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Flow Generated by Double Vortices Part 1. Ouside Creeping Flows

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Abstract

This the first part of a two-part study of flow generated by a double vortices. In this paper, the two-dimensional flow outside a circular cylinder induced by a double vortices is studied. Analytical solutions for the flow field are obtained by application of the Fourier method. The streamline patterns are sketched for a number of special cases where the cylinder is either stationary or rotating about its own axis. In particular, some interesting flow patterns are observed in the parameter space which may have potential significance in studies of various flows. We also investigate into the way the streamline topologies change as the parameters are varied.

1 Introduction

Double-vortex problems involving application of point forces and torques are of considerable interest in continuum mechanics. These solutions can be used in the representation of solutions of more complicated and physically realizable problems. The vortices of creeping flow have been known from the works of Lorentz ref[15], El-Bashir ref[8] and Burgers ref[6]. Some elegant application of these vortices in three-dimensional boundary value problems may be found in the works of Batchelor ref[5], Blake ref[3], Blake and Chwang ref[4], among many others.

In the case of two-dimensional creeping flows, a study of singularity solutions has been the topic of many researchers. One of the interesting and unusual phenomena associated with the plane creeping flow is the Stokes paradox which is a consequence of the fact that there is no solution to the biharmonic equation that represents slow streaming flow past a finite body. The cause and resolution of this paradox was explained by Kaplun and Lagerstrom ref[13] and Proudman and Pearson ref[17]. However, Jeffery ref[12], showed that two cylinders of equal radius rotating with equal but opposite angular velocities produce a uniform flow at large distances. Jeffery's work was the catalyst for many investigations concerning locally generated two-dimensional creeping flows. Dorrepaal et al. ref[7] found a uniform stream in situations when a vortex is located in front of a circular cylinder. The vortex model has also been used in the stirring mechanism inside a corrugated boundary ref[10]. The potential flow vortices such as a source, a sink, a doublet, etc., when placed in front of a cylinder also produce a uniform flow at large distances as shown by Avudainayagam et al. ref[1]. Furthermore, the image solutions for a vortex is also used in the interpretation of the results for two cylinders rotating in a viscous fluid.

In section (3) we study creeping flows outside a circular cylinder generated by a double-vortex. The corresponding problem for creeping flow with single vortex in the presence of a circular as well as an elliptic cylinder was investigated by El-Bashir. In ref[8], the vortices were located at different distance from the cylinder. One of the main conclusions in ref[8] was that a uniform flow always exists for four points. However, this fact is not true for double vortices if the vortices either have unequal strengths or are located non equidistant from the cylinder. We illustrate this by choosing double-vortex of different strengths in the present study. In particular, we show that the vortices-cylinder combination produces interesting flow patterns in the presence of cylinder rotation. These interesting features of the flow fields do not seem to have been noticed before.

The paper is organized as follows. In section (2), we formulate the problem of creeping flow in two-dimensions in terms of stream function. In section (3), we provide the solution for a double vortices in the presence of a circular cylinder. Finally, main results of the paper are summarized in section (4).

2 Formulation of the problem

We consider the creeping flow of a viscous incompressible fluid past a finite circular cylinder of radius a. The governing equations are the linearized steady Navier-Stokes equations given by

$$\mu\nabla^2 u = \nabla p, \tag{1}$$

$$\nabla . u = 0. \tag{2}$$

Here u is the two-dimensional velocity vector with components (u_r, u_θ) in the radial and transverse directions (r, θ) , respectively, p the pressure, and μ the coefficient of viscosity of the fluid . It is well-known that the equations (1) and

(2) (in two-dimensions) when expressed in terms of stream function $\psi(r,\theta)$, reduce to

$$\nabla^4\psi = 0, \tag{3}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

The velocity components in r and θ directions are given by

$$u_r = -\frac{1}{r}\frac{\partial\psi}{\partial\theta}, \tag{4}$$

$$u_\theta = \frac{\partial\psi}{\partial r}. \tag{5}$$

We assume that the cylinder is impenetrable and there is no slip on the surface. In terms of stream function these boundary conditions become

$$\psi = \frac{\partial\psi}{\partial r} = 0, \quad \texttt{on} \quad r = a. \tag{6}$$

The far-field boundary condition could make the problem ill-posed in several situations. The simplest example of this kind is the "Stokes paradox" which illustrates that there is no solution to Eq. (3) subject to the boundary conditions Eq. (6) and a uniform flow at infinity. However, in the case of singularity driven flows, the ill-posedness disappears and one could obtain the solutions of the two-dimensional creeping equations with finite velocities at large distances from the cylinder. In other words, the singularity driven creeping flow problems in two-dimensions are well-posed. For singularity driven flows, one must also have

$$\psi \approx \psi_s, \quad \texttt{as} \quad R \longrightarrow 0, \tag{7}$$

where ψ_s corresponds to the stream function only due to the singularity and R is the distance of the field point measured from the singularity. We now proceed to present solutions for two-dimensional creeping flows outside/inside a circular cylinder due to a double vortices. The exact solutions have been used to depict the flow topologies for a number of cases where the cylinder is either stationary or rotating about its own axis. If the cylinder rotates in the presence of the vortices, then the stream function is taken to be

$$\Psi(r,\theta) = \psi(r,\theta) - k\ln(\frac{r}{a}), \quad k > 0.$$

Here, $\psi(r,\theta)$ refers to the stream function when the cylinder is stationary. The second term arises when the cylinder rotates about its axis. Below we refer to k as the rotation parameter. The negative sign in front of the second term indicates the rotation in the counterclockwise direction for our purposes below. We provide the discussion for outside flows in Sec. (3) and for inside flows in Sec. (4).

3 Outside flows

The solutions for various singularity driven flows may be obtained by the Fourier expansion method. We skip the details and present the exact solutions in each case. The where closed form solutions are then used to illustrate the flow topologies in each case. For our purposes, the center of the cylinder is taken to be the origin of the coordinate system and the primary vortices are assumed to lie on the x axis but outside the cylinder $r > a$. Next, we consider various singularity induced flows past a cylinder.

We consider a double vortices of strengths F and $\hat{F}$ positioned at $(r,\theta) = (c,0)$, and $(r,\theta) = (\hat{c},\pi)$, respectively. Here $c, \hat{c} > a$. The stream function corresponding to these vortices in an unbounded flow is

$$\psi_0 = F(\ln R) + \hat{F}(\ln \hat{R}) \tag{8}$$

where

$$R^2 = r^2 - 2cr\cos\theta + c^2,$$
$$\hat{R}^2 = r^2 + 2\hat{c}r\cos\theta + \hat{c}^2.$$

Now the solution of Eq. (3) satisfying Eqs. (6) and (7) in the presence of two vortices may be obtained by either the singularity method ref[7] or ref[1] or the Fourier series method ref[8]. The solution can also be derived using the boundary integral equation method ref[14]. Since the derivation is straightforward, we omit the details for the sake of brevity and give the final solution for the stream function:

$$\begin{aligned}
\Psi \;=\; & F[\ln R - \ln(\frac{cR_1}{a}) + \frac{a^2(a^2-c^2)}{c^4}\frac{(rc\cos\theta - a^2)}{R_1^2} \\
& + \frac{a^2}{c^4 R_1^2}(r^2c^2\cos 2\theta - 2a^2 rc\cos\theta + a^4) + \ln(\frac{r}{a}) + \frac{r}{c}\cos\theta] \\
& + \hat{F}[\ln \hat{R} - \ln\frac{\hat{c}\hat{R}_1}{a} - \frac{a^2(a^2-\hat{c}^2)}{\hat{c}^4}\frac{(r\hat{c}\cos\theta + a^2)}{\hat{R}_1^2} \\
& + \frac{a^2}{\hat{c}^4 \hat{R}_1^2}(r^2\hat{c}^2\cos 2\theta + 2a^2 r\hat{c}\cos\theta + a^4) + \ln(\frac{r}{a}) - \frac{r}{\hat{c}}\cos\theta] \\
& - k\ln(r/a),
\end{aligned} \tag{9}$$

where

$$R_1^2 = r^2 - 2\frac{a^2}{c}r\cos\theta + \frac{a^4}{c^2},$$
$$\hat{R}_1^2 = r^2 + 2\frac{a^2}{\hat{c}}r\cos\theta + \frac{a^4}{\hat{c}^2}.$$

vortex, a potential-dipole, and a Stokes dipole at $(a^2/c, 0)$ together with a vortex at the origin and a uniform flow at infinity. The image system for the other vortex located at $(\hat{c}, \pi)$ also consists of the same set of vortices as the previous one except that the direction of the uniform flow at infinity is reversed here. The image solution for a single vortex was given independently by Dorrepaal ref[7] and Avudainayagam ref[1]. If we take $\hat{F} = 0$ and $a = 1$ in Eq. (9), we obtain the solution for a single vortex derived by these authors. It is evident from Eq. (9) that for large r,

$$\psi \approx (\frac{F}{c} - \frac{\hat{F}}{\hat{c}}) r \cos\theta + (F + \hat{F}) \ln(\frac{r}{a}) + O(1), \tag{10}$$

which shows that the far-field behavior is that of uniform flow with speed $(\frac{F}{c} - \frac{\hat{F}}{\hat{c}})$. It is interesting to note that for two vortices with equal strengths and with $c = \hat{c}$, the far-field uniform flow vanishes. In this case the flow behavior changes significantly. We return to the discussion of flow patterns later in this section.

Since expression (9) corresponds to a force-free representation for a two line vortices, the force acting on the cylinder is zero. However, the torque acting on the cylinder need not be zero. The torque may be calculated from the fact that it is $4\pi\mu$ times the strength of the image vortex at the origin. Therefore, it follows from Eq. (9) that the torque in the present case is $4\pi\mu(F + \hat{F})$. The torque vanishes if the vortices have opposite strengths.

It is worth mentioning that the corresponding solution for a plane boundary can be obtained from Eq. (9) in the limit of large radius. The no-slip plane boundary in this case becomes $x = 0$ and the fluid occupies the region $x > 0$. We first derive the limiting case for a single vortex located in the vicinity of the plane boundary $x = 0$. If a, c both approach infinity while $(c - a) \longrightarrow h_1$, then Eq. (9), with $\hat{F} = 0$, after some simplification reduces to

$$\psi(x, y) = F(\ln R - \ln R_1 + \frac{2x(x + h_1)}{R_1^2}) \tag{11}$$

where $R^2 = (x - h_1)^2 + y^2$ and $R_1^2 = (x + h_1)^2 + y^2$. The above solution corresponds to the case of a single vortex situated near a rigid plane boundary. If we take $h = 1$ and replace x by y in the expression (11), we recover the solution due to Ranger ref[18]. Similar limiting procedure with $\hat{F} \neq 0$ in Eq. (9) leads to a solution which is not physical. This is because the vortex with strength $\hat{F}$ in this case is located in the region $(x > 0)$ not occupied by the fluid . However, by a suitable transformation of the y coordinate (after taking the limits), the solution for a two vortices located in the vicinity of a plane boundary may be obtained. To this end, we first let $a, c, \hat{c}$ all approach infinity while $(c - a) \longrightarrow h_1$ and $(\hat{c} - a) \longrightarrow \hat{h}_1$ in Eq. (9). In the resulting expression, we make the transformation $y \longrightarrow (y - a_1)$ in the terms multiplied by F and $y \longrightarrow (y + \hat{a}_1)$ in the terms multiplied by $\hat{F}$ and replace $\hat{h}_1$ by $-\hat{h}_1$. This yields

$$\begin{aligned}\psi(x, y) &= F(\ln R_t - \ln R_{1t} + \frac{2x(x + h_1)}{R_{1t}^2}) \\ &\quad + \hat{F}(\ln \hat{R}_t - \ln \hat{R}_{1t} + \frac{2x(x + \hat{h}_1)}{\hat{R}_{1t}^2}),\end{aligned} \tag{12}$$

where $R_t^2 = (x - h_1)^2 + (y - a_1)^2$ and $R_{1t}^2 = (x + h_1)^2 + (y - a_1)^2$. By replacing h_1, a_1 by $\hat{h}_1, -\hat{a}_1$ in R_t, R_{1t}, the expressions for $\hat{R}_t, \hat{R}_{1t}$ may be written down in a similar fashion. The above solution corresponds to a two vortices located at $(h_1, a_1), (\hat{h}_1, -\hat{a}_1)$ near a plane boundary.

We now turn our attention to the flow streamlines for a two vortices in the presence of a cylinder. The streamlines are sketched using Eq. (9) for different values of the parameters $F, \hat{F}, c, \hat{c}$, and k (rotation parameter). Here and in subsequent sections, we use the terminology "opposite vortices" to refer to two vortices of equal strengths but of different sign. We also use the terminology "equal vortices" to refer to two vortices of equal strengths and of same sign. In this case, both the vortices can have either positive or negative sign. We have considered only the case with positive sign in the present study. The counterclockwise rotation is considered in the case of a cylinder rotating in the presence of vortices. The term "equidistance" is used to refer to two vortices located at equal distance ($c = \hat{c}$) from the center of the cylinder along the x-axis. The cases with different strengths of the primary vortices are also considered and explained at appropriate places. Below we discuss the streamline patterns for a pair of vortices in the presence of a cylinder.

In Fig. 1 we have plotted the streamlines for opposite vortices ($\hat{F} = -F = 1$) for various locations of the primary vortices. In the absence of rotation, the flow streamlines are closed in the neighborhood of the vortices and are parallel to the y-axis at distances far from them. In other words, the flow is uniform in the y-direction [see Figs. 1(a) - 1(b)] far from the cylinder. The locations of vortices influence the flow patterns very little. However, rotation of the cylinder changes the flow pattern noticeably as evident from Figs. 1(c) - 1(d). When $k = 1.1$, eddies of semicircular shape appear around the vortex which is farther from the cylinder. The flow is uniform far away from the cylinder as before and is not affected by the rotation.

Figure 2 illustrates the cases of a pair of vortices of (i) equal strengths, i.e., equal vortices [Figs. 2(a) - 2(c)] and (ii) different strengths ($\hat{F} = 2F$) and of same sign [Figs. 2(d)-2(f)]. In the case of vortices having equal strengths and for $c = \hat{c}$, the terms due to uniform flow cancels and the far-field is no longer uniform. The flow streamlines are closed and appear as a single set of eddies enclosing the cylinder and vortices [Fig. 2(a)]. The enclosing streamline

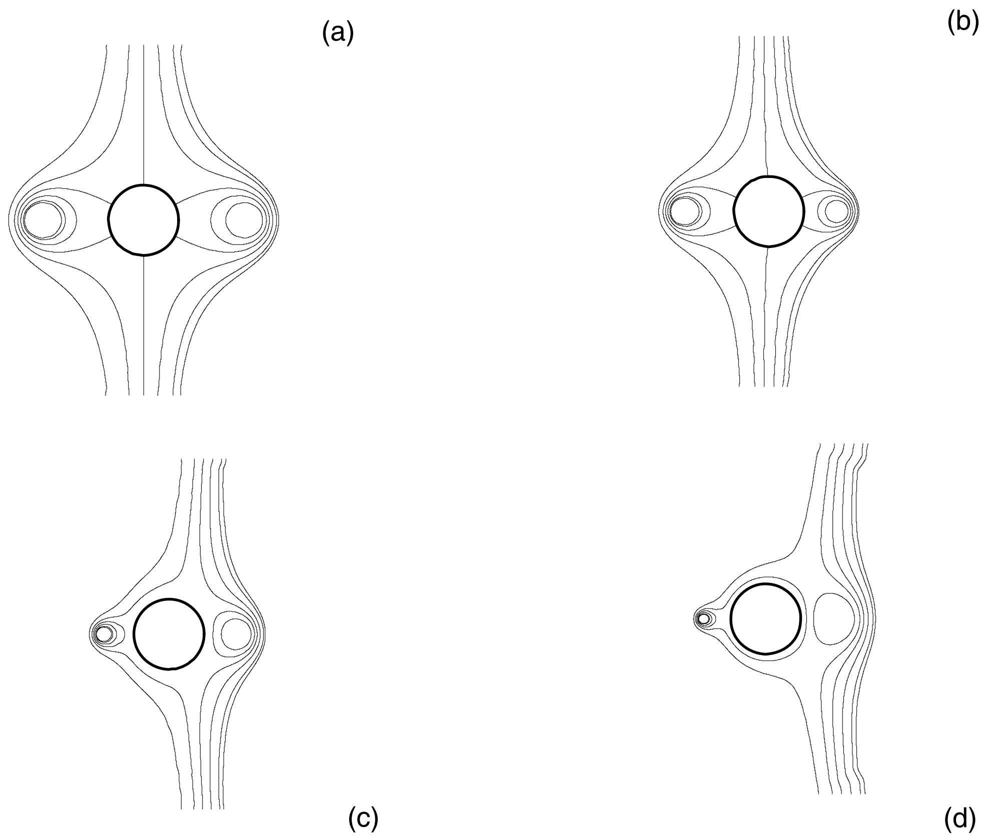

Figure 1: Outside flows: Streamlines for a double vortices with $F = \hat{F} = -1$ (a)$C = \hat{C} = 3, k = 0$ (b) $C = 2, \hat{C} = 2.5, k = 0$ (c) $C = 2, \hat{C} = 1.9, k = 0.5$ (d) $C = 2, \hat{C} = 1.8, k = 1.1$.

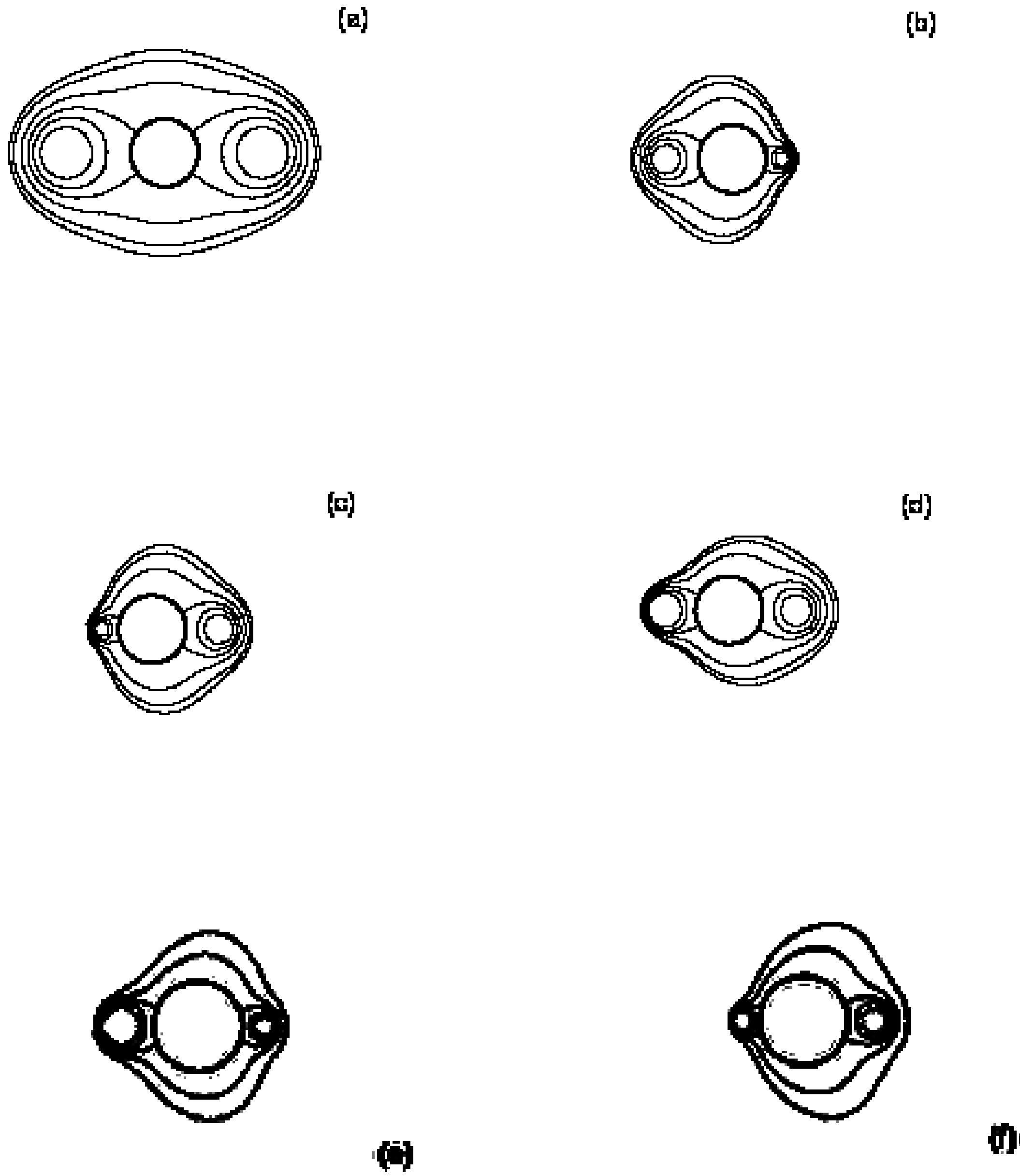

Figure 2: Outside flows: Streamlines for a double vortices with equal strengths $F = \hat{F} = 1$ (a)$C = \hat{C} = 3, k = 0$ (b) $C = 1.5, \hat{C} = 2, k = 0$ (c) $C = 2, \hat{C} = 1.5, k = 0$ and with different strengths $F = 1, \hat{F} = 2$(d) $C = 2, \hat{C} = 2, k = 0$ (e)$C = 1.6, \hat{C} = 1.4, k = 0$ (f) $C = 1.6, \hat{C} = 1.4, k = 0$.

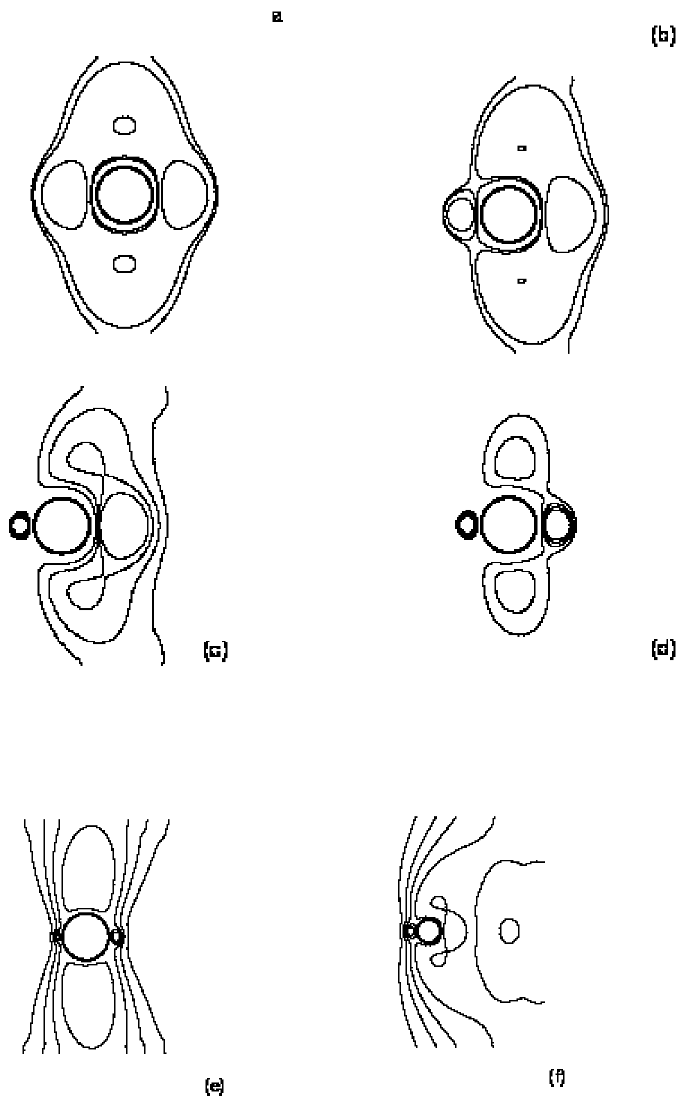

Figure 3: Outside flows: Streamlines for a double vortices with equal strengths $F = \hat{F} = 1$ and rotation parameter $k = 1.1$ (a)$C = \hat{C} = 2$ (b) $C = 2, \hat{C} = 1.7$ (c) $C = 2, \hat{C} = 1.5$ (d) $C = 1.7, \hat{C} = 1.5$ (e)$C = 1.3, \hat{C} = 1.2$ (f) $C = 1.5, \hat{C} = 2$.

pattern was also noticed earlier in the case of two rotating cylinders ref[7]. For $c > \hat{c}$ or $\hat{c} > c$, the closed streamline pattern surrounding the cylinder and vortices changes its size and shape as shown in [Figs. 2(b) - 2(c)]. The existence of the eddy and the uniform flow in the far-field are due to the interaction of the two vortices. In the case of vortices having different strengths and of same sign, the uniform flow always exists for all values of c and $\hat{c}$ [Figs. 2(d) - 2(f)]. Here again, shape of the eddy structure changes considerably with various vortex locations.

The streamlines for equal vortices with cylinder rotation are sketched in Fig. 3 for various locations of the vortices. The cylinder rotates with $k = 1.1$. In this case the streamlines show very interesting flow patterns. If the vortices are located equidistant from the cylinder, eddies of different shapes appear in the flow field, as can be seen from Fig. 3(a). Two sets of eddies surrounding the vortices, one surrounding cylinder, and the other one surrounding the cylinder and vortices, appear in the flow field. On the other hand, if the vortices, are located at nonequidistant positions, two sets of eddies are formed of which one is nearly circular in shape and the other has unusual shape [Figs. 3(b) - 3(d)]. The sizes and shapes of these eddies change significantly with c and $\hat{c}$. Figure 3(d) further shows that there are two stagnation points near the vortex which is farther from the cylinder. The closed separatrix through these two stagnation points enclose three eddies. These interesting features disappear if the locations of the primary vortices are changed. For instance, if the vortices are moved much closer to the cylinder, the unusual eddy structure disappears and a pair of symmetrical eddies enclosed in a single larger eddy appears [Fig. 3(e)]. The unusual eddies seem to appear near the vortex that is farther from the cylinder [see Figs. 3(c), 3(f)].

4 Conclusion

Singularity induced two-dimensional creeping flows inside and outside a circular cylinder are studied by careful investigation of the level sets of stream functions of these flows. The types of vortices considered here include vortices outside a circular cylinder. The exact expression for the stream function of these flows is obtained by using Fourier expansion method. In all of these flows, the axes of the line vortices are assumed to be parallel to the axis of the cylinder and all of these axes lie in one plane. In the plane of flow, the x-axis contains all of these vortices and the center of the cylinder.

Section (3) of this paper investigates flows outside a cylinder for various combinations (strengths and locations) of the vortices. The far-field behavior in all of these cases is that of uniform flow with speed and flow direction depending on the primary vortices and their locations. In the case of vortices with their axes along the y-direction, the far-field uniform flow vanishes if the primary vortices have equal strengths. In the presence of rotation of the cylinder, eddies of unusual sizes and shapes appear. The rotation parameter also significantly influences the eddy structure.

Forcing time dependence in these flows in an appropriate manner so to alternate between these steady creeping flows with different homoclinic and heteroclinic orbits may generate interesting Lagrangian chaotic flows.

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