Adv. Theor. Appl. Mech., Vol. 1, 2008, no. 1, 39 - 44

Soliton Perturbation Theory for the Generalized Kawahara Equation

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Abstract

The adiabatic parameter dynamics of solitons, due to the generalized Kawahara equation are obtained in this paper. The soliton perturbation theory is exploited to obtain the results. Also, the change in the velocity is obtained in presence of these perturbation terms.

1 INTRODUCTION

The dimensionless form of the generalized Kawahara equation (gKE) is given by

$$q_t + aq^p q_x + bq_{xxx} - cq_{xxxxx} = 0 \tag{1}$$

where a, b and c are constant parameters and $p > 0$. This equation is studied in the context of shallow water waves in fluid dynamics, particularly for $p = 1$ and $p = 2$.

It needs to be noted that (1) will fail the Painleve test of integrability for arbitrary p. Thus the classical methods of studying this gKE namely Inverse Scattering Transform, Backlund Transform or Hirota's bilinear method will not work. However, (1) supports 1-soliton solution that is given by [1]

$$q(x,t) = \frac{A}{\cosh^{\frac{4}{p}} B\left(x - \bar{x}(t)\right)} \tag{2}$$

where

$$A = \left[\frac{b^2}{ac}\frac{(p+1)\,(p+2)^2\,(p+4)}{\left(p^2+4p+8\right)^2}\right]^{\frac{1}{p}} \tag{3}$$

$$B = \frac{p}{2}\sqrt{\frac{b}{c\left(p^2+4p+8\right)}} \tag{4}$$

where A is the amplitude of the soliton while B is the inverse width of the soliton and $\bar{x}$ is the center position of the soliton. Thus, from (3) and (4), the amplitude and the inverse width are related as

$$A^p = \frac{16cB^4}{ap^4}(p+1)(p+2)^2(p+4) \tag{5}$$

The velocity of the soliton is given by

$$v = \frac{d\bar{x}}{dt} \tag{6}$$

2 MATHEMATICAL PROPERTIES

Equation (1) has at least two integrals of motion that are known as linear momentum (M) and energy (E) [1, 4, 10]. These are respectively given by

$$M = \int_{-\infty}^{\infty} q\,dx = \frac{A}{B}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{2}{p}\right)}{\Gamma\left(\frac{1}{2}+\frac{2}{p}\right)} \tag{7}$$

and

$$E = \int_{-\infty}^{\infty} q^2 dx = \frac{A^2}{B} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{4}{p}\right)}{\Gamma\left(\frac{1}{2}+\frac{4}{p}\right)} \tag{8}$$

These conserved quantities are calculated by using the 1-soliton solution given by (2). Also, in (6) and (7), $\Gamma(x)$ is the usual gamma function that is defined as

$$\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1} dt \tag{9}$$

The center of the soliton $\bar{x}$ is given by the definition

$$\bar{x} = \frac{\int_{-\infty}^{\infty} xq dx}{\int_{-\infty}^{\infty} q dx} = \frac{\int_{-\infty}^{\infty} xq dx}{M} \tag{10}$$

where M is defined in (7). Thus, the velocity of the soliton is given by

$$v = \frac{d\bar{x}}{dt} = \frac{\int_{-\infty}^{\infty} xq_t dx}{\int_{-\infty}^{\infty} q dx} = \frac{\int_{-\infty}^{\infty} xq_t dx}{M} \tag{11}$$

On using (1), (2) and (6), the velocity of the soliton reduces to

$$v = -\frac{4b^2}{c} \frac{p^2+4p+4}{\left(p^2+4p+8\right)^2} \tag{12}$$

3 PERTURBATION TERMS

The perturbed fKdV equation that is going to be studied in this paper is given by

$$q_t + aq^p q_x + bq_{xxx} - cq_{xxxxx} = \epsilon R \tag{13}$$

where, in (12), ϵ is the perturbation parameter and $0 < \epsilon \ll 1$, while R gives the perturbation terms. In presence of perturbation terms, the momentum and the energy of the soliton do not stay conserved. Instead, they undergo adiabatic changes that lead to the adiabatic deformation of the soliton amplitude, width and a slow change in the velocity [3]. Using (8), the law of adiabatic deformation of the soliton energy is given by

$$\frac{dE}{dt} = 2\epsilon \int_{-\infty}^{\infty} qR dx \tag{14}$$

while the adiabatic law of change of the velocity of the soliton, from (10), is given by

$$v = -\frac{4b^2}{c}\frac{p^2+4p+4}{\left(p^2+4p+8\right)^2} + \frac{\epsilon}{M}\int_{-\infty}^{\infty} xR dx \tag{15}$$

In this paper, the perturbation terms that are going to be considered are

$$\begin{aligned} R = {} & \alpha q + \beta q_{xx} + \gamma q_x q_{xx} + \delta q^m q_x + \lambda q_{xxx} + \nu q q_x q_{xx} \\ & + \sigma q_x^3 + \xi q_x q_{xxxx} + \eta q_{xx} q_{xxx} + \rho q_{xxxx} + \psi q_{xxxxx} + \kappa q q_{xxxx} \end{aligned} \tag{16}$$

In R, dissipation gives rise to the first two terms and so α and β are small dissipative coefficients [4]. Also, δ or ψ represent the coefficient of higher order nonlinear dispersive term [4] and m is a positive integer with $1 \leq m \leq 4$ [7]. The coefficient of ρ provide a higher order stabilizing term and must therefore be taken into account [4]. The perturbation term given by coefficient of η was recently considered [6] while the remaining perturbation terms arise in the context of extended version of integrable equations [7].

3.1 APPLICATIONS

In presence of these perturbation terms, the adiabatic variation of the energy of the soliton is given by

$$\begin{aligned} \frac{dE}{dt} = {} & \frac{2\epsilon A^2}{Bp^2}\Gamma\left(\frac{1}{2}\right)\left[\frac{256\kappa(p^2+4p-72)AB^4}{p(p+4)(p+12)}\frac{\Gamma\left(\frac{6}{p}\right)}{\Gamma\left(\frac{1}{2}+\frac{6}{p}\right)}\right. \\ & \left.\frac{1}{p(p+8)(3p+8)}\left\{\alpha p^2(p+8)(3p+8) - 16\beta B^2 p(3p+8) + 256\rho(p+3)B^4\right\}\frac{\Gamma\left(\frac{4}{p}\right)}{\Gamma\left(\frac{1}{2}+\frac{4}{p}\right)}\right] \end{aligned} \tag{17}$$

The law of the change of velocity for the given perturbation terms in (16) is given by

$$\begin{aligned} v = {} & -\frac{2aA^2(p+2)^2}{b\left(p^2+4p+8\right)^2} + \epsilon\frac{\Gamma\left(\frac{2}{p}+\frac{1}{2}\right)}{\Gamma\left(\frac{2}{p}\right)}\left[\frac{\delta A^m}{m+1}\frac{\Gamma\left(\frac{2m+2}{p}\right)}{\Gamma\left(\frac{1}{2}+\frac{2m+2}{p}\right)}\right. \\ & + \frac{8AB^2}{p^2(p+8)(3p+8)}\left\{p(3p+8)(\gamma-2\lambda) - 16B^2(p+3)(3\xi-\eta)\right\}\frac{\Gamma\left(\frac{4}{p}\right)}{\Gamma\left(\frac{1}{2}+\frac{4}{p}\right)} \end{aligned}$$

$$+\frac{8A^2B^2}{3p(p+6)(p+12)}\left\{\nu p - 2\sigma(p+18)\right\}\frac{\Gamma\left(\frac{6}{p}\right)}{\Gamma\left(\frac{1}{2}+\frac{6}{p}\right)}\Bigg] \tag{18}$$

4 CONCLUSIONS

In this paper, soliton perturbation theory is used to study the gKE. In future, it is possible to extend these perturbation terms to include other perturbation terms that include the non-local ones too. The quasi-stationary aspects of the perturbed soliton in presence of such perturbation terms will be studied and reported in future publications.

ACKNOWLEDGEMENT

This research of the first author (AB) was fully supported by NSF-CREST Grant No: HRD-970668 and the support is very thankfully appreciated.

The research work of the second author is financially supported by the Army Research Office (ARO) under the award number W911NF-05-1-0451 and this support is thankfully appreciated.

References

[1] A. Biswas. “Solitary wave solution of the generalized Kawahara equation”. Submitted.

[2] E. Fan & Y. C. Hon. “Applications of the extended tanh method to ’special’ types of nonlinear equations”. *Applied Mathematics and Computations*. Vol 141, Issues 2-3 351-358. (2003).

[3] V. I. Karpman. “Stabilization of soliton instabilities by higher order dispersion: KdV type equations. *Physics Letters A*. Vol 210, Issues 1-2, 77-84. (1996).

[4] Y. Kivshar & B. A. Malomed. "Dynamics of solitons in nearly integrable systems" *Reviews of Modern Physics* Vol 61, No 4, 763-915. (1989).

[5] W. Malfliet. "Solitary wave solutions of nonlinear wave equations". *American Journal of Physics*. Vol 60, Number 7, 650-654. (1992).

[6] X. Niu & Z. Pan. "New approximate solution of the perturbed KdV equation". *Physics Letters A*. Vol 349, 192-197. (2006).

[7] A. R. Osborne. "Approximate asymptotic integration of a higher order water-wave equation using the inverse scattering transform. *Nonlinear Processes in Geophysics*. Vol 4, Number 1, 29-53. (1997).

[8] E. J. Parkes & B. R. Duffy. "An automated tanh function method for finding solitary wave solutions to non-linear evolution equations". *Computer Physics Communications*. Vol 98, Issue 3, 288-300. (1996).

[9] A. M. Wazwaz. "Analytical study on the generalized fifth-order KdV equation: New solitons and periodic solutions". *Communications in Nonlinear Science and Numerical Simulations*. Vol 12, Issue 7, 1172-1180. (2007).

[10] P. E. Zhidkov. *Korteweg-de Vries and Nonlinear Schrödinger's Equations: Qualitative Theory*. Springer Verlag. New York, NY. (2001).

Received: December 21, 2007