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Hamiltonian System and Classical Mechanics

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Abstract

In this paper the formalism of Hamiltonian system (X, ω, H) on the symplectic manifold due to Reeb [2] given in Abraham and Marsden [4] and Arnold[5] is used to derive the equations of motion (1) for a particle in a line with a bona fide solution (2) for a free particle in three-space with Hamiltonian and non-Hamiltonian flow, it is also shown that the non-Hamiltonian flow can be converted into a Hamiltonian flow by changing symplectic form and the phase space.

Keywords: Symplectic Manifold, Hamiltonian System, Phase-Space, Hamiltonian Flow

(1.1) Introduction

The use of differential form in mechanics and its eventual formulation in terms of symplectic manifolds has been slowly evolving since Cartan [1].

In a closed system the energy is constant as the physical system moves in time. For any given physical system, each known conserved quantity provides an equation that can be very useful in the analysis of the system. But energy is more than a conserved quantity, it determine the equation of motion. All conceivable information about the system is hidden in the energy formula. Unpacking this information is a mathematical problem. In this paper we will use the Hamiltonian system to derive the equations of motion in classical mechanics from the energy function and the kinetics of phase space. In other words, we will show how a real-valued function on a symplectic manifold determines a vector field. The energy function is known as Hamiltonian function and is denoted by *H*. The corresponding vector field on phase space is called a Hamiltonian vector field and is denoted by X_H .This vector field corresponds naturally to a system of first-order differential equations on phase space, which in physical system is equivalent to Newton's second law (Force = mass × acceleration). The solution of the differential equations is the Hamiltonian flow. Physically the Hamiltonian flow gives us possible physical motions.

(1.2) Hamiltonian System

A general Hamiltonian system consists of a manifold X, possibly infinite dimensional together with a (weakly) non-degenerate closed two-form ω on X (i.e. ω is an alternating bilinear form on each tangent space $T_x X$ of X, $d\omega = 0$, and for $x \in X, \omega_x(u,v) = 0$ for all $u \in T_x X$ implies $v = 0$) and a Hamiltonian function $H : X \to R$.Then X, H, ω determine in nice cases, a vector field X_H called the Hamiltonian vector field.

Let X be a Banach manifold and let $X_\circ$ be a manifold domain of X. Let H be a C^∞ function on $X_\circ$ into R called the Hamiltonian or energy function, then for each $x \in X_\circ$, $d_x H : T_x X_\circ \to R$ is a continuous linear function. Let ω be a C^∞ covariant tensor field of order two on a C^∞ manifold domain $X_\circ$ such that for each $x \in X_\circ$, $\omega_x : T_x X_\circ \times T_x X_\circ \to R$ is non-degenerate, for each $x \in X_\circ$, let $\lambda_x : T_x X_\circ \to T_x^* X_\circ$ be the isomorphism induced by ω_x and is defined by

$$\lambda(u).(v) = \omega_x(u,v) \quad \forall\, x \in X_\circ; u,v \in T_x X_\circ$$

Denote $\lambda : TX_\circ \to T^* X_\circ$ where $\lambda_x = \lambda \upharpoonright_{T_x X_\circ}$ for each $x \in X_\circ$, this λ is a vector bundle isomorphism of $TX_\circ$ onto $T^* X_\circ$. Then ω and H induces a C^∞ vector field $X_H = \lambda^{-1} \circ dH$ on $X_\circ$, determined by the condition

$$C_{X_H}\omega = dH \tag{1.2.1}$$

(1.3) Flow

Let X be a smooth manifold. A smooth function $F : R \times X \to X$ is called a flow for the vector field v if $F_{\cdot x} : R \to X$ is an integral solution for v i.e.

$$\frac{d}{dt} F_{\cdot x}(t) = v \circ F_{\cdot x}(t)$$

or

$$\frac{d}{dt} F(t,x) = v \circ F(t,x)$$

and

$$F_{\cdot x}(0) = F(0,x) = x \quad \forall\ t \in R, x \in X$$

(1.4) Hamiltonian Flow

Let (X, H, ω) be a Hamiltonian system. A flow F is called a Hamiltonian flow if it preserves the symplectic form and Hamiltonian function (i.e. $F_t^*\omega = \omega$ and $F_t^* H = H$ for $t \in R$) see Abraham & Marsden[4].

Here, it should be noted that the symplectic form plays a crucial role i.e. without changing the Hamiltonian function but changing the symplectic form we can get different flows i.e. vector fields.

(2.1) The motion of a particle on a line in a plane

Here we consider the example of physical system whose phase space is the simplest non-trivial symplectic manifold, the two-dimensional plane $X = R^2 = \{(r, p) : r \in R, p \in R\}$ with the area two-form $\omega = dr \wedge dp$. Consider a particle of mass m moving on a line, subject to no forces, such a particle is called a free particle, any free particle travels with constant speed. Thus the Hamiltonian function for such a particle is $H = \frac{1}{2m} p^2$. The equation $C_{X_H}\omega = dH$ gives the Hamiltonian vector field X_H for any $r \in R$, $v \in T_x R$ and taking $X_H = x_r \frac{\partial}{\partial r} + x_p \frac{\partial}{\partial p}$ and $v = v_r \frac{\partial}{\partial r} + v_p \frac{\partial}{\partial p}$ as arbitrary vector field, we find

$$(dr \wedge dp)(x_r \frac{\partial}{\partial r} + x_p \frac{\partial}{\partial p}, v_r \frac{\partial}{\partial r} + v_p \frac{\partial}{\partial p}) = (\frac{p}{m} dp)(v_r \frac{\partial}{\partial r} + v_p \frac{\partial}{\partial p})$$

or $$x_r v_p - x_p v_r = \frac{p}{m} v_p$$

$$\Rightarrow \quad x_r = \frac{p}{m} \text{ and } x_p = 0.$$

Thus, we have

$$X_H = \frac{p}{m}\frac{\partial}{\partial r}$$

since r and p are functions of time t(along a particular trajectory), taking the vector field $X_H = \frac{dr}{dt}.\frac{\partial}{\partial r} + \frac{dp}{dt}.\frac{\partial}{\partial p}$ as time derivative along trajectories on the plane, we have

$$\frac{p}{m}\frac{\partial}{\partial r} = \frac{dr}{dt}.\frac{\partial}{\partial r} + \frac{dp}{dt}.\frac{\partial}{\partial p}$$

Since, $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial p}$ are linearly independent, we have

$$\frac{p}{m} = \frac{dr}{dt} \text{ and } \frac{dp}{dt} = 0 . \qquad (2.1.1)$$

which shows that a free particle travels with constant momentum along the line. This is equivalent to $\frac{d^2r}{dt^2} = 0$, the equation of motion for the free particle.

Since the Hamiltonian flow preserves the Hamiltonian and the symplectic form we can use this conservation of the Hamiltonian by the Hamiltonian flow to draw useful pictures, because it implies that the orbits of the system must lie inside level sets of H . (an orbit a set of all points in phase space that the system passes through , during on particular motion. In other words it is the set of all points on one particular trajectory). The beautiful features of Hamiltonian systems is that we can get information about orbits of the differential equations of motion by solving the algebraic equation $H =$ constant, which is easy to solve. For example, if we consider the motion of a free particle on the line then the conservation of the Hamiltonian by the Hamiltonian flow tells us that orbits must lie inside sets of the form $H = \frac{p^2}{2m} =$ constant. Since the motion is continuous, it follows that each orbit is contained in a line $p =$ constant (see figure 1). Here it should be noted that not every orbit is an entire line. The r-axis $(p = 0)$ is made up of single-point orbit representing motionless particles. All other orbits are entire lines representing particles moving at constant velocities.

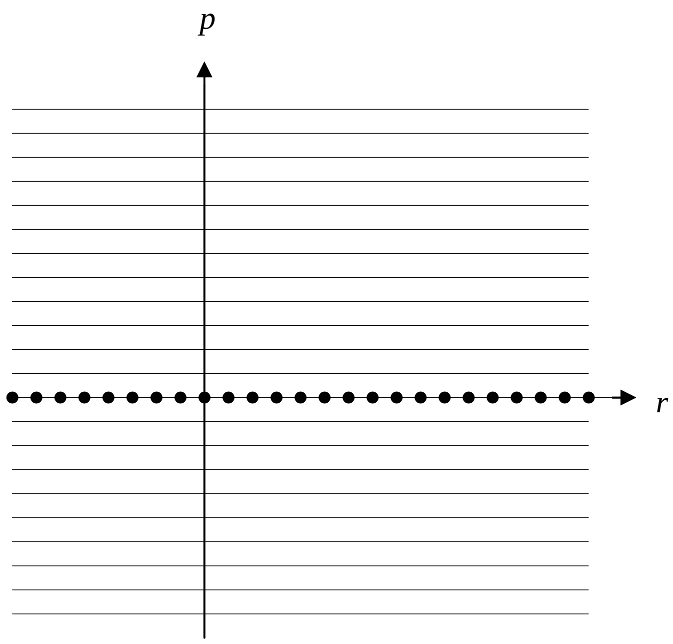

Figure 1 : phase space of the particle on the line with level sets of the force free Hamiltonian.

If we desire to find bonafide solutions to our differential equation, i.e., we desire to know not only the orbit of a trajectory but the trajectory itself (i.e., the position as a function of time). We have to solve one differential equation than solving the original system of differential equations ($\frac{dr}{dt}=\frac{p}{m}$ and $\frac{dp}{dt}=0$).

Let

$$H_0=\frac{p^2}{2m}$$

$$\Rightarrow \quad p=\pm\sqrt{2mH_0}$$

$$\text{or} \quad \frac{dr}{dt}=\pm\sqrt{\frac{2H_0}{m}} \tag{2.1.2}$$

We take plus sign if the orbit lies on a line above the r-axis, otherwise we take the minus sign. Now we can easily integrate the above equation (2.1.2) to get

$$r(t)=\pm\sqrt{\frac{2H_0}{m}}\,t+r(0)$$

initially when $t=0$, $p(t)=p(0)$, so

$$r(t)=\frac{t}{m}p(0)+r(0)$$

for any fixed t, the flow map $f_t:R^2\to R^2$ defined by

$$\begin{pmatrix} r \\ p \end{pmatrix}\mapsto\begin{pmatrix} r+\frac{p}{m}t \\ p \end{pmatrix}$$

preserves symplectic form $\omega=dr\wedge dp$. For,

$$f_t^*(dr\wedge dp)=f_t^*(dr)\wedge f_t^*(dp)$$

$$=(dr+\frac{t}{m}dp)\wedge dp$$

$$=dr\wedge dp=\omega.$$

(2.2) Motion of a free particle in Three-Space

Consider the motion of a free particle in three space. Let $\mathbf{r}=(r_1,r_2,r_3)$ be the position vector of the particle and $\mathbf{p}=(p_1,p_2,p_3)$ be the corresponding momentum of the particle. Then the phase space of the particle is the manifold $X=\{(r_1,r_2,r_3,p_1,p_2,p_3):r_1,r_2,r_3,p_1,p_2,p_3\in R\}$ with the symplectic form $\omega=dr_1\wedge dp_1+dr_2\wedge dp_2+dr_3\wedge dp_3$. The Hamiltonian function of the system is

$H = \frac{1}{2m}(p_1^2 + p_2^2 + p_3^2)$. Then (X, ω, H) determine vector field X_H by the condition (1.2.1).

Let $X_H = a_1 \frac{\partial}{\partial r_1} + a_2 \frac{\partial}{\partial r_2} + a_3 \frac{\partial}{\partial r_3} + b_1 \frac{\partial}{\partial p_1} + b_2 \frac{\partial}{\partial p_2} + b_3 \frac{\partial}{\partial p_3}$ and $v = \sum_{i=1}^{i=3} a_i' \frac{\partial}{\partial r_i} + b_i' \frac{\partial}{\partial p_i}$ be arbitrary vector fields, then using (1.2.1), we have

$$\omega(X_H, v) = dH(v)$$

or $$(\sum_{i=1}^{i=3} dr_i \wedge dp_i)(\sum_{i=1}^{i=3} a_i \frac{\partial}{\partial r_i} + b_i \frac{\partial}{\partial p_i}, \sum_{i=1}^{i=3} a_i' \frac{\partial}{\partial r_i} + b_i' \frac{\partial}{\partial p_i})$$

$$= [\frac{1}{m}(\sum_{i=1}^{i=3} p_i dp_i)](\sum_{i=1}^{i=3} a_i' \frac{\partial}{\partial r_i} + b_i' \frac{\partial}{\partial p_i})$$

or $$\sum_{i=1}^{i=3}(a_i b_i' - a_i' b_i) = \frac{1}{m}(\sum_{i=1}^{i=3} p_i b_i')$$

This gives,

$$a_i = \frac{p_i}{m} \quad \text{and} \quad b_i = 0, \qquad \text{i=1,2,3.}$$

Thus the vector field is given by

$$X_H = \frac{p_1}{m}\frac{\partial}{\partial r_1} + \frac{p_2}{m}\frac{\partial}{\partial r_2} + \frac{p_3}{m}\frac{\partial}{\partial r_3} \tag{2.2.1}$$

Taking the vector field,

$$X_H = \frac{dr_1}{dt}\frac{\partial}{\partial r_1} + \frac{dr_2}{dt}\frac{\partial}{\partial r_2} + \frac{dr_3}{dt}\frac{\partial}{\partial r_3} + \frac{dp_1}{dt}\frac{\partial}{\partial p_1} + \frac{dp_2}{dt}\frac{\partial}{\partial p_2} + \frac{dp_3}{dt}\frac{\partial}{\partial p_3} \tag{2.2.2}$$

as time derivative along trajectories, we have

$$\frac{p_1}{m} = \frac{dr_1}{dt}, \frac{p_2}{m} = \frac{dr_2}{dt}, \frac{p_3}{m} = \frac{dr_3}{dt} \quad \text{and} \quad \frac{dp_i}{dt} = 0 \;, \qquad \text{i = 1,2,3.}$$

This gives,

$$m\frac{d^2}{dt^2}(\mathbf{r}) = 0 \tag{2.2.3}$$

This is the required equation of motion of the free particle in three- space.
It is evident from above that for any fixed time t, the map

$$f_t : R^3 \times (R^3)^* \to R^3 \times (R^3)^*$$

defined by $$\begin{pmatrix} \mathbf{r} \\ \mathbf{p} \end{pmatrix} \mapsto \begin{pmatrix} \mathbf{r} + \frac{\mathbf{p}}{m}t \\ \mathbf{p} \end{pmatrix}$$

is a Hamiltonian flow as it preserves the symplectic form $\omega = dr_1 \wedge dp_1 + dr_2 \wedge dp_2 + dr_3 \wedge dp_3$ and the Hamiltonian function $H = \frac{1}{2m}(p_1^2 + p_2^2 + p_3^2)$, for

$$f_t^* \omega = f_t^* (\sum_{i=1}^{i=3} dr_i \wedge dp_i)$$
$$= \sum_{i=1}^{i=3} (f_t^* dr_i \wedge f_t^* dp_i)$$
$$= \sum_{i=1}^{i=3} (dr_i + \frac{t}{m} dp_i) \wedge dp_i$$
$$= \sum_{i=1}^{i=3} (dr_i \wedge dp_i) = \omega$$

and $f_t^* H = H$, as f_t changes only the values of **r**, while H depends only on **p**. So f_t preserves the Hamiltonian.

Now to show that every flow is not a Hamiltonian flow. Consider an example of the motion of a free particle in space having the flow,

$$g_t : R^3 \times (R^3)^* \to R^3 \times (R^3)^*$$

defined by
$$\begin{pmatrix} \mathbf{r} \\ \mathbf{p} \end{pmatrix} \mapsto \begin{pmatrix} \mathbf{r}\, e^t \\ \mathbf{p}\, e^t \end{pmatrix} \qquad (2.2.4)$$

for any $t \in R$, then $g_t^* \omega = e^{2t} \omega$ shows that the flow g_t does not preserves the symplectic form. Hence it is not the Hamiltonian flow of a Hamiltonian system with the canonical symplectic form on R^6.

Now if we define the symplectic form on $R^6 \sim \{0\}$ as $\omega = \sum_{i=1}^{i=3} \frac{1}{r_i p_i} (dr_i \wedge dp_i)$ then the flow g_t defined by (2.2.4) preserves ω, for

$$g_t^* \omega = g_t^* (\sum_{i=1}^{i=3} \frac{1}{r_i p_i} (dr_i \wedge dp_i))$$
$$= \sum_{i=1}^{i=3} g_t^* (\frac{1}{r_i p_i} dr_i \wedge dp_i)$$
$$= \sum_{i=1}^{i=3} \frac{1}{(r_i e^t \, p_i e^t)} (d(r_i e^t) \wedge d(p_i e^t))$$
$$= \sum_{i=1}^{i=3} \frac{1}{(r_i p_i e^{2t})} e^{2t} (dr_i \wedge dp_i)$$
$$= \sum_{i=1}^{i=3} \frac{1}{(r_i p_i)} (dr_i \wedge dp_i)$$
$$= \omega .$$

In order to find the Hamiltonian function for this system, let $X_H = r_1 \frac{\partial}{\partial r_1} + r_2 \frac{\partial}{\partial r_2} + r_3 \frac{\partial}{\partial r_3} + p_1 \frac{\partial}{\partial p_1} + p_2 \frac{\partial}{\partial p_2} + p_3 \frac{\partial}{\partial p_3}$ and $v = \sum_{i=1}^{i=3} a_i' \frac{\partial}{\partial r_i} + b_i' \frac{\partial}{\partial p_i}$ be an arbitrary vector field then using (1.2.1), we have

$$\omega (X_H , v) = dH(v)$$

or $$\sum_{i=1}^{i=3}(\frac{1}{r_i p_i} dr_i \wedge dp_i)(\sum_{i=1}^{i=3} r_i \frac{\partial}{\partial r_i} + p_i \frac{\partial}{\partial p_i}, \sum_{i=1}^{i=3} a_i' \frac{\partial}{\partial r_i} + b_i' \frac{\partial}{\partial p_i}) = dH(v)$$

or $$\sum_{i=1}^{i=3}(\frac{1}{r_i p_i}(r_i b_i' - a_i' p_i)) = (\sum_{i=1}^{i=3} \frac{\partial H}{\partial r_i} dr_i + \sum_{i=1}^{i=3} \frac{\partial H}{\partial p_i} dp_i)(\sum_{i=1}^{i=3} a_i' \frac{\partial}{\partial r_i} + b_i' \frac{\partial}{\partial p_i})$$

This gives,

$$\frac{\partial H}{\partial r_i} = -\frac{1}{r_i} \quad , \quad \frac{\partial H}{\partial p_i} = \frac{1}{p_i}$$

which on integration yields,

$$H = \log\left(\frac{p_1 p_2 p_3}{r_1 r_2 r_3}\right) + c \tag{2.2.5}$$

Now,

$$\begin{aligned} g_t^* H &= g_t^*\left(\log\left(\frac{p_1 p_2 p_3}{r_1 r_2 r_3}\right) + c \right) \\ &= \log\left(\frac{(p_1 e^t)(p_2 e^t)(p_3 e^t)}{(r_1 e^t)(r_2 e^t)(r_3 e^t)}\right) + c \\ &= H \end{aligned}$$

Hence g_t also preserves H. Thus g_t defined by (2.2.4) is a Hamiltonian flow for the Hamiltonian system (M, ω, H), where $M = R^6 - \{0\}$, $\omega = \sum_{i=1}^{i=3} \frac{1}{r_i p_i}(dr_i \wedge dp_i)$ and H is given by (2.2.5).

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