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Application of He's Homotopy Perturbation and Variational Iteration Methods to Solving Propagation of Thermal Stresses Equation in an Infinite Elastic Slab

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1. Abstract

In this Letter, He's homotopy perturbation method (HPM) and variational iteration method (VIM), are implemented for solving the thermal stresses equation in an infinite elastic slab. Comparison of the results with exact solution method has led us to significant consequences. The results reveal that the HPM and VIM are very effective, convenient and quite accurate to solve partial differential equations. It is predicted that the HPM and VIM can be found widely applicable in engineering.

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2. Introduction

Most scientific problems do not have analytical solution. Therefore, these equations should be solved using other methods. Some of them are solved by using numerical techniques and some are solved by using the analytical method of perturbation. In the numerical method, stability and convergence should be considered so as to avoid divergence or inappropriate results. In the analytical perturbation method, we should exert the small parameter in the equation. Therefore, finding the small parameter and exerting it into the equation are difficulties of this method. Since there are some limitations with the common perturbation method, and also because the basis of the common perturbation method is upon the existence of a small parameter, developing the method for different applications is very difficult.

Therefore, many different methods have recently introduced some ways to eliminate the small parameter, such as artificial parameter method introduced by He [1], the homotopy perturbation method by He [2, 3], the variation iteration method by He [4,5].

3. Mathematical Modeling of the problem [6,7]

Let us consider an infinite elastic slab bounded by the planes $x=0$ and $x=1$ as the case study. Suppose that a time dependent temperature distribution $T(x,t)$ is acting on the face of the slab. It is also assumed that the displacement in the y and z directions is zero.

The only non-vanishing strains are the principal strains, which are:

$$e_{xx} = \frac{\partial u}{\partial x} - \alpha T \tag{1}$$

$$e_{yy} = e_{zz} = -\alpha T \tag{2}$$

$$e_{xy} = e_{yz} = e_{zx} = 0 \tag{3}$$

Where α is the coefficient of linear expansion.

The stresses in terms of displacement are in the following form

$$\tau_{xx} = \frac{2\mu}{1-2\sigma}[(1-\sigma)\frac{\partial u}{\partial x} - (1+\sigma)\alpha T] \tag{4}$$

$$\tau_{yy} = \tau_{zz} = \frac{2\mu}{1-2\sigma}[\sigma\frac{\partial u}{\partial x} - (1+\sigma)\alpha T] \tag{5}$$

$$\tau_{xx} = \tau_{yy} = \tau_{zx} = 0 \tag{6}$$

where μ is Lame's constant and σ is poison's ratio. In the absence of body forces, the equations of motion reduce to a single equation as follows:

$$\frac{\partial \tau_{xx}}{\partial x} = Q\frac{\partial^2 u}{\partial t^2} \tag{7}$$

Where Q is the density.
Substituting Eq. (4) into Eq. (7).eventually yields to:

$$\frac{2\mu(1-\sigma)}{1-2\sigma}(\frac{\partial^2 u}{\partial x^2}) - \frac{2\mu(1+\sigma)}{1-2\sigma}\alpha(\frac{\partial T}{\partial x}) = Q(\frac{\partial^2 u}{\partial t^2}) \tag{8}$$

Where "c" is the velocity of the irrotational waves, transforms Eq. (8) into:

$$\frac{Q}{c^2}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + \eta\frac{\partial T}{\partial x} = 0 \tag{9}$$

Where: $c^2 = \frac{2(1-\sigma)\mu}{1-2\sigma}$, $\eta = \frac{1+\sigma}{1-\sigma}\alpha$

Assuming that the temperature distribution obeys the uncoupled heat conduction equation, we obtain

$$\frac{\partial T}{\partial t} - k_1\frac{\partial^2 T}{\partial x^2} = 0 \tag{10}$$

Where k_1 is the thermal diffusivity.

4. Initial and Boundary Conditions [7]

The initial and Boundary Conditions must be satisfied by the temperature distribution are:

$$T(x,0) = T_0(1-x^2) \tag{11}$$

$$T(1,t) = 0 \ , \ \frac{\partial T(0,t)}{\partial x} = 0 \tag{12}$$

We then impose the initial conditions on displacement equation as follows:

$$u(x,0) = x(1-x^2) \tag{13}$$

$$\frac{\partial u(x,0)}{\partial t} = 0 \tag{14}$$

The boundary conditions are:

$$u(0,t) = 0 \tag{15}$$

$$u(1,t) = 0 \tag{16}$$

5. Basic idea of He's homotopy perturbation method [8]

The combination of the perturbation method and the homotopy method is called the HPM, which lacks the limitations of the traditional perturbation methods, although this technique can have full advantages of the traditional perturbation techniques. However, the convergence rate depends on the nonlinear operator. [9]

To illustrate the basic idea of this method, we consider the following nonlinear differential equation:

$$A(u) - f(r) = 0, \quad r \in \Omega \tag{17}$$

Considering the boundary conditions of:

$$B(u, \partial u/\partial n) = 0, \quad r \in \Gamma \tag{18}$$

Where "*A*" is a general differential operator, "*B*" a boundary operator, $f(r)$ a known analytical function and Γ is the boundary of the domain Ω.

The operator "*A*" can be divided into two parts of "*L*" and "*N*", where "*L*" is the linear part, while "*N*" is a nonlinear one. Eq. (17) can, therefore, be rewritten as:

$$L(u) + N(u) - f(r) = 0, \tag{19}$$

By the homotopy technique, we construct a homotopy as $v(r, p): \Omega \times [0,1] \to \Re$ which satisfies:

$$H(v, p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0, \quad P \in [0,1], \quad r \in \Omega \tag{20}$$

Where $p \in [0,1]$ is an embedding parameter and u_0 is an initial approximation of Eq. (17), which satisfies the boundary conditions. Obviously, considering Eq. (20) we will have:

$$\begin{cases} H(v,0) = L(v) - l(u_0) = 0, \\ H(v,1) = A(v) - f(r) = 0, \end{cases} \tag{21}$$

The changing process of *P* from zero to unity is just that of *v(r,p)* from $u_0(r)$ to $u(r)$. In topology, this is called deformation, and $L(v) - L(u_0)$ and $A(v) - f(r)$ are called homotopy. According to HPM, we can first use the embedding parameter *p* as "small parameter", and assume that the solution of Eq. (20) can be written as a power series in *P*:

$$v = v_0 + pv_1 + p^2 v_2 + \cdots, \tag{22}$$

Setting *p=1* results in the approximate solution of Eq. (17):

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \ldots., \tag{23}$$

The following opinions are suggested by He:

(1) The second derivative of N (V) with respect to V must be small because the parameter may be relatively large, i.e., p→1.

(2) The norm of L−1∂N/∂V must be smaller than one so that the series converges.

6. **Basic idea of He's variational iteration method. [10]**

To illustrate its basic concepts of the new technique, we consider following general differential equation

$$Lu + Nu = g(x) \tag{24}$$

Where L is a linear operator, and N a nonlinear operator, g(x) an inhomogeneous

or forcing term.
According to the variational iteration method, we can construct a correct functional as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(Lu_n(\xi) + N\tilde{u}_n(\xi) - g(\xi))d\xi \tag{25}$$

Where λ is a general Lagrange multiplier, which can be identified optimally Via the variational theory, the subscript n denotes the nth approximation, $\tilde{u}_n$ is considered as a restricted variation, i.e. $\delta\tilde{u}_n = 0$
For linear problems, its exact solution can be obtained by only one iteration step due to the fact that the Lagrange multiplier can be exactly identified.

7. Solution of the problem
7-1- The application of HPM:
7-1-1- Temperature distribution:

We can readily construct a homotopy which satisfies

$$H(T,P) = (1-p)(\frac{\partial T(x,t)}{\partial t} - \frac{\partial \mathrm{T}_0(x,t)}{\partial t}) + p(\frac{\partial T(x,t)}{\partial t} - k_1\frac{\partial^2 T(x,t)}{\partial x^2}) \tag{26}$$

One may now try to obtain a solution of Eq. (10) in the form of:

$$T(x,t) = T_0(x,t) + pT_1(x,t) + p^2T_2(x,t) \tag{27}$$

Where the $T_i(x,t)$, i=0, 1, 2,… are functions yet to be determined. The substitution of Eq. (27) into Eq. (26) yields:

$$\begin{cases} p^0 \Rightarrow \frac{\partial}{\partial t}T_0(x,t) - \frac{\partial}{\partial t}\mathrm{T}_0(x,t) = 0 \\ p^1 \Rightarrow \frac{\partial}{\partial t}T_1(x,t) - k_1(\frac{\partial^2 T_0(x,t)}{\partial x^2}) + \frac{\partial}{\partial t}\mathrm{T}_0(x,t) = 0 \\ p^2 \Rightarrow \frac{\partial}{\partial t}T_2(x,t) - k_1(\frac{\partial^2 T_1(x,t)}{\partial x^2}) = 0 \\ p^3 \Rightarrow -k_1(\frac{\partial^2 T_2(x,t)}{\partial x^2}) = 0 \end{cases} \tag{28}$$

The initial approximation $\mathrm{T}_0(x,t)$ or $T_0(x,t)$ can be freely chosen, here we set

$$\mathrm{T}_0(x,t) = T(x,0) = T_0(1-x^2) \tag{29}$$

With solving system of equation (28) then we obtain

$$T(x,t) = T_0(1-x^2) - 2pk_1T_0t \tag{30}$$

Setting *p=1* results in the approximate solution of Eq. (30):

$$T(x,t) = T_0 - T_0x^2 - 2k_1T_0t \tag{31}$$

You can see the results and comparison in the graphical plot.

7-1-2- Displacement equation:

We should rewrite a displacement equation (9) in term of $T(x,t)$ from previous stage:

$$\frac{Q}{c^2}\frac{\partial^2 u(x,t)}{\partial t^2}-\frac{\partial^2 u(x,t)}{\partial x^2}-2\eta T_0 x=0 \tag{32}$$

A homotopy can be redialy constructed as follows:

$$H(u,p)=(1-p)\frac{Q}{c^2}\{\frac{\partial^2}{\partial t^2}u(x,t)-\frac{\partial^2}{\partial t^2}U_0(x,t)\} \\ +p\{\frac{Q}{c^2}\frac{\partial^2}{\partial t^2}u(x,t)-\frac{\partial^2}{\partial x^2}u(x,t)-2\eta T_0 x\} \tag{33}$$

By the same manipulation as solution of temperature distribution the initial approximation $U_0(x,t)$, here we set:

$$U_0(x,t)=u(x,0)=x(1-x^2) \tag{34}$$

One may now try to obtain a solution of Eq. (32) in the form

$$u(x,t)=u_0(x,t)+pu_1(x,t)+p^2u_2(x,t) \tag{35}$$

Where the $u_i(x,t)$, i=0, 1, 2,… are functions yet to be determined. The substitution of Eq.(35) into Eq. (33) yields:

$$\begin{cases} p^0 \Rightarrow \frac{Q}{c^2}\frac{\partial^2}{\partial t^2}u_0(x,t)-\frac{Q}{c^2}\frac{\partial^2}{\partial t^2}U_0(x,t)=0 \\ p^1 \Rightarrow \frac{Q}{c^2}\frac{\partial^2}{\partial t^2}u_1(x,t)-2\eta T_0 x-\frac{\partial^2}{\partial x^2}u_0(x,t)+\frac{Q}{c^2}\frac{\partial^2}{\partial t^2}U_0(x,t)=0 \\ p^2 \Rightarrow -\frac{\partial^2}{\partial x^2}u_1(x,t)+\frac{Q}{c^2}\frac{\partial^2}{\partial t^2}u_2(x,t)=0 \\ p^3 \Rightarrow -\frac{\partial^2}{\partial x^2}u_2(x,t)=0 \end{cases} \tag{36}$$

With solving system of equation (36) then we obtain

$$u(x,t)=x(1-x^2)+\frac{1}{2}\frac{(2\eta T_0 x-6x)c^2t^2p}{Q} \tag{37}$$

Setting *p=1* results in the approximate solution of Eq.(37):

$$u(x,t)=\frac{x(1-x^2)Q+xc^2t^2\eta T_0-3xc^2t^2}{Q} \tag{38}$$

You can see the results and comparison in the graphical plot.

7-2- The application of VIM

7-2-1- Temperature distribution:

We can write the correction function as follows:

$$T_{n+1}(x,t) = T_n(x,t) + \int_0^x \lambda(\xi)\{-k_1 \frac{\partial^2 T_n(\xi,t)}{\partial \xi^2} + \frac{\partial \widetilde{T}_n(\xi,t)}{\partial t}\}d\xi \quad (39)$$

Where $\widetilde{T}_n(x,t)$ is considered as restricted variation.
Its stationary conditions can be obtained as follows:

$$\begin{cases} -k_1\lambda(x) = 0 \\ 1 + k_1\lambda'(x) = 0 \\ \lambda''(\xi) = 0 \end{cases} \quad (40)$$

The multiplier, therefore, can be identified as:

$$\lambda(\xi) = \frac{1}{k_1}(x-\xi) \quad (41)$$

And the following variational iteration formula can be obtained:

$$T_{n+1}(x,t) = T_n(x,t) + \int_0^x \frac{1}{k_1}(x-\xi)\{-k_1 \frac{\partial^2 T_n(\xi,t)}{\partial \xi^2} + \frac{\partial T_n(\xi,t)}{\partial t}\}d\xi \quad (42)$$

Now we begin with an arbitrary initial approximation: $T_0(x,t) = A(t)x + B(t)$, where *A* and *B* are constant to be determined, by the variational iteration formula in *x*-direction , we have

$$T_1(x,t) = A(t)x + B(t) - \frac{1}{3}\frac{\left(\frac{d}{dt}A(t)\right)x^3}{k_1} + \frac{1}{2}\frac{\left(\left(\frac{d}{dt}A(t)\right)x - \frac{d}{dt}B(t)\right)x^2}{k_1} + \frac{x^2\left(\frac{d}{dt}B(t)\right)}{k_1} \quad (43)$$

By imposing the boundary conditions at $x = 0$ and $x = 1$ yields $A(t) = 0$ and $B(t) = T_0 e^{(-2k_1 t)}$, thus we have:

$$T_1(x,t) = -T_0 e^{(-2k_1 t)}(-1 + x^2) \quad (44)$$

You can see the results and comparison in the graphical plot.

7-2-2- Displacement equation:

We should rewrite a displacement equation in term of $T(x,t)$ as same as Eq.(32):
Its correction functional can be written down as follows

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\tau)\{\frac{Q}{c^2}\frac{\partial^2 u_n(x,\tau)}{\partial \tau^2} - \frac{\partial^2 \widetilde{u}_n(x,\tau)}{\partial x^2} + \eta \frac{\partial \widetilde{T}(x,\tau)}{\partial x}\}d\tau \quad (45)$$

Where $\widetilde{u}_n(x,\tau)$ is a restricted variation.
Its stationary conditions can be obtained as follows:

$$\begin{cases} \frac{Q}{c^2}\lambda(t)=0 \\ 1-\frac{Q}{c^2}\lambda'(t)=0 \\ \lambda''(\tau)=0 \end{cases} \tag{46}$$

The Lagrange multiplier can be readily identified as:

$$\lambda(\tau)=\frac{c^2}{Q}(\tau-t) \tag{47}$$

And the following iteration formula can be obtained:

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^t \frac{c^2}{Q}(\tau-t)\{\frac{Q}{c^2}\frac{\partial^2 u_n(x,\tau)}{\partial\tau^2}-\frac{\partial^2\tilde{u}_n(x,\tau)}{\partial x^2}-2\eta T_0 e^{(-2k_1 t)}x\}d\tau \tag{48}$$

Assuming that the trial-function has the form;

$$u_0(x,t)=u(x,0)=x(1-x^2) \tag{49}$$

With substitution of Eq. (49) to Eq. (48) we will have:

$$u_1(x,t)=-\frac{x(-Q+Qx^2+3c^2t^2-c^2t^2\eta T_0 e^{(-2k_1t)})}{Q} \tag{50}$$

By the same manipulation and by Eq. (48) we obtain its second approximation:

$$u_2(x,t)=u_1(x,t)=-\frac{x(-Q+Qx^2+3c^2t^2-c^2t^2\eta T_0 e^{(-2k_1t)})}{Q} \tag{51}$$

7-3- The theorical exact solutions;

In order to compare the approximate solutions obtained by the Hpm and Vim with the exact solution of Eqs (9) and (10) we must have the exact solution.

7-3-1- Temperature distribution:

The exact solution of Eq (10) is obtained as:

$$T(x,t)=4T_0\sum_{n=0}^{\infty}(-1)^n\frac{1}{\beta_n{}^3}e^{-k_1\beta_n{}^2 t}\cos(\beta_n x) \tag{52}$$

$$\beta_n=\frac{(2n+1)\pi}{2} \tag{53}$$

7-3-2- Displacement Equation:

The exact solution of heterogeneous equation of displacement, i.e., Eq. (9), can be written in the form of:

$$u(x,t)=\sum_{k=1}^{\infty}G_k(t)\sin(k\pi x) \tag{54}$$

Function $G_k(t)$ are the solution of following initial value problem:

$$\ddot{G}_k(t)+a^2k^2\pi^2G_k(t)=2\int_0^1 F(x,t)\sin(k\pi x)dx \tag{55}$$

Subject to the initial conditions of:

$$G_k(0)=2\int_0^1 f(x)\sin(k\pi x)dx \tag{56}$$

$$\dot{G}_k(0)=2\int_0^1 g(x)\sin(k\pi x)dx \tag{57}$$

$$a^2=\frac{c^2}{Q},F(x,t)=-\frac{c^2}{Q}\eta\frac{\partial T}{\partial x},f(x)=u(x,0),g(x)=u_t(x,0) \tag{58}$$

It has to be noted that by backward substitution of displacement equation of the proposed method and exact solutions into equations (4) and (5), stress component distributions throughout the slab thickness are obtained accordingly.

8. Result and Discussions

The results presented here are based on using the physical constant listed in Table1 for a 18-8 stainless steel slab in the CGS system. In this section, the results obtained by the HPM & VIM method are presented and compared with those of exact solution.

The variation of temperature at given times and different values of "*x*" is also shown in Figure 1 and 2, where it is apparent that even the approximate solution by the HPM & VIM method is in reasonably good agreement with the exact solution. The same turned of agreement has been observed at any other given time.

The variation of $u(x,t)$ at a fixed time and different values of "*x*" obtained by the HPM & VIM method are shown in Figure 3 and 4.It is apparently seen that the variation of displacement obtained by the HPM & VIM method is in relatively good agreement with the exact solution.

Finally, thermal stresses found by the HPM & VIM method are shown in Figure 5,6 and compared with the stresses based on the exact solution .A good agreement between the two is seen that verifies the application of proposed methods for those cases with no exact solution.

9. Conclusion

The HPM & VIM has been successfully used to obtain to propagation of thermal stresses throughout the slab thickness. Displacement temperature and stress distribution in the slab thickness have been presented and compared with those of exact solution. This shows very good agreement. This verifies the applicability of the HPM & VIM methods for those problems with no exact solutions.

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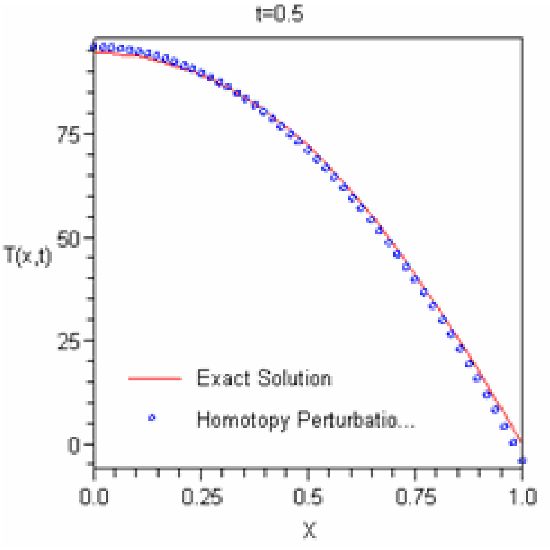
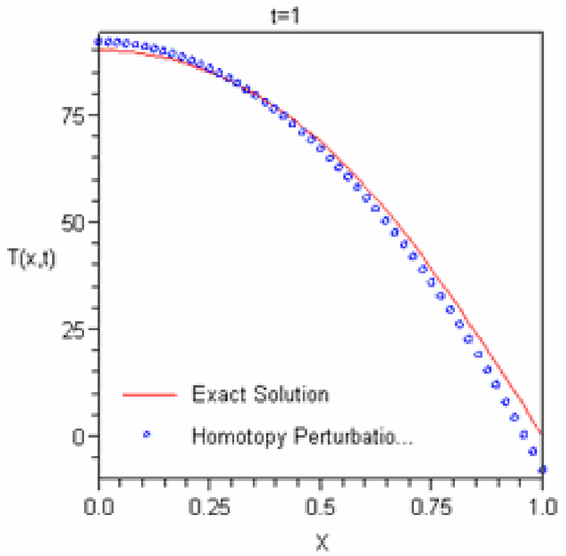
Table 1.Physical Constant for 18-8 stainless steel in CGS system			
Parameter	Name	value	Unit
$\boldsymbol{E}$	Young Modulus	18×10^{11}	Dyn/cm^2
$\boldsymbol{k_1}$	Thermal Diffusivity	0.039	Cm^2/s
$\boldsymbol{\sigma}$	Poison's Ratio	0.305	
$\boldsymbol{\mu}$	Lame's Constant	6.89×10^{11}	Dyn/cm^2
$\boldsymbol{\alpha}$	Linear Expansion coefficient	16.4×10^{-6}	$1/^{o}c$
$\boldsymbol{Q}$	Density	7.93	g/cm^3
$\boldsymbol{T_0}$	Initial Temprature	100	^{o}c

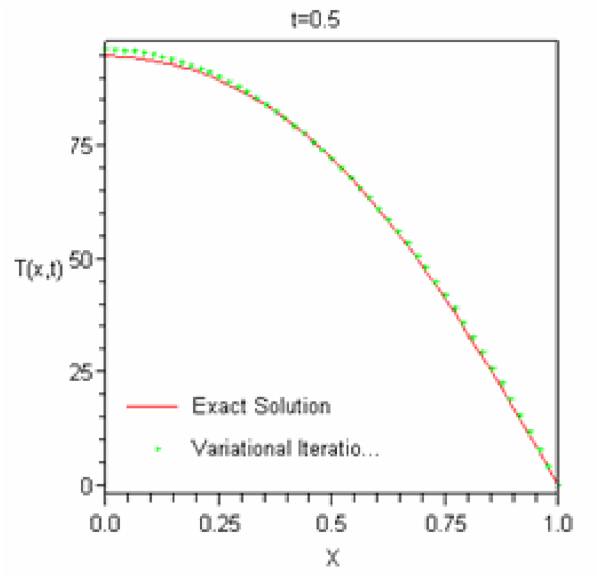


(a)

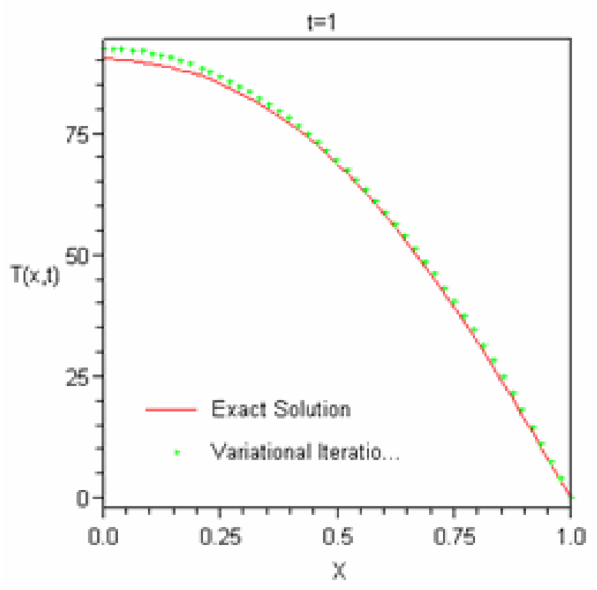


(b)

Figure 1. Variation of temperature thought the slab thickness by HPM at: (a) *t=0.5*s (b) *t=1*s



(a)



(b)

Figure 2. Variation of temperature thought the slab thickness by VIM at: (a) $t=0.5$s (b) $t=1$s

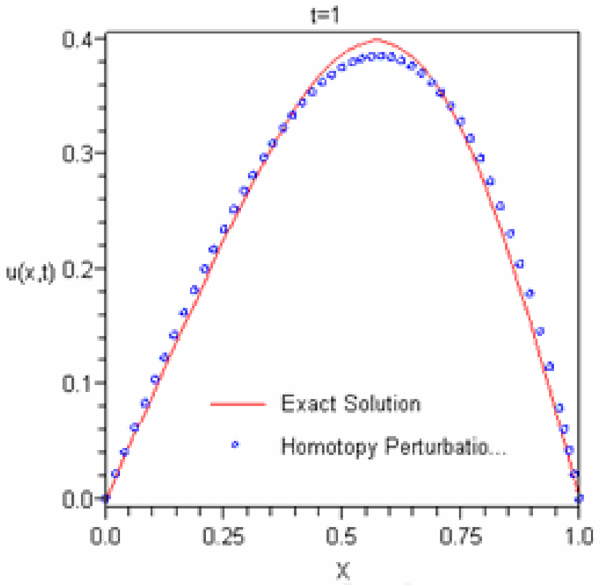


Figure 3. Variation of Displacement thought the slab thickness by HPM at: $t=1$s

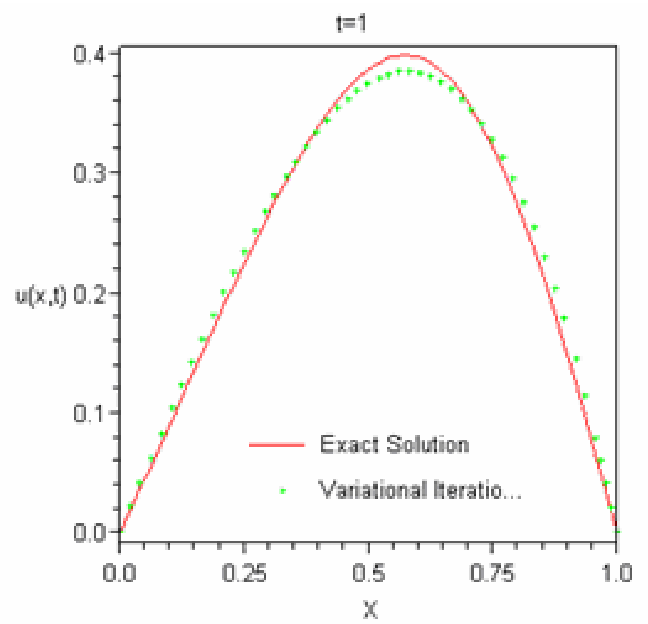


Figure 4. Variation of Displacement thought the slab thickness by VIM at: $t=1$s

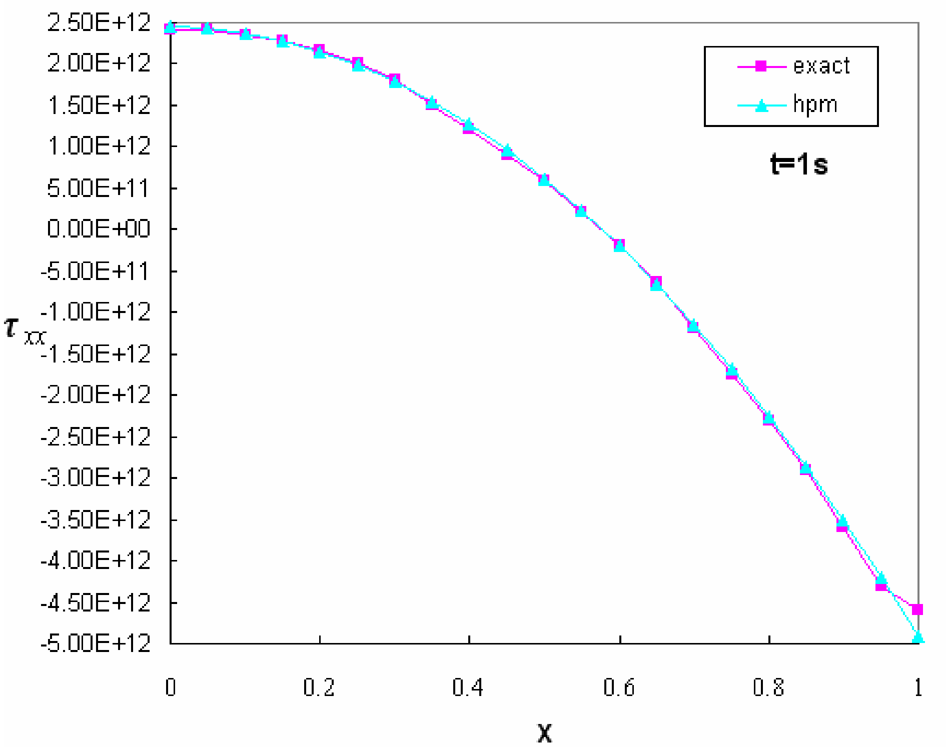


Figure 5. Variation of thermal stress in x direction thought the slab thickness by HPM at: *t=1*s

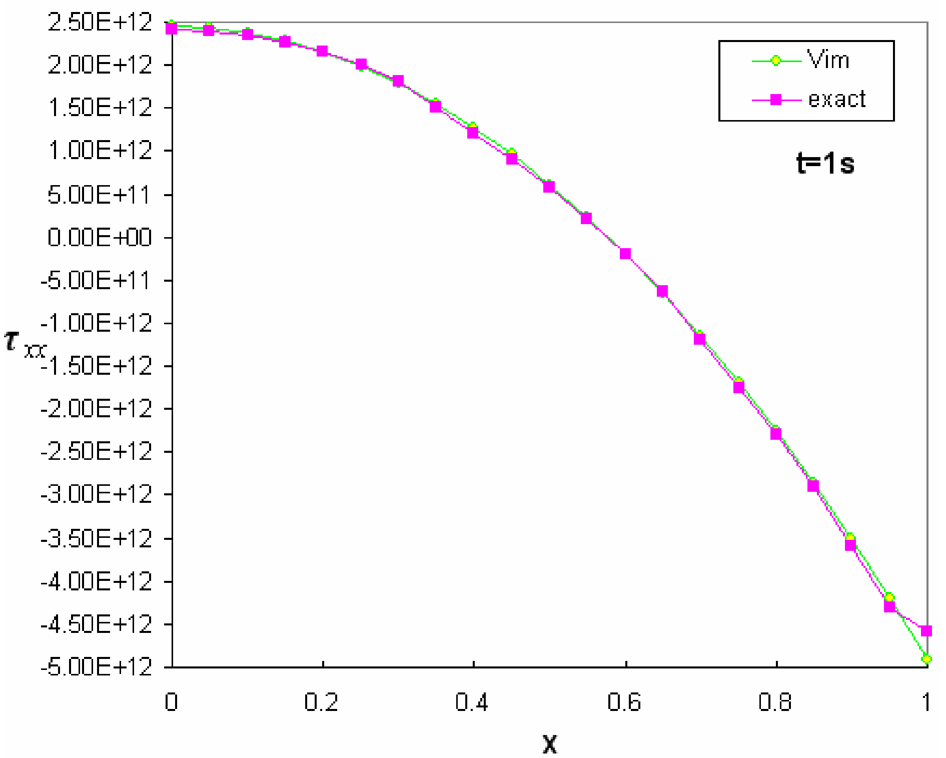


Figure 6. Variation of thermal stress in x direction thought the slab thickness by VIM at: *t=1*s

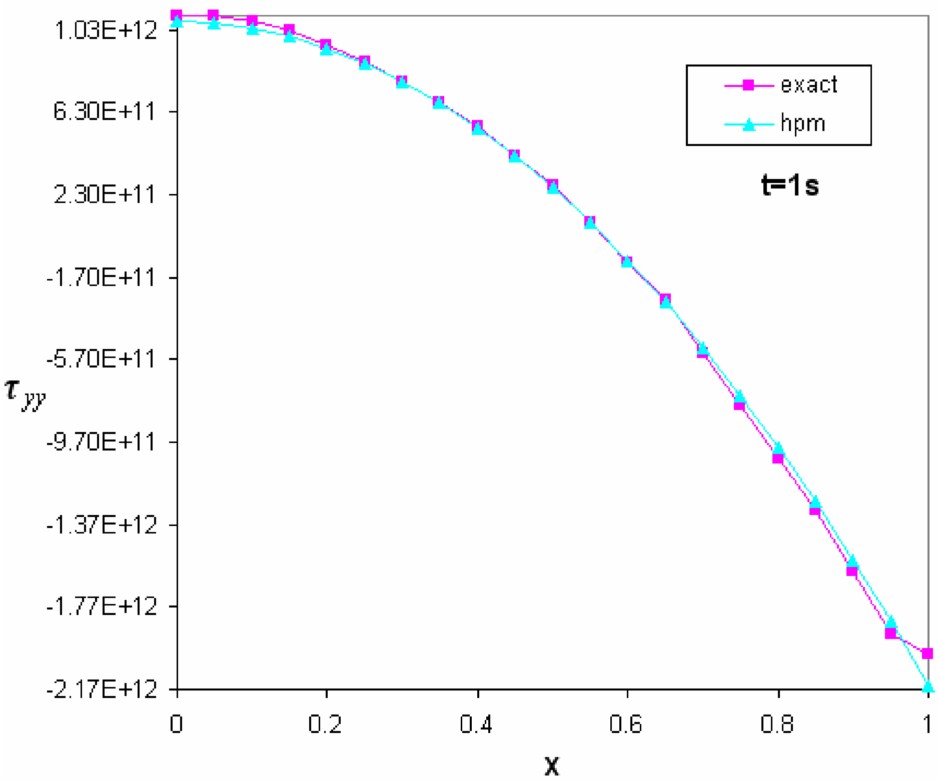


Figure 4. Variation of thermal stress in *y,z* direction thought the slab thickness by HPM at: *t=1*s

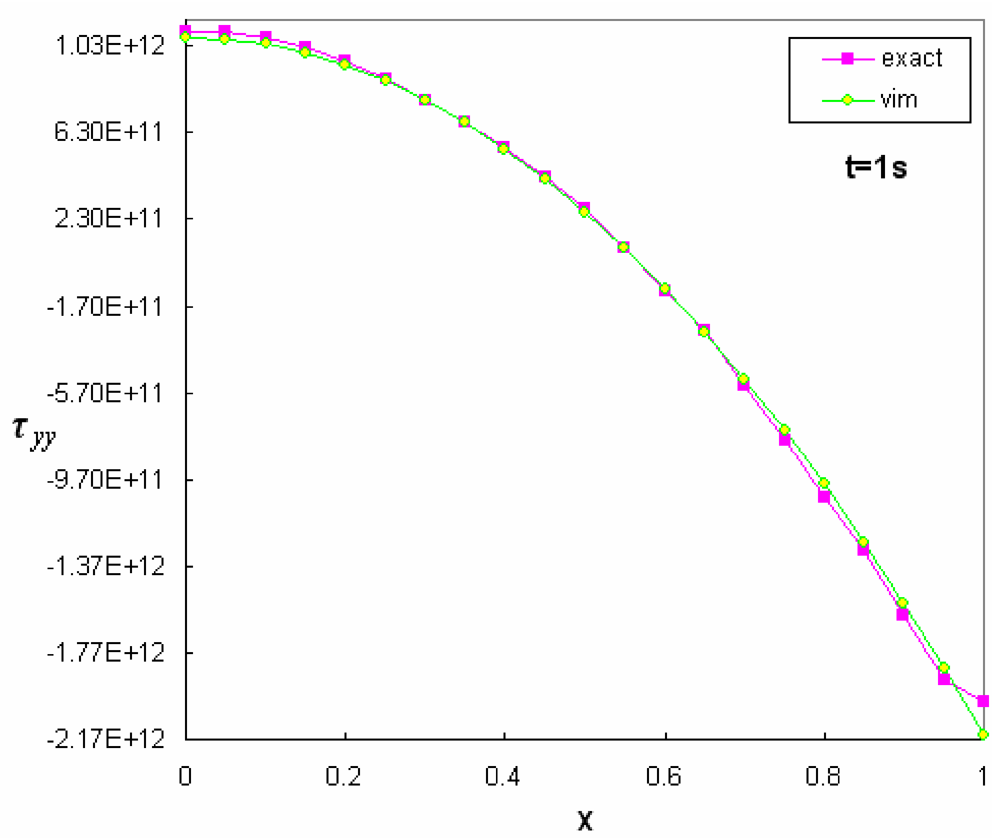


Figure 4. Variation of thermal stress in *y,z* direction thought the slab thickness by VIM at: *t=1*s