	:	Remanufacturing goal rate,
$\bar{P}_d$	:	Disposal goal rate,
h_1	:	Inventory holding cost coefficient in the first store,
h_2	:	Inventory holding cost coefficient in the second store,
c_m	:	Manufacturing cost,
c_r	:	Remanufacturing cost,
c_d	:	Disposal cost,
θ_1	:	Deterioration rate in first store
θ_2	:	Deterioration rate in second store
$x_1(t)$	:	Inventory level in the first store at time t,
$x_2(t)$	:	Inventory level in the second store at time t,
$P_m(t)$	:	Rate of manufacturing at time t,
$P_r(t)$	:	Rate of remanufacturing at time t,
$P_d(t)$	:	Rate of disposal at time t.

The inventory goal levels are safety stock levels that the company wants to keep on hand. Also, the manufacturing, remanufacturing and disposal goal rates are the most efficient rates at which they are described to run the factory.

2.1 Mathematical form of the problem

The main objective of this section is to Describe the mathematical model of the problem and Derive the optimal inventory levels, the optimal manufacturing and remanufacturing rates, and the optimal disposal rate in the case of finite time.

The optimal control problem is to determine the manufacturing, remanufacturing and disposal rates which minimize the following total cost:

$$J = \min_{P_m, P_r, P_d \geq 0} \frac{1}{2} \int_0^T \left\{ h_1(x_1 - \bar{x}_1)^2 + h_2(x_2 - \bar{x}_2)^2 + c_m(P_m - \bar{P}_m)^2 + c_r(P_r - \bar{P}_r)^2 + c_d(P_d - \bar{P}_d)^2 \right\} dt,$$

(1)

subject to
$$\dot{x}_1 = P_m(t) + P_r(t) - S(t) - \theta_1 x_1(t),$$

$$\dot{x}_2 = -P_r(t) - P_d(t) + R(t) - \theta_2 x_2(t), \tag{2}$$

and
$$x_1(t) \geq 0, \ \ x_2(t) \geq 0, \ \ P_m(t) \geq 0, \ \ P_r(t) \geq 0, \ \ P_d(t) \geq 0, \tag{3}$$

where $h_1 > 0, \ h_2 > 0, \ c_m > 0, \ \ c_r > 0$ and $c_d > 0$

The economic interpretation of the objective function (1) is that we want to keep the inventory levels $(x_1(t), x_2(t))$ of the two stores as close as possible to their goal levels $(\bar{x}_1, \bar{x}_2)$ and the manufacturing, the remanufacturing, and disposal rates $(P_m(t), P_r(t), P_d(t))$ as close as possible to their goal rates $(\bar{P}_m, \bar{P}_r, \bar{P}_d)$, respectively.

The integral in (1) represents the total cost of the model, and the quadratic terms inside this integral represent the imposed penalties for having either $(x_1(t), x_2(t))$ or $(P_m(t), P_r(t), P_d(t))$ not being close to their corresponding goal levels $(\bar{x}_1, \bar{x}_2)$ and goal rates $(\bar{P}_m, \bar{P}_r, \bar{P}_d)$ respectively.

Then we can apply PMP to get the following system of linear differential equations.

$$\left. \begin{array}{l} \dot{x}_1 = -\theta_1 x_1(t) + \dfrac{c_m + c_r}{c_m c_r}\lambda_1(t) - \dfrac{\lambda_2(t)}{c_r} + \bar{P}_m + \bar{P}_r - S(t) \\[3mm] \dot{x}_2 = -\theta_2 x_2(t) + \dfrac{c_r + c_d}{c_r c_d}\lambda_2(t) - \dfrac{\lambda_1(t)}{c_r} - \bar{P}_d - \bar{P}_r + R(t) \\[3mm] \dot{\lambda}_1 = \theta_1 \lambda_1(t) + h_1(x_1(t) - \bar{x}_1) \\[3mm] \dot{\lambda}_2 = \theta_2 \lambda_2(t) + h_2(x_2(t) - \bar{x}_2) \end{array} \right\}, \tag{4}$$

We can put the system (4) in the form

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{\lambda}_1 \\ \dot{\lambda}_2 \end{pmatrix} = \begin{pmatrix} -\theta_1 & 0 & \dfrac{c_m + c_r}{c_m c_r} & -\dfrac{1}{c_r} \\[2mm] 0 & -\theta_2 & -\dfrac{1}{c_r} & \dfrac{c_r + c_d}{c_r c_d} \\[2mm] h_1 & 0 & \theta_1 & 0 \\[2mm] 0 & h_2 & 0 & \theta_2 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ \lambda_1(t) \\ \lambda_2(t) \end{pmatrix} + \begin{pmatrix} \bar{P}_m + \bar{P}_r - S(t) \\ -\bar{P}_d - \bar{P}_r + R(t) \\ -h_1 \bar{x}_1 \\ -h_2 \bar{x}_2 \end{pmatrix},$$

$$\tag{5}$$

with initial and terminal conditions

$$\begin{pmatrix} x_1(0) \\ x_2(0) \\ \lambda_1(T) \\ \lambda_2(T) \end{pmatrix} = \begin{pmatrix} x_{10} \\ x_{20} \\ 0 \\ 0 \end{pmatrix}.$$

Then, the equations (5) can be presented in the matrix form as:

$$\dot{X} = AX + F, \tag{6}$$

The optimal inventory levels:

$$
\begin{aligned}
x_1^*(t) &= \sum_{i=1}^{4} e^{\gamma_i t} \Big(\int_0^t e^{-\gamma_i s} (b_{i1}[\bar{P}_m + \bar{P}_r - S(s)] + \\
&\quad b_{i2}[-\bar{P}_d - \bar{P}_r + R(s)] - b_{i3} h_1 \bar{x}_1 - b_{i4} h_2 \bar{x}_2) ds + c_i \Big),
\end{aligned} \tag{7}
$$

$$
\begin{aligned}
x_2^*(t) &= \sum_{i=1}^{4} -\frac{-h_1(c_r + c_d + c_m) + c_m c_r \gamma_i^2 + c_m c_d \gamma_i^2 - c_m c_r \theta_1^2 - c_m c_d \theta_1^2}{c_m c_d(-\gamma_i \theta_1 - \theta_1 \theta_2 + \gamma_i^2 + \gamma_i \theta_2)} \\
&\quad e^{\gamma_i t} \Big(\int_0^t e^{-\gamma_i s} (b_{i1}[\bar{P}_m + \bar{P}_r - S(s)] + \\
&\quad b_{i2}[-\bar{P}_d - \bar{P}_r + R(s)] - b_{i3} h_1 \bar{x}_1 - b_{i4} h_2 \bar{x}_2) ds + c_i \Big),
\end{aligned} \tag{8}
$$

$$
\begin{aligned}
\lambda_1(t) &= \sum_{i=1}^{4} \frac{h_1}{\gamma_i - \theta_1} e^{\gamma_i t} \Big(\int_0^t e^{-\gamma_i s} (b_{i1}[\bar{P}_m + \bar{P}_r - S(s)] + \\
&\quad b_{i2}[-\bar{P}_d - \bar{P}_r + R(s)] - b_{i3} h_1 \bar{x}_1 - b_{i4} h_2 \bar{x}_2) ds + c_i \Big),
\end{aligned} \tag{9}
$$

$$
\begin{aligned}
\lambda_2(t) &= \sum_{i=1}^{4} -\frac{-h_1(c_r + c_m) + c_m c_r \gamma_i^2 - c_m c_r \theta_1^2}{c_m(\gamma_i - \theta_1)} \\
&\quad e^{\gamma_i t} \Big(\int_0^t e^{-\gamma_i s} (b_{i1}[\bar{P}_m + \bar{P}_r - S(s)] + \\
&\quad b_{i2}[-\bar{P}_d - \bar{P}_r + R(s)] - b_{i3} h_1 \bar{x}_1 - b_{i4} h_2 \bar{x}_2) ds + c_i \Big),
\end{aligned} \tag{10}
$$

where the arbitrary constants $c_i(i = 1, 2, 3, 4)$ are determined using initial conditions $x_1(0) = x_{10}, \ x_2(0) = x_{20}$ and terminal conditions $\lambda_1(T) = 0, \ \lambda_2(T) = 0$. These constraints should enable us to find the optimal solution to our problem.

The optimal manufacturing, remanufacturing, and disposal rates :

$$P_m^*(t) = \max\left[0, \ \frac{\lambda_1(t)}{c_m} + \bar{P}_m\right], \tag{11}$$

$$P_r^*(t) = \max\left[0, \ \frac{\lambda_1(t) - \lambda_2(t)}{c_r} + \bar{P}_r\right], \tag{12}$$

and

$$P_d^*(t) = \max\left[0, \ -\frac{\lambda_2(t)}{c_d} + \bar{P}_d\right], \tag{13}$$

3   Numerical examples

In this section, we present the numerical example for logistic demand and return rates

- $S(t) = w_1 + t(k_1 - t)$, $R(t) = \alpha + t(k_2 - t)$.
  Where w_1, w_2, k_1, k_2 and α are positive constants.

And compare these rates with another three rates:

1. Demand and return rates are constants:
   $S(t) = w_1$, $R(t) = \alpha$,

2. Demand and return rates are linear functions of time:
   $S(t) = w_1 + w_2 t$, $R(t) = rw_2(t - \tau) + \alpha$,

3. Demand and return rates are periodic functions of time:
   $S(t) = 1 + \sin(t)$, $R(t) = r[1 + \sin(t - \tau)] + \alpha$.

Table(1) presents the values of system parameters and the initial states which are used in the numerical example:

parameter	c_m	c_r	c_d	P_m	P_r	P_d	h_1	h_2	α	r
value	9	10	15	15	13	11	10	12	15	0.8
parameter	T	$\bar{x}_1$	$\bar{x}_2$	x_{10}	x_{20}	θ_1	θ_2	w_1	w_2	τ
value	5	40	25	10	15	0.02	0.03	5	1	1

We summarize the values of the optimal inventory levels, the optimal manufacturing rate, the optimal remanufacturing rate and the optimal disposal rate at the end of the planning horizon period, and the objective function in table(2):

The demand and return rates	$x_1^*(T)$	$x_2^*(T)$	$P_m^*(T)$	$P_r^*(T)$	$P_d^*(T)$	J^*
Constant	55	22	15	13	11	6467
Linear	52	23	15	13	11	5718
Periodic	59	23	15	13	11	7463

The results of logistic rates shown in table (3):

The demand and return rates	$x_1^*(T)$	$x_2^*(T)$	$P_m^*(T)$	$P_r^*(T)$	$P_d^*(T)$	J^*
Logistic $k_1 = k_2 = 1$	63	12	15	13	11	9270
Logistic $k_1 = k_2 = 5$	54	23	15	13	11	4843
Logistic $k_1 = k_2 = 10$	42	36	15	13	11	3951

Comparing the results obtained from table (2) and table (3) we found that:

- We get the minimum value of the total cost when use logistic rates at same period.

- The total costs are decreasing as the planning period increasing, also the optimal inventory level of store (1) decreasing while the optimal inventory level of the store (2) increasing.

4   Conclusions

- In this paper, The problem of the optimal control of reverse logistics model of a production inventory system with deteriorating items is studied.

- Optimal inventory levels, optimal manufacturing and remanufacturing rates, and optimal disposal rates are obtained by minimizing the total costs.

- Logistic rates of the demand and return rates it is preferred when we use long planning period.

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