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# A Decomposition Method for Volume Flux and Average Velocity of Thin Film Flow of a Third Grade Fluid Down an Inclined Plane

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**Abstract**

Perturbation methods depend on a small parameter which is difficult to be found for real-life nonlinear problems. To overcome this shortcoming, a powerful analytical method is introduced to solve the thin film flow problem with a third grade fluid on an inclined plane. Here, Adomian Decomposition method is applied to solve nonlinear equation of the velocity field. The results obtained by this method are then compared with the traditional perturbation method to illustrate the effectiveness of this method. Finally volume flux and average film velocity is given graphically.

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## 1. Introduction

Most scientific problems and phenomena in different fields of science and engineering occur nonlinearly. Except in a limited number of these problems, we encounter difficulties in finding their exact analytical solutions.

Perturbation method provides the most versatile tools available in nonlinear analysis of engineering problems, but its limitations hamper its application:

1. Perturbation method is based on assuming a small parameter. An overwhelming majority of nonlinear problems, especially those having strong nonlinearity, have no small parameters at all.
2. The approximate solutions obtained by the perturbation methods, in most cases, are valid only for the small values of the small parameter. The perturbation solutions are generally uniformly valid as long as a specific system parameter is small. However, we cannot rely fully on the approximations, because there is no criterion on which the small parameter should exist. Thus, it is essential to check the validity of the approximations numerically and/or experimentally.

To overcome these difficulties, approximate analytical solutions, such as the tanh method [1,3], the sine-cosine method [2,3], the homogeneous balance method [4,5], the variational iteration method [6-8], the homotopy-perturbation method [9-12] and the Adomian decomposition method [13-15] are introduced, among which Adomian decomposition method [13-15] is the most effective and convenient one for both weakly and strongly nonlinear problems. This method has been shown to effectively and accurately solve a large class of nonlinear problems with approximations converging rapidly to accurate solutions.

There are few exact solutions of the Navier-Stokes equations because of their highly nonlinearity and these become rare when non-Newtonian fluids equations are used. Perturbation techniques [16,17] are widely applied for obtaining approximate solutions to these equations involving a small parameter $\varepsilon$. But they have limitations that mentioned above.

In this paper, we apply Adomian Decomposition method to study the thin film flow problem with a third grade fluid on an inclined plane. The capability and effectiveness of this method are revealed by obtaining the analytical solutions of the model and comparing with perturbation method.

## 2. Fundamentals of Adomian decomposition method

Let us discuss a brief outline of the Adomian Decomposition method. For this, we consider a general nonlinear equation in the form [18]

$$Lu + Ru + Nu = g \tag{1}$$

where $L$ is the highest order derivative which is assumed to be easily invertible, $R$ the linear differential operator of less order than $L$, $Nu$ presents the nonlinear terms and $g$ is the source term. Applying the inverse operator $L^{-1}$ to the both sides of Eq. (1), and using the given conditions we obtain:

$$u = f(x) - L^{-1}(Ru) - L^{-1}(Nu) \tag{2}$$

where the function $f(x)$ represents the terms arising from integration the source term $g(x)$, using given conditions. For nonlinear differential equations, the nonlinear operator $Nu = F(u)$ is represented by an infinite series of the so-called Adomian polynomials

$$F(u) = \sum_{m=0}^{\infty} A_m \tag{3}$$

The polynomials $A_m$ are generated for all kind of nonlinearity so that $A_0$ depends only on $u_0$ , $A_1$ depends on $u_0$ and $u_1$, and so on. The Adomian polynomials introduced above show that the sum of subscripts of the components of $u$ for each term of $A_m$ is equal to $n$ [19].

The Adomian method defines the solution $u(x)$ by the series

$$u = \sum_{m=0}^{\infty} u_m \tag{4}$$

In the case of $F(u)$, the infinite series is a Taylor expansion about $u_0$, as follows:

$$F(u) = F(u_0) + F'(u_0)(u - u_0) + F''(u_0)\frac{(u-u_0)}{2!} + F'''(u_0)\frac{(u-u_0)^2}{3!} + \cdots \tag{5}$$

By rewriting Eq. (4) as $u - u_0 = u_1 + u_2 + u_3 + \ldots$, substituting it into Eq. (5) and then equating two expressions for $F(u)$ found in Eq. (5) and Eq. (3), defines formulas for the Adomian polynomials in the form of [18]

$$F(u) = A_1 + A_2 + \ldots = F(u_0) + F'(u_0)(u_1 + u_2 + \ldots) + F''(u_0)\frac{(u_1 + u_2 + \ldots)^2}{2!} + \ldots. \tag{6}$$

By equating terms in Eq. (6), the first few Adomian's polynomials $A_0$, $A_1$, $A_2$, $A_3$ and $A_4$ are given:

$$A_0 = F(u_0) \tag{7}$$

$$A_1 = u_1 F'(u_0) \tag{8}$$

$$A_2 = u_2 F'(u_0) + \frac{1}{2!}u_1^2 F''(u_0) \tag{9}$$

$$A_3 = u_3 F'(u_0) + u_1 u_2 F''(u_0) + \frac{1}{3!}u_1^3 F'''(u_0) \tag{10}$$

$$A_4 = u_4 F'(u_0) + \left(\frac{1}{2!}u_2^2 + u_1 u_3\right)F''(u_0) + \frac{1}{2!}u_1^2 u_2 F'''(u_0) + \frac{1}{4!}u_1^4 F^{(iv)}(u_0) \tag{11}$$

$\vdots$

Now that the $A_m$ are known , Eq. (3) can be substituted in Eq.(2) to specify the terms in the expansion for the solution of Eq. (4).

## 3. Mathematical modeling of the problem

The basic equations governing the motion of an incompressible fluid, neglecting the thermal effects, are [20]:

$$\nabla \cdot V = 0 \tag{12}$$

$$\rho \frac{DV}{Dt} = -\nabla \mathrm{p} + \rho \mathrm{f} + \mathrm{div}\,\tau \tag{13}$$

where $\rho$ the constant density, $V$ the velocity vector, p the pressure, $\tau$ the stress tensor and $D/Dt$ denoting the material derivative.

The stress tensor defining a third grade fluid is given by [20]

$$\tau = \sum_{i=1}^{3} S_i \tag{14}$$

where

$$S_1 = \mu A_1 \tag{15}$$

$$S_2 = \alpha_1 A_2 + \alpha_2 A_1^2 \tag{16}$$

$$S_3 = \beta_1 A_3 + \beta_2 \left(A_1 A_2 + A_2 A_1\right) + \beta_3 \left(trA_2\right) A_1 \tag{17}$$

Here $\mu$ is the coefficient of viscosity and $\alpha_1, \alpha_2, \beta_1, \beta_2$ and $\beta_3$, are material constants [20]. The Rivilin-Ericksen tensors, $A_n$ are defined by $A_0 = I$, the identity tensor, and

$$A_n = \frac{DA_{n-1}}{Dt} + A_{n-1}(\nabla V) + (\nabla V)^T A_{n-1},\ n \geq 1 \tag{18}$$

We consider a thin film of an incompressible fluid of third grade flow down an inclined plane. The ambient air is assumed stationary so that the flow is due to gravity alone. We assume that the surface tension of the fluid is negligible and the film is of uniform thickness $\delta$. We seek a velocity field of the form

$$v = (u(y), 0, 0) \tag{19}$$

Substituting for $v$ and $\tau$ in Eqs. (12) and (13) and assuming the absence of pressure gradient we obtain

$$\mu \frac{d^2u}{dy^2} + 6(\beta_2 + \beta_3)\left(\frac{du}{dy}\right)^2 \frac{d^2u}{dy^2} + \rho g \sin\alpha = 0 \tag{20}$$

Subject to the boundary conditions of

$$u(y) = 0, \text{ at } y = 0 \tag{21}$$

$$\frac{du}{dy} = 0, \text{ at } y = \delta \tag{22}$$

## 4. Solution of the problem

In this section, we study the velocity field and find expressions for $u(y)$ by traditional perturbation method and Adomian decomposition method.

### *4.1. Solution by perturbation method*

In Eq. (20), we take $\varepsilon = \frac{\beta}{\mu}$, where $\beta = \beta_2 + \beta_3$. So Eq. (20) transforms to [20]:

$$\frac{d^2u}{dy^2} + 6\varepsilon\left(\frac{du}{dy}\right)^2 \frac{d^2u}{dy^2} + \frac{\rho g \sin\alpha}{\mu} = 0 \tag{23}$$

Let us assume $\varepsilon$ as a small parameter. In order to solve Eq. (23) by traditional perturbation method, we expand $u(y)$ in the form of

$$u(y,\varepsilon) = u_0(y) + \varepsilon u_1(y) + \varepsilon^2 u_2(y) + \cdots \tag{24}$$

Substituting Eq. (24) into Eq. (23) and rearranging based on powers of $\varepsilon$-terms, we can obtain:

$$\varepsilon^0 \quad : \frac{d^2u_0}{dy^2} + \frac{\rho g \sin\alpha}{\mu} = 0 \tag{25}$$

$$u_0(0) = 0\,, \frac{du_0(\delta)}{dy} = 0 \tag{26}$$

$$\varepsilon^1 \quad : \frac{d^2u_1}{dy^2} + 6\left(\frac{du_0}{dy}\right)^2 \frac{d^2u_0}{dy^2} = 0 \tag{27}$$

$$u_1(0) = 0\,, \frac{du_1(\delta)}{dy} = 0 \tag{28}$$

$$\varepsilon^2 \quad : \frac{d^2u_2}{dy^2} + 6\left(\frac{du_0}{dy}\right)^2 \frac{d^2u_1}{dy^2} 0 + 12\left(\frac{du_0}{dy}\right)\left(\frac{du_1}{dy}\right)\frac{d^2u_0}{dy^2} = 0 \tag{29}$$

$$u_2(0) = 0\,, \frac{du_2(\delta)}{dy} = 0 \tag{30}$$

Solving Eq. (25)-(30), we obtain:

$$u_0(y) = \frac{\rho g \sin\alpha}{\mu}\left(\delta y - \frac{y^2}{2}\right) \tag{31}$$

$$u_1(y)=6\frac{\rho^3 g^3\sin^3\alpha}{\mu^3}\left(\frac{y^4}{12}-\frac{\delta y^3}{3}+\frac{\delta^2 y^2}{2}-\frac{\delta^3 y}{3}\right) \tag{32}$$

$$u_2(y)=36\frac{\rho^5 g^5\sin^5\alpha}{\mu^5}\left(\frac{\delta^5 y}{3}-\frac{5}{6}\delta^4 y^2+\frac{10}{9}\delta^3 y^3-\frac{5}{6}\delta^2 y^4+\frac{\delta y^5}{3}-\frac{y^6}{18}\right) \tag{33}$$

And approximate solution obtained by perturbation method will be as follows:

$$u(y)=\frac{\rho g\sin\alpha}{\mu}\left(\delta y-\frac{y^2}{2}\right)+6\varepsilon\frac{\rho^3 g^3\sin^3\alpha}{\mu^3}\left(\frac{y^4}{12}-\frac{\delta y^3}{3}+\frac{\delta^2 y^2}{2}-\frac{\delta^3 y}{3}\right)$$
$$+36\varepsilon^2\frac{\rho^5 g^5\sin^5\alpha}{\mu^5}\left(\frac{\delta^5 y}{3}-\frac{5}{6}\delta^4 y^2+\frac{10}{9}\delta^3 y^3-\frac{5}{6}\delta^2 y^4+\frac{\delta y^5}{3}-\frac{y^6}{18}\right) \tag{34}$$

### *4.2. Solution by Adomian Decomposition method*

Following the Adomian decomposition analysis, the linear operator is defined as:

$$L=\frac{d^2}{dy^2} \tag{35}$$

Consequently, Eq. (20) can be written as follows:

$$Lu=-6\frac{\beta}{\mu}\left(\frac{du}{dy}\right)^2\frac{d^2u}{dy^2}-\frac{\rho g\sin\alpha}{\mu}=-6\frac{\beta}{\mu}Nu-\frac{\rho g\sin\alpha}{\mu} \tag{36}$$

where $\beta=\beta_2+\beta_3$. The nonlinear term is

$$Nu=\left(\frac{du}{dy}\right)^2\frac{d^2u}{dy^2}=\sum_{m=0}^{\infty}A_m \tag{37}$$

Hence, using Eqs. (7) - (11) gives:

$$A_0=\left(\frac{du_0}{dy}\right)^2\frac{d^2u_0}{dy^2} \tag{38}$$

$$A_1=\left(\frac{du_0}{dy}\right)^2\frac{d^2u_1}{dy^2}+2\frac{du_0}{dy}\frac{du_1}{dy}\frac{d^2u_0}{dy^2} \tag{39}$$

$$A_2=\left(\frac{du_0}{dy}\right)^2\frac{d^2u_2}{dy^2}+2\frac{du_0}{dy}\frac{du_1}{dy}\frac{d^2u_1}{dy^2}+\left(\frac{du_1}{dy}\right)^2\frac{d^2u_0}{dy^2} \tag{40}$$

$$A_3 = 2\frac{du_0}{dy}\frac{du_1}{dy}\frac{d^2u_2}{dy^2} + 2\frac{du_0}{dy}\frac{du_2}{dy}\frac{d^2u_1}{dy^2} + \left(\frac{du_1}{dy}\right)^2\frac{d^2u_1}{dy^2} + 2\frac{du_1}{dy}\frac{du_2}{dy}\frac{d^2u_0}{dy^2} \tag{41}$$

$$A_4 = 2\frac{du_0}{dy}\frac{du_2}{dy}\frac{d^2u_2}{dy^2} + \left(\frac{du_1}{dy}\right)^2\frac{d^2u_2}{dy^2} + 2\frac{du_1}{dy}\frac{du_2}{dy}\frac{d^2u_1}{dy^2} + \left(\frac{du_2}{dy}\right)^2\frac{d^2u_0}{dy^2} \tag{42}$$

$\vdots$

Applying the inverse operator $L^{-1}$ to both sides of Eq. (36), we obtain:

$$L^{-1}Lu = -6\frac{\beta}{\mu}L^{-1}Nu - L^{-1}\frac{\rho g \sin\alpha}{\mu} \tag{43}$$

If $L$ is a second-order operator, $L^{-1}$ is a twofold indefinite integral. Performing the indicated operations we obtain:

$$u - u(0) - y\frac{du(0)}{dy} = -6\frac{\beta}{\mu}L^{-1}Nu - \frac{1}{2}\frac{\rho g \sin\alpha}{\mu}y^2 \tag{44}$$

thus,

$$u_0(y) = u(0) + y\frac{du(0)}{dy} - \frac{1}{2}\frac{\rho g \sin\alpha}{\mu}y^2 \tag{45}$$

Applying boundary conditions given in Eqs. (21) and (22), we obtain:

$$u_0(y) = \frac{\rho g \sin\alpha}{\mu}\left(\delta y - \frac{y^2}{2}\right) \tag{46}$$

The next iterates are determined recursively by

$$u_{m+1} = -6\frac{\beta}{\mu}L^{-1}A_m \tag{47}$$

Using above iteration formula, we obtain:

$$u_1(y) = 6\frac{\rho^3 g^3 \sin^3\alpha}{\mu^3}\left(\frac{\beta}{\mu}\right)\left(\frac{y^4}{12} - \frac{\delta y^3}{3} + \frac{\delta^2 y^2}{2} - \frac{\delta^3 y}{3}\right) \tag{48}$$

$$u_2(y) = 36\frac{\rho^5 g^5 \sin^5\alpha}{\mu^5}\left(\frac{\beta}{\mu}\right)^2\left(\frac{\delta^5 y}{3} - \frac{5}{6}\delta^4 y^2 + \frac{10}{9}\delta^3 y^3 - \frac{5}{6}\delta^2 y^4 + \frac{\delta y^5}{3} - \frac{y^6}{18}\right) \tag{49}$$

and so on. In the same manner the rest of the components of the iteration formula can be obtained. Upon summing above iterations, the second order approximation is expressed as

$$u(y)=\frac{\rho g\sin\alpha}{\mu}\left(\delta y-\frac{y^2}{2}\right)+6\frac{\rho^3 g^3\sin^3\alpha}{\mu^3}\left(\frac{\beta}{\mu}\right)\left(\frac{y^4}{12}-\frac{\delta y^3}{3}+\frac{\delta^2 y^2}{2}-\frac{\delta^3 y}{3}\right)$$
$$+36\varepsilon^2\frac{\rho^5 g^5\sin^5\alpha}{\mu^5}\left(\frac{\beta}{\mu}\right)^2\left(\frac{\delta^5 y}{3}-\frac{5}{6}\delta^4 y^2+\frac{10}{9}\delta^3 y^3-\frac{5}{6}\delta^2 y^4+\frac{\delta y^5}{3}-\frac{y^6}{18}\right) \quad (50)$$

which is the same as that obtained by perturbation method.

Now let us do the following change of parameters:

$$u^*=\frac{u\nu}{\delta},\ Y=\frac{y}{\delta},\ \beta^*=\frac{\beta\nu}{\rho\delta^4},\ m^*=\frac{\delta^3 g\sin\alpha}{\nu^2} \quad (51)$$

where $\nu$ is kinematic viscosity, $u^*$ is dimensionless velocity and $\beta^*$ is non-Newton parameter. After parameter change, Eq. (50) transforms into

$$u^*(Y)=m^*\left(Y-\frac{Y^2}{2}\right)+6m^{*3}\beta^*\left(\frac{Y^4}{12}-\frac{Y^3}{3}+\frac{Y^2}{2}-\frac{Y}{3}\right)+36m^{*5}\beta^{*2}$$
$$\left(\frac{Y}{3}-\frac{5}{6}Y^2+\frac{10}{9}Y^3-\frac{5}{6}Y^4+\frac{Y^5}{3}-\frac{Y^6}{18}\right) \quad (52)$$

It is worth pointing out that if we set the non-Newton parameter equal to zero, i.e. $\beta^*=0$, non-Newtonian solution, i.e. Eq. (52), transforms into Newtonian solution.

## 5. Flow rate and average velocity

After finding expression for velocity profile, the flow rate per unit width is given by [20]

$$\frac{Q}{W}=\int_0^\delta u(y)dy \quad (53)$$

where $W$ is width of the film. Using Eq. (50) we obtain the following expression for the flow rate:

$$\frac{Q}{W}=\frac{\rho g\sin\alpha}{3\mu}\delta^3-\frac{2}{5}\left(\frac{\beta}{\mu}\right)\left(\frac{\rho g\sin\alpha}{\mu}\right)^3\delta^5+\frac{12}{7}\left(\frac{\beta}{\mu}\right)^2\left(\frac{\rho g\sin\alpha}{\mu}\right)^5\delta^7 \quad (54)$$

Introducing

$$\varphi = \frac{\rho g \sin\alpha}{\mu}, \psi = \frac{\beta}{\mu} \tag{55}$$

transforms Eq. (54) into

$$\frac{Q}{W} = \frac{1}{3}\varphi\delta^3 - \frac{2}{5}\psi\varphi^3\delta^5 + \frac{12}{7}\psi^2\varphi^5\delta^7 \tag{56}$$

The average velocity over the cross section of the film is

$$\bar{u}(y) = \frac{Q}{W\delta} = \frac{1}{3}\varphi\delta^2 - \frac{2}{5}\psi\varphi^3\delta^4 + \frac{12}{7}\psi^2\varphi^5\delta^6 \tag{57}$$

## 6. Results and discussions

Fig. 1 shows the dimensionless velocity profile with different values of $\beta^*$, as non-Newton parameter, and given $m^*$. It is apparent that the velocity profile converges to the Newtonian fluid velocity profile as the non-Newton parameter decreases.

The velocity profile for different values of $m^*$ and given $\beta^*$ is depicted in Fig. 2 where it is obvious that when we decrease $m^*$, the velocity profile meets the Newtonian fluid velocity profile.

Fig.3 represents the average velocity over the cross section of the film. As it is seen when we equal $\psi$ to zero, the fluid shows the Newtonian fluid behavior.

## 7. Conclusion

In this paper Adomian decomposition method has been successfully used to obtain the velocity profile of thin film flow of a third grade fluid down an inclined plane. The results obtained by decomposition method are in excellent agreement with perturbation method. But using the common perturbation method is based upon the existence of a small parameter, so developing the method for different applications is not easy and finding this small parameter is also difficult. But Adomian decomposition method does not need any small parameter and can be applied to wide class of nonlinear problems, whether or not with small parameter.

In conclusion, Adomian decomposition method provides highly accurate numerical solutions for nonlinear problems. It also has many merits in comparison with other methods, such as:

1. Adomian decomposition method does not require small parameters which are needed by perturbation method.
2. Adomian decomposition method avoids linearization and physically unrealistic assumptions.

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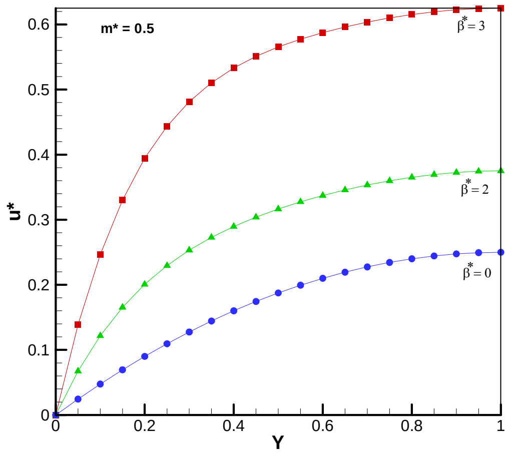


Fig. 1. The dimensionless velocity profile with different values of $\beta^*$ and given $m^*$.

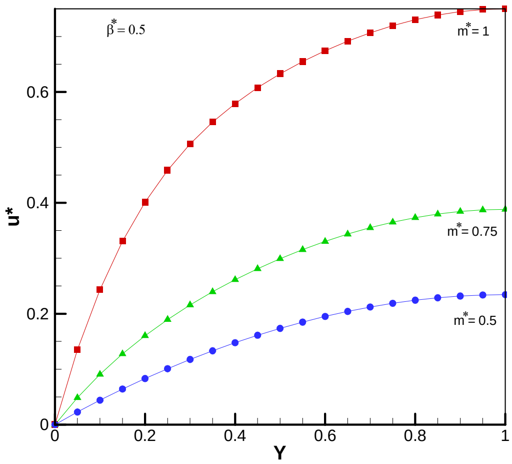


Fig. 2. The dimensionless velocity profile for different values of $m^*$ and given $\beta^*$

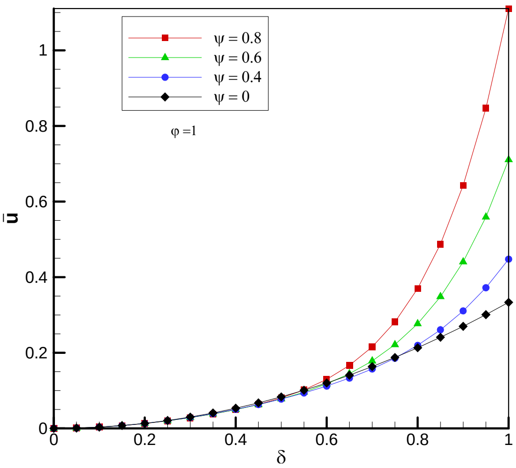


Fig.3. The average velocity over the cross section of the film for given amounts of $\psi$ .