

The Expansions Approach for Solving Cauchy Integral Equation of the First Kind

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Abstract

In this paper we expand the kernel of Cauchy integral equation of first kind as a series of Chebyshev polynomials of the second kind times some unknown functions. These unknown functions are determined by applying the orthogonality of the Chebyshev polynomial. Whereas the unknown function in the integral is expanded using Chebyshev polynomials of the first kind with some unknown coefficients. These two expansions in the integral can be simplified by the used of the property of orthogonality. The advantage of this approach is that the unknown coefficients are stability computed.

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1 Introduction

The Cauchy type singular integral equations of the form

$$\frac{1}{\pi} \int_{-1}^1 \frac{g(t)dt}{t-x} = f(x), \quad -1 < x < 1 \quad (1)$$

has four complete analytical solutions [5]. Consider the bounded solution of (1) i.e.

$$g(x) = \frac{1}{\pi} \sqrt{1-x^2} \int_{-1}^1 \frac{f(t)}{\sqrt{1-t^2}(x-t)} dt, \quad (2)$$

provided that

$$\int_{-1}^1 \frac{f(t)}{\sqrt{1-t^2}} dt = 0. \quad (3)$$

Many researchers constructed the quadrature formulas for evaluating the singular integral (2). Dagnino and Lamberti [4] proposed a numerical evaluation of the Cauchy principal value integrals based on local spline approximation operators. Dagnino and Santi [3] deliberated on the convergence of spline product quadratures for Cauchy principal value integrals. They also proposed the quadrature rules, based on the cubic spline interpolation for the numerical evaluation of Cauchy singular integrals and obtained an error bound [2]. Diethelm [7] proposed a method for the numerical evaluation of the weighted Hilbert transform over the entire real line, where the integral is taken in the sense of Cauchy principal value. Chakrabarti and Berghe [1] developed an approximate method for solving singular integral equations of the first kind, over a finite interval. Maleknejad and Arzhang [8] presented a Taylor-series expansion method for the class of Fredholm singular integro-differential equation with Cauchy kernel. They utilised used the truncated Taylor-series polynomial of the unknown function and transforms the integro-differential equation into an n th order linear ODE with variable coefficients. They used the orthogonal Legendre polynomials as a basis for finding the approximate solution of n th order differetial equation. method.

In this paper we expanded both kernel and unknown function in the Cauchy type singular integral equation in terms of Chebyshev polynomials. In Section 2 we expand the kernel in terms of Chebyshev polynomials of the second kind, with we consider some unknown functions as the coefficients. These unknown functions are determined using Galerkin method. We expand the unknown function in terms of Chebyshev polynomials of the second kind as well. In Section 3 we introduce some form of known function that its corresponding solution is exact. In Section 4 we show the accuracy and efficiency of the method by considering some problems.

2 The expansion of the kernel and the unknown function

The orthogonality of the Chebyshev polynomials of second kind is defined as [6]

$$\int_{-1}^1 \sqrt{1-t^2} U_i(t) U_j(t) dt = \frac{\pi}{2} \delta_{ij}, \quad i, j = 0, 1, \dots, \quad (4)$$

where δ_{ij} is Kronecker delta. It is also known that

$$\int_{-1}^1 \frac{\sqrt{1-t^2}}{x-t} U_i(t) dt = \pi T_{i+1}(x) \quad (5)$$

where $T_i(x)$ is Chebyshev polynomials of first kind.

Write the kernel and the unknown function of equation (1) as

$$\frac{1}{t-x} = k_0(x)U_0(t) + k_1(x)U_1(t) + \dots + k_n(x)U_n(t) + \dots, \quad (6)$$

$$g(t) = \sqrt{1-t^2}(c_0U_0(t) + c_1U_1(t) + \dots + c_nU_n(t) + \dots) \quad (7)$$

where $U_i(x)$ are Chebyshev polynomials of the second kind and $k_i(x)$ are unknown functions. Multiply both sides of Equation (6) by $U_i(x)$ and integrate over -1 and 1 , we obtain

$$\int_{-1}^1 \frac{U_i(t)\sqrt{1-t^2}}{t-x} dt = \sum_{j=0}^{\infty} k_j(x) \int_{-1}^1 \sqrt{1-t^2} U_j(t) U_i(t) dt, \quad i = 0, 1, \dots, n. \quad (8)$$

Applying (4), on the right hand side of Equation (8), and From (8)-(5) it follows that

$$k_i(x) = -2T_{i+1}(x). \quad (9)$$

Substitute (9) in Equation (6) gives

$$\frac{1}{t-x} = -2 \sum_{i=0}^{\infty} T_{i+1}(x) U_i(t). \quad (10)$$

Substitute (10) and (7) in (1) and use (4) to have

$$-(c_0T_1(x) + c_1T_2(x) + \dots + c_nT_{n+1}(x) + \dots) = f(x) \quad (11)$$

Apply Galerkin method in equation (11) and gives

$$c_i = -\frac{2}{\pi} \int_{-1}^1 \frac{f(x)T_{i+1}(x)}{\sqrt{1-x^2}} dx, \quad i = 0, 1, \dots \quad (12)$$

Thus the bounded approximate solution of Equation (1) is

$$g_n(t) = \sqrt{1-t^2} \sum_{j=0}^n c_j U_j(t),$$

where the unknown coefficients c_j are defined by (12).

3 Numerical results

If the known function, $f(x)$ be a polynomial of any degree, we can easily obtain the exact solution.

Now consider the known function $f(x) = \frac{x}{1+x^2}$. The exact solution of (1) is $g(x) = -\frac{\sqrt{1-x^2}}{(1+x^2)\sqrt{2}}$. Table 1 shows the absolute difference between exact and approximate solutions for $n = 5$ and $n = 10$ and indicate the efficiency of this approach. From the figures it is observed that the proposed method is good even when singular point x tends close to the end points of the interval $[-1, 1]$, and the convergence is very fast.

4 Conclusion

In this study we have used the expansion approach for Cauchy singular kernel and unknown function. Then using Galerkin method the unknown coefficients are defined. We have developed a method for finding the coefficients of the unknown function. These coefficients are independent. The accuracy of the solution can be increased by having more terms without recalculating the previous coefficients.

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x	$n = 5$	$n = 10$
-1	0.0	0.0
-0.9	0.00027949743	0.00000855674581
-0.8	0.002606264150	0.00001481942705
-0.7	0.00226587368	0.00000907120711
-0.6	0.00011190978	0.00001152424306
-0.5	0.00247428565	0.00001077421363
-0.4	0.003292247387	0.00001252148573
-0.3	0.002139841544	0.00000790398766
-0.2	0.000285484091	0.00001466607112
-0.1	0.00261895787	0.00000283749429
0	0.00357132205	0.00001554597368
0.1	0.00261895787	0.00000283749429
0.2	0.000285484091	0.00001466607112
0.3	0.002139841544	0.00000790398766
0.4	0.003292247387	0.00001252148573
0.5	0.00247428565	0.00001077421363
0.6	0.00011190978	0.00001152424306
0.7	0.00226587368	0.00000907120711
0.8	0.002606264150	0.00001481942705
0.9	0.00027949743	0.00000855674581
1	0.0	0.0

Table 1: Absolute errors for $n = 5$ and $n = 10$.

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