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# Wave Properties of the Elastic Half-Space Loaded by a Periodic Distribution of Vibrating Punches: An Analytical Approach

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**Abstract**

An analytical approach is developed to study the wave properties of an elastic half-space subjected to harmonic vibrations applied on its free surface by a periodic array of rigid punches. In the frequency range ensuring the so-called *one-mode* (far-field) propagation, both the *anti-plane* and *in-plane* problems are reduced to integral equations which are solved analytically. The explicit formulas obtained for the wave field are reflected through some figures in order to discuss the peculiar physical properties of the structure.

**Mathematics Subject Classification:** 74J99

**Keywords:** Wave properties, Vibrating punches, Analytical results

## 1 Introduction

It is well known that force transfer from one elastic body to another is realized through some contact interactions. This type of problems is of a high importance in mechanical or civil engineering, as well as in various other fields of the elasticity theory and practice [7]. Mathematically, such problems are related to the so-called class of problems with *mixed boundary conditions* : this means that typically the unknown contact stresses arise on certain parts of the boundary surface, while on other parts of the same surface these contact stresses are known, representing the *natural* boundary conditions [8,17]. As a rule, the strict mathematical formulation of such contact problems reduces them to some sort of integral equations.

In the case of the dynamic theory of elasticity, the most important application is probably connected with seismic and seismological problems, more precisely in the theory of seismic vibrations of the foundation and in the seismological practice of *exploration* activity [2]. As an example of the latter topic, the researchers use to apply a vibrator on the ground surface to generate a certain structure of waves inside the ground, in order to evaluate the mechanical properties of soils and/or discover expected deposits of coal, oil or natural gas [3]. Initially, one massive vibrator (so-called *punch*) was used for this purpose; however, some recent results in published works seem to show that the efficiency of energy transmission into the soil can be improved by using several vibrating punches simultaneously. Actually, it is demonstrated that multiple sources and receivers arrays provide an efficient high-resolution seismological technique [11]. This can be explained by the fact that in this case one can arrange an optimal control of the amplitude and phase distribution between adjacent vibrators, in order to achieve some desirable wave structure [18-20]. Even a pseudo-random sequence in amplitude and phase modulation can be applied to make higher the efficiency of such systems.

The problems in concern are also important in different contexts such as microcontact problems arising in *micro-technologies* [5, 6], where multiple arrays of punches are used for various technical aims.

It should be noted that, from the mathematical point of view, the case of any periodic wave structure can be formulated within the framework established by the Floquet theory, which was originally developed to study differential equations with periodic coefficients. An interesting approach to a periodic contact problem based on the Floquet analysis is presented in [4]. Some similar ideas are applicable even in the case of structures of non-constant periodicity [16].

Based on the above consideration, the main goal of the present paper is to investigate the wave propagation through an elastic half-space originated by the vibration of a periodic array of (identical) punches applied on its free surface. We will treat both the *anti-plane* and *in-plane* problems in a linear context, and reduce them to integral equations holding over the basis of one (typical) punch. Then, following the guidelines of some previous papers of ours devoted to scattering problems in acoustic and/or elastic context $[13-15]$, we will apply an analytical solution of such equations valid in the so-called *one-mode* regime of far-field propagation (which is related to not very high frequencies). The explicit representation of the wave field obtained in this way will be finally reflected in several figures, from which the physical properties of the structure can be deduced and commented. A direct numerical treatment has been also applied to the main equations to control the precision of the analytical solution.

# 2 Formulation of the problem and reduction to integral equations

Let us consider a periodic distribution of absolutely rigid coplanar *punches* vibrating harmonically with given frequency and (same) phase above the free surface of an elastic half-space $y \geq 0$. The punches are infinitely long (in the $z-$direction), while the common width of their bases is $2b$; the period of the array is $2a$ $(a > b)$. The total contact area will be denoted by $S \times \{-\infty < z < +\infty\}$; it holds here

$$S = \bigcup_{n=-\infty}^{+\infty} (-b + 2an, b + 2an). \tag{2.1}$$

Figure 1 shows the section of the structure with (any) normal plane $xy$.

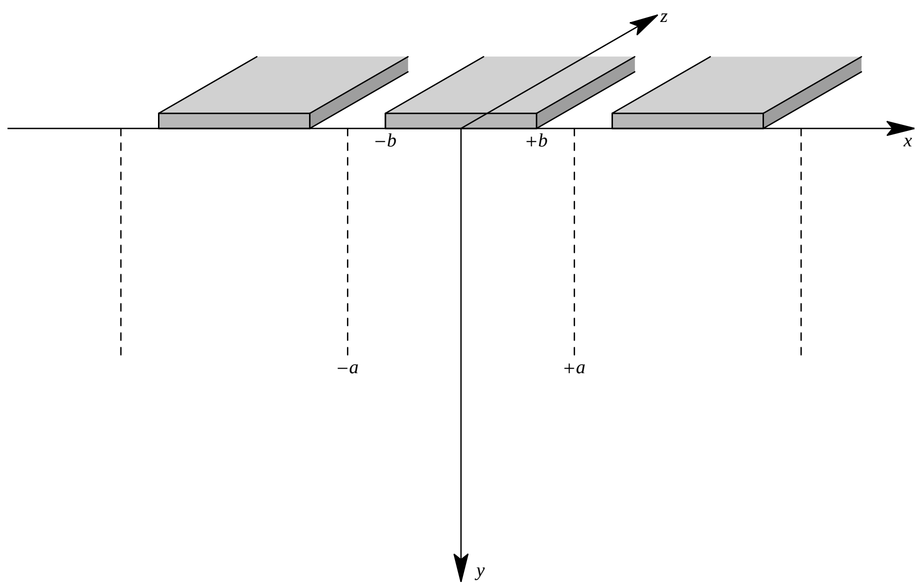


Figure 1: A periodic array of rigid punches (of width $2b$) vibrating above the free surface of an elastic half-space. The period is $2a$.

In the *anti-plane* problem, we assume that each punch is perfectly joined with the half-space, and the applied forces $Pe^{-i\omega t}$ also are directed along $z-$axis. In the *in-plane* problem, we assume that the contact between punch and half-space is frictionless, and the applied forces $Pe^{-i\omega t}$ are directed along $y-$axis, namely, normally onto the punches. $P$ denotes the amplitude per unit length in the $z-$direction. In both cases, this clearly implies a displacement of given amplitude at $y = 0$ (in the contact zones), which will be involved as boundary datum in our problems.

Of course, the linear model here adopted implies that time dependence is harmonic with frequency $\omega$ in the whole structure; thus, the common factor $e^{-i\omega t}$ should be present (but actually omitted) throughout.

## 2.1 The anti-plane problem

In the given geometry, the displacement field $\mathbf{u}$ in the elastic half-space has non-trivial only its $z-$component $u_z = u_z(x, y, t)$, clearly independent on $z$; thus, the only unknown of the problem is the stationary *wave field*

$$w(x, y) = u_z(x, y, t)\, e^{i\omega t}, \tag{2.2}$$

and the governing equation is Helmholtz equation

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + k^2 w = 0, \tag{2.3}$$

where $k = \omega\sqrt{\rho/\mu}$ denotes the (transverse) wave number, while $\rho$ and $\mu$ are mass density and shear modulus of the elastic material, respectively.

The stress tensor has non-trivial only the tangential components given by

$$\tau_{xz} = \mu\, \frac{\partial w}{\partial x}, \quad \tau_{yz} = \mu\, \frac{\partial w}{\partial y}; \tag{2.4}$$

moreover, solution $w$ to Eq.(2.3) should satisfy a radiation condition as $y \to +\infty$.

If we assume that all punches produce the same amplitude of vibration $w_0$ along $z-$axis and the elastic surface $y = 0$ is free where not joined with punches, the relevant boundary conditions for the problem at hand are

$$w(x, 0) = w_0\ , \quad x \in S; \qquad \frac{\partial w}{\partial y}(x, 0) = 0\ , \quad x \in (-\infty, +\infty) - S. \quad \text{(2.5a, 2.5b)}$$

We now submit the main equations to Fourier transformation $f \to \hat{f}$ along $x-$axis [9] . As a consequence, Eq.(2.3) becomes an ordinary differential equation (with respect to $y$) for the *transformed* wave field

$$\hat{w}(\alpha, y) = \int_{-\infty}^{+\infty} w(x, y)\, e^{i\alpha x}\, dx \quad , \tag{2.6}$$

which can be easily solved to give

$$\hat{w}(\alpha, y) = A(\alpha)\, e^{\gamma(\alpha) y} + B(\alpha)\, e^{-\gamma(\alpha) y} \quad , \quad y \geq 0 \, , \tag{2.7a}$$

$$\gamma(\alpha) = \sqrt{\alpha^2 - k^2} \quad , \quad \alpha \in (-\infty, +\infty) \tag{2.7b}$$

(the branch in the square-root is chosen so that $\sqrt{-1} = -i$). The radiation condition implies $A(\alpha) = 0$, thus we can deduce

$$\hat{w}(\alpha, y) = \hat{w}(\alpha, 0)\, e^{-\gamma(\alpha) y} \tag{2.8}$$

Let us now denote the tangential stress $\tau_{yz}$ over the surface $y = 0$ as $\tau(x)$, $x \in (-\infty, +\infty)$ (of course, $\tau \equiv 0$ outside $S$); by Eq.(2.4), we get

$$\hat{\tau}(\alpha) = \mu \, \frac{d\hat{w}}{dy}(\alpha, 0). \tag{2.9}$$

As follows from (2.8, 2.9),

$$\hat{\tau}(\alpha) = -\mu \, \gamma(\alpha) \, \hat{w}(\alpha, 0),$$

hence it finally holds

$$\hat{w}(\alpha, y) = -\frac{e^{-\gamma(\alpha) y}}{\mu \, \gamma(\alpha)} \, \hat{\tau}(\alpha) = -\frac{e^{-\gamma(\alpha) y}}{\mu \, \gamma(\alpha)} \int\limits_{-\infty}^{+\infty} \tau(\xi)\, e^{i\alpha\xi}\, d\xi \ , \quad y \geq 0. \tag{2.10}$$

Therefore, by inverse transformation, the wave field is given by the representation formula

$$\begin{aligned} w(x, y) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{w}(\alpha, y)\, e^{-i\alpha x}\, d\alpha = \\ &= -\frac{1}{2\pi\mu} \int_S \tau(\xi) \left[ \int_{-\infty}^{+\infty} \frac{e^{-\gamma(\alpha) y}}{\gamma(\alpha)}\, e^{-i\alpha(x-\xi)}\, d\alpha \right] d\xi \quad , \quad y \geq 0, \end{aligned} \tag{2.11}$$

taking also into account Eq.(2.5b).

An (integral) equation to be imposed on function $\tau(x)$, $x \in S$, can be derived by using Eqs.(2.5a) in Eq.(2.11), as follows

$$\int_S \tau(\xi) \left[ \int\limits_{-\infty}^{+\infty} \frac{e^{-i\alpha(x-\xi)}}{\gamma(\alpha)}\, d\alpha \right] d\xi = -2\pi\mu\, w_0 \ , \quad x \in S. \tag{2.12}$$

Once solved this integral equation, Eq.(2.11) gives the full structure of the wave field throughout the elastic half-space. It is worth noting that the kernel (in square brackets) above is nothing but the well known Green's function (properly scaled) for the 2-dim wave equation [1] .

## 2.2 The in-plane problem

In the given geometry, the displacement (or wave) field $\mathbf{u}$ has non-trivial only the components $u_x(x,y,t),\ u_y(x,y,t)$. By using Green-Lamè representation,

$$u_x = \frac{\partial\varphi}{\partial x} + \frac{\partial\psi}{\partial y}, \quad u_y = \frac{\partial\varphi}{\partial y} - \frac{\partial\psi}{\partial x} \tag{2.13}$$

and omitting - henceforth - the harmonic time dependence, the governing equations are the (uncoupled) Helmholtz equations for the *potentials* $\varphi(x,y)$ and $\psi(x,y)$ :

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + k_1^2\varphi = 0, \quad \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + k_2^2\psi = 0, \tag{2.14}$$

where $k_1 = \omega\sqrt{\rho/(\lambda+2\mu)}$ and $k_2 = \omega\sqrt{\rho/\mu}$ $(> k_1)$ denote the longitudinal and transverse wave numbers, respectively ($\rho, \mu$ as before; $\lambda$ is the Lamè modulus of the elastic material). The potentials should satisfy a radiation condition as $y \to +\infty$.

The relevant components of the stress tensor are given by

$$\tau_{xy} = \mu\left(2\frac{\partial^2\varphi}{\partial x\partial y} + \frac{\partial^2\psi}{\partial y^2} - \frac{\partial^2\psi}{\partial x^2}\right), \tag{2.15a}$$

$$\sigma_{yy} = \lambda\left(\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2}\right) + 2\mu\left(\frac{\partial^2\varphi}{\partial y^2} - \frac{\partial^2\psi}{\partial x\partial y}\right). \tag{2.15b}$$

If we assume that all punches produce the same amplitude of vibration $u_0$ along $y-$axis, and the elastic surface $y=0$ is free where not pushed by punches (recalling that contact is frictionless), the boundary conditions for the problem at hand are

$$u_y(x,0) = u_0\ ,\ \ x \in S; \qquad \tau_{xy}(x,0) = 0\ ,\ \ x \in (-\infty,+\infty); \tag{2.16a, b}$$

$$\sigma_{yy}(x,0) = 0\ ,\ \ x \in (-\infty,+\infty) - S\ . \tag{2.16c}$$

As before, by applicating Fourier transformation to Eqs.(2.14), we get as transformed solutions

$$\hat{\varphi}(\alpha,y) = A(\alpha)\,e^{-\gamma_1(\alpha)y}\ , \quad \hat{\psi}(\alpha,y) = B(\alpha)\,e^{-\gamma_2(\alpha)y}\ , \quad y \ge 0 \tag{2.17a}$$

$$\gamma_1(\alpha) = \sqrt{\alpha^2 - k_1^2}\ , \quad \gamma_2(\alpha) = \sqrt{\alpha^2 - k_2^2}\ , \quad \alpha \in (-\infty,+\infty) \tag{2.17b}$$

where the radiation condition has been already used (and correct branch implied in square roots - see after Eq.(2.7)).

On the other hand, Eqs.(2.15a) and (2.16b) imply

$$\frac{d^2\hat{\psi}}{dy^2}(\alpha,0) + \alpha^2\hat{\psi}(\alpha,0) - 2i\alpha\frac{d\hat{\varphi}}{dy}(\alpha,0) = 0,$$

so that by Eqs.(2.17) we get

$$\left(2\alpha^2 - k_2^2\right) B(\alpha) + 2i\alpha\gamma_1(\alpha)\, A(\alpha) = 0. \tag{2.18}$$

Let us now denote the normal stress $\sigma_{yy}$ over the surface $y = 0$ as $\sigma(x)$, $x \in (-\infty, +\infty)$ (of course, $\sigma \equiv 0$ outside $S$); by (2.15b), we get

$$\left(2\alpha^2 - k_2^2\right) A(\alpha) - 2i\alpha\gamma_2(\alpha)\, B(\alpha) = \hat{\sigma}(\alpha)/\mu. \tag{2.19}$$

The linear system (2.18, 2.19) gives

$$A(\alpha) = \frac{(2\alpha^2 - k_2^2)\ \hat{\sigma}(\alpha)/\mu}{\Delta(\alpha)}\ , \quad B(\alpha) = -\frac{2i\alpha\gamma_1(\alpha)\, \hat{\sigma}(\alpha)/\mu}{\Delta(\alpha)}\ , \tag{2.20}$$

where the determinant $\Delta(\alpha) = (2\alpha^2 - k_2^2)^2 - 4\alpha^2\gamma_1(\alpha)\,\gamma_2(\alpha)$ has some similarity with the well-known *Rayleigh function* [1].

By substituting Eqs.(2.20) into Eqs.(2.17) and using (transformed) Eqs.(2.13), we deduce

$$\hat{u}_x(\alpha, y) = \frac{i\alpha}{\mu\Delta(\alpha)} \left[(k_2^2 - 2\alpha^2)\, e^{-\gamma_1(\alpha)y} + 2\gamma_1(\alpha)\gamma_2(\alpha)\, e^{-\gamma_2(\alpha)y}\right] \int_S \sigma(\xi)\, e^{i\alpha\xi}\, d\xi\ , \tag{2.21a}$$

$$\hat{u}_y(\alpha, y) = \frac{\gamma_1(\alpha)}{\mu\Delta(\alpha)} \left[(k_2^2 - 2\alpha^2)\, e^{-\gamma_1(\alpha)y} + 2\alpha^2\, e^{-\gamma_2(\alpha)y}\right] \int_S \sigma(\xi)\, e^{i\alpha\xi}\, d\xi\ , \tag{2.21b}$$

so that, by inverse transformation, the (stationary) wave field $\mathbf{u}$ is given by the representation formulas

$$u_x(x, y) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{u}_x(\alpha, y)\, e^{-i\alpha x}\, d\alpha = \frac{i}{2\pi\mu} \int_S \sigma(\xi) \times$$
$$\times \left\{ \int_{-\infty}^{+\infty} \frac{\alpha}{\Delta(\alpha)} \left[(k_2^2 - 2\alpha^2)\, e^{-\gamma_1(\alpha)y} + 2\gamma_1(\alpha)\gamma_2(\alpha)\, e^{-\gamma_2(\alpha)y}\right] e^{-i\alpha(x-\xi)}\, d\alpha \right\} d\xi\ , \quad y \geq 0, \tag{2.22a}$$

$$u_y(x, y) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{u}_y(\alpha, y)\, e^{-i\alpha x}\, d\alpha = \frac{1}{2\pi\mu} \int_S \sigma(\xi) \times$$
$$\times \left\{ \int_{-\infty}^{+\infty} \frac{\gamma_1(\alpha)}{\Delta(\alpha)} \left[(k_2^2 - 2\alpha^2)\, e^{-\gamma_1(\alpha)y} + 2\alpha^2\, e^{-\gamma_2(\alpha)y}\right] e^{-i\alpha(x-\xi)}\, d\alpha \right\} d\xi\ , \quad y \geq 0. \tag{2.22b}$$

Like in the anti-plane problem, an (integral) equation to be imposed on function $\sigma(x)$, $x \in S$, can be derived by using Eqs.(2.16a) in Eq.(2.22b), as follows

$$\int_S \sigma(\xi) \left[ \int_{-\infty}^{+\infty} \frac{\gamma_1(\alpha)}{\Delta(\alpha)} e^{-i\alpha(x-\xi)} \, d\alpha \right] d\xi = \frac{2\pi\mu}{k_2^2} u_0 \,, \quad x \in S. \tag{2.23}$$

Once solved this integral equation, Eqs.(2.22) give the full structure of the wave field throughout the elastic half-space.

## 2.3 Fully periodic (anti-plane and in-plane) problems

As it can be easily recognized, the basic equations (2.11, 12) for anti-plane problem and (2.22, 23) for in-plane problem, have been obtained for an arbitrary contact area $S$. In the full-periodic case we are treating, where $S$ is given by (2.1) and the punches vibrate with the same phase, we can assume that the distribution of tangential or normal stress $\tau(x)$ or $\sigma(x)$ is the same over each interval $(-b + 2an, b + 2an)$, being here an even function. As a consequence,

$$\int_S \begin{pmatrix} \tau(\xi) \\ \sigma(\xi) \end{pmatrix} e^{i\alpha\xi} \, d\xi = \sum_{n=-\infty}^{+\infty} \int_{-b}^{b} \begin{pmatrix} \tau(\xi) \\ \sigma(\xi) \end{pmatrix} e^{i\alpha(\xi + 2an)} \, d\xi = \tag{2.24}$$

$$= \frac{\pi}{a} \sum_{m=-\infty}^{+\infty} \int_{-b}^{b} \begin{pmatrix} \tau(\xi) \\ \sigma(\xi) \end{pmatrix} e^{i\alpha\xi} \, \delta(\alpha - \frac{\pi m}{a}) \, d\xi \quad ,$$

where well-known properties of Dirac function $\delta$ have been used [9] :

$$\sum_{n=-\infty}^{+\infty} e^{int} = \sum_{m=-\infty}^{+\infty} \delta\left(\frac{t}{2\pi} - m\right), \qquad \delta(\beta t) = \frac{1}{\beta}\delta(t). \tag{2.25}$$

Thus, the representation formulas (2.11) and (2.22), along with integral equations (2.12) and (2.23), respectively, can be accordingly rewritten in the following forms:

$$w(x,y) = -\frac{1}{2a\mu} \sum_{m=-\infty}^{+\infty} \frac{e^{-q_m y}}{q_m} \left( \int_{-b}^{b} \tau(\xi) \, e^{-i\pi m(x-\xi)/a} d\xi \right) =$$

$$= \frac{1}{2aik\mu} e^{iky} \left( \int_{-b}^{b} \tau(\xi) d\xi \right) - \frac{1}{a\mu} \sum_{m=1}^{+\infty} \frac{e^{-q_m y}}{q_m} \left( \int_{-b}^{b} \tau(\xi) \cos\frac{\pi m \xi}{a} \, d\xi \right) \cos\frac{\pi m x}{a} \quad , \quad y \geq 0; \tag{2.26}$$

$$\begin{cases} \int\limits_{-b}^{b} \tau(\xi)\, K^\tau(x-\xi)\, d\xi = -\mu w_0 \pi/2 \equiv C^\tau \quad , \quad x \in (-b,b), \\ K^\tau(x) = \dfrac{\pi}{4a} \sum\limits_{m=-\infty}^{+\infty} \dfrac{e^{-i\pi m x/a}}{q_m} = -\dfrac{\pi}{4aik} + \dfrac{\pi}{2a} \sum\limits_{m=1}^{+\infty} \dfrac{\cos(\pi m x/a)}{q_m}, \end{cases} \tag{2.27}$$

where $q_m = q_{-m} = \sqrt{(\pi m/a)^2 - k^2}$ $(q_0 = -ik)$, for the anti-plane problem, - and

$$u_x(x,y) = \frac{i}{2a\mu} \sum_{m=-\infty}^{+\infty} \frac{\pi m/a}{\Delta_m} \left\{ \left[ k_2^2 - 2\left(\frac{\pi m}{a}\right)^2 \right] e^{-q_m y} + 2 q_m r_m\, e^{-r_m y} \right\} \times$$
$$\times \left( \int\limits_{-b}^{b} \sigma(\xi)\, e^{-i\pi m(x-\xi)/a}\, d\xi \right) = \frac{1}{a\mu} \sum_{m=1}^{+\infty} \frac{\pi m/a}{\Delta_m} \times$$
$$\times \left\{ \left[ k_2^2 - 2\left(\frac{\pi m}{a}\right)^2 \right] e^{-q_m y} + 2 q_m r_m\, e^{-r_m y} \right\} \left( \int\limits_{-b}^{b} \sigma(\xi) \cos\frac{\pi m \xi}{a}\, d\xi \right) \sin\frac{\pi m x}{a} \ , \quad y \geq 0; \tag{2.28a}$$

$$u_y(x,y) = \frac{1}{2a\mu} \sum_{m=-\infty}^{+\infty} \frac{q_m}{\Delta_m} \left\{ \left[ k_2^2 - 2\left(\frac{\pi m}{a}\right)^2 \right] e^{-q_m y} + 2\left(\frac{\pi m}{a}\right)^2 e^{-r_m y} \right\} \times$$
$$\times \left( \int\limits_{-b}^{b} \sigma(\xi)\, e^{-i\pi m(x-\xi)/a}\, d\xi \right) = -\frac{i k_1}{2a k_2^2 \mu} e^{i k_1 y} \left( \int\limits_{-b}^{b} \sigma(\xi) d\xi \right) + \frac{1}{a\mu} \sum_{m=1}^{+\infty} \frac{q_m}{\Delta_m} \times$$
$$\times \left\{ \left[ k_2^2 - 2\left(\frac{\pi m}{a}\right)^2 \right] e^{-q_m y} + 2\left(\frac{\pi m}{a}\right)^2 e^{-r_m y} \right\} \left( \int\limits_{-b}^{b} \sigma(\xi) \cos\frac{\pi m \xi}{a}\, d\xi \right) \cos\frac{\pi m x}{a} \ , \quad y \geq 0; \tag{2.28b}$$

$$\begin{cases} \int\limits_{-b}^{b} \sigma(\xi)\, K^\sigma(x-\xi)\, d\xi = \mu\, u_0\, \pi (k_1^2 - k_2^2)/k_2^2 \equiv C^\sigma \quad , \quad x \in (-b,b), \\ K^\sigma(x) = \dfrac{\pi(k_1^2 - k_2^2)}{2a} \sum\limits_{m=-\infty}^{+\infty} \dfrac{q_m}{\Delta_m} e^{-i\pi m x/a} = \\ \qquad = -\dfrac{i\pi k_1 (k_1^2 - k_2^2)}{2a k_2^4} + \dfrac{\pi(k_1^2 - k_2^2)}{a} \sum\limits_{m=1}^{+\infty} \dfrac{q_m}{\Delta_m} \cos(\pi m x/a), \end{cases} \tag{2.29}$$

where $q_m = q_{-m} = \sqrt{(\pi m/a)^2 - k_1^2}$ , $r_m = r_{-m} = \sqrt{(\pi m/a)^2 - k_2^2}$ , $\Delta_m = \Delta_{-m} = \left[2(\pi m/a)^2 - k_2^2\right]^2 - 4(\pi m/a)^2 q_m r_m$ $(q_0 = -ik_1,\ r_0 = -ik_2,\ \Delta_0 = k_2^4)$, for the in-plane problem.
Eveness of $\tau, \sigma$ has been used.

# 3 Solution in the one-mode approximation

Of course, both integral equations (2.27), (2.29) could be directly submitted to standard numerical algorithms for arbitrary values of all parameters involved. However, in this paper we prefer to remain in an analytical context, and to this end we accept to put an upper (*cut-off*) bound to possible frequencies. Thus, for both anti-plane and in-plane problems here treated, we now assume that frequency belong to the range such that

$$k,\ (k_1 <)\ k_2 < \pi/a \tag{3.1}$$

This trivially implies $q_m,\ r_m > 0 \ \ \forall m \geq 1$, and therefore guarantees the so-called *one-mode* far-field propagation, in the sense that at large distance from the surface $y = 0$, only the terms extracted from summations (of order zero) remain as propagating waves (with the given wave number $k$ or $k_1$)[1] in Eqs.(2.26), (2.28b).

Positions (3.1) entitles us to put

$$q_m \approx r_m \approx \frac{\pi m}{a}, \quad \Delta_m \approx 2\left(\frac{\pi m}{a}\right)^2 (k_1^2 - k_2^2) \quad \forall m \geq 2, \tag{3.2}$$

keeping exact the values for $m = 1$ (cf. [13, 14]). Looking at integral equations (2.27) and (2.29), by this approximation the kernels become

$$K^{\tau,\sigma}(x) = -A^{\tau,\sigma} - B^{\tau,\sigma} \cos\frac{\pi x}{a} - \frac{1}{2} \ln|2\sin\frac{\pi x}{2a}| \,, \tag{3.3a}$$

where we put

$$A^{\tau} = \frac{\pi}{4aik},\ A^{\sigma} = \frac{i\pi k_1(k_1^2 - k_2^2)}{2ak_2^4},\ B^{\tau} = \frac{aq_1 - \pi}{2aq_1},\ B^{\sigma} = \frac{1}{2} - \frac{\pi(k_1^2 - k_2^2)q_1}{a\Delta_1} \,, \tag{3.3b}$$

and used summation $\sum_{m=1}^{+\infty}(1/m)\cos(\pi m x/a) = -\ln|2\sin(\pi x/2a)|$. As a consequence, those equations attain the forms

$$-\frac{1}{2}\int_{-b}^{b} \ln|2\sin\frac{\pi(x-\xi)}{2a}| \begin{pmatrix} \tau(\xi) \\ \sigma(\xi) \end{pmatrix} d\xi = \begin{pmatrix} A^{\tau}H_0^{\tau} + C^{\tau} \\ A^{\sigma}H_0^{\sigma} + C^{\sigma} \end{pmatrix} + \begin{pmatrix} B^{\tau}H_1^{\tau} \\ B^{\sigma}H_1^{\sigma} \end{pmatrix} \cos\frac{\pi x}{a} \,,\ |x| < b, \tag{3.4a}$$

[1]In the in-plane problem, the far-field wave is clearly (only) longitudinal, according to the applied vibration of punches.

where

$$\begin{pmatrix} H_0^\tau \\ H_0^\sigma \end{pmatrix} = \int_{-b}^{b} \begin{pmatrix} \tau(\xi) \\ \sigma(\xi) \end{pmatrix} d\xi, \quad \begin{pmatrix} H_1^\tau \\ H_1^\sigma \end{pmatrix} = \int_{-b}^{b} \begin{pmatrix} \tau(\xi) \\ \sigma(\xi) \end{pmatrix} \cos\frac{\pi\xi}{a}\, d\xi\,. \tag{3.4b}$$

Now, it is clear that, if $g_0(\xi),\ g_1(\xi)$ solve the *auxiliary* equations

$$-\frac{1}{2}\int_{-b}^{b} \ln|2\sin\frac{\pi(x-\xi)}{2a}|\, g_\nu(\xi)\, d\xi = \cos\frac{\pi\nu x}{a}\ ,\ |x|<b,\ \nu=0,1, \tag{3.5}$$

not containing wave numbers, then by linearity it holds

$$\begin{pmatrix} \tau(\xi) \\ \sigma(\xi) \end{pmatrix} = \begin{pmatrix} A^\tau H_0^\tau + C^\tau \\ A^\sigma H_0^\sigma + C^\sigma \end{pmatrix} g_0(\xi) + \begin{pmatrix} B^\tau H_1^\tau \\ B^\sigma H_1^\sigma \end{pmatrix} g_1(\xi)\ ,\ \ |\xi|<b. \tag{3.6}$$

Unknown constants $H_0,\ H_1$ can be calculated by (twice) integrating Eq.(3.6) as it is and after multiplying by $\cos\pi\xi/a$; one gets a linear system as follows

$$\begin{cases} (1-A^{\tau,\sigma}G_0^0)\, H_0^{\tau,\sigma} - B^{\tau,\sigma}G_1^0\, H_1^{\tau,\sigma} = C^{\tau,\sigma}G_0^0 \\ -A^{\tau,\sigma}G_0^1\, H_0^{\tau,\sigma} + (1-B^{\tau,\sigma}G_1^1)\, H_1^{\tau,\sigma} = C^{\tau,\sigma}G_0^1 \end{cases}, \tag{3.7}$$

where new constants $G$ are given by

$$G_\nu^m = \int_{-b}^{b} g_\nu(\xi)\cos\frac{\pi m\xi}{a}\, d\xi\ ,\quad \nu=0,1;\quad m=0,1,2,... \tag{3.8}$$

(here for only $m=0,1$) and are clearly free of any frequency parameter.

By substituting Eqs.(3.6) into representation formulas (2.26), (2.28), we get explicit expressions of such formulas holding in the whole structure, as follows

$$w(x,y) = \frac{1}{2aik\mu}e^{iky}\, H_0^\tau - \\ -\frac{1}{a\mu}\sum_{m=1}^{+\infty}\frac{e^{-q_m y}}{q_m}\left[(A^\tau H_0^\tau + C^\tau)\, G_0^m + B^\tau H_1^\tau G_1^m\right]\cos\frac{\pi m x}{a}\quad ,\quad y\geq 0; \tag{3.9}$$

$$u_x(x,y) = \frac{1}{a\mu}\sum_{m=1}^{+\infty}\frac{\pi m/a}{\Delta_m}\left\{\left[k_2^2 - 2\left(\frac{\pi m}{a}\right)^2\right]e^{-q_m y} + 2q_m r_m\, e^{-r_m y}\right\}\times \\ \times\left[(A^\sigma H_0^\sigma + C^\sigma)\, G_0^m + B^\sigma H_1^\sigma G_1^m\right]\sin\frac{\pi m x}{a}\ ,\quad y\geq 0; \tag{3.10a}$$

$$u_y(x,y) = -\frac{ik_1}{2ak_2^2\mu} e^{ik_1 y} H_0^\sigma + \\ +\frac{1}{a\mu}\sum_{m=1}^{+\infty}\frac{q_m}{\Delta_m}\left\{\left[k_2^2 - 2\left(\frac{\pi m}{a}\right)^2\right] e^{-q_m y} + 2\left(\frac{\pi m}{a}\right)^2 e^{-r_m y}\right\}\times \\ \times\left[\left(A^\sigma H_0^\sigma + C^\sigma\right) G_0^m + B^\sigma H_1^\sigma G_1^m\right]\cos\frac{\pi m x}{a}\ , \quad y \geq 0. \tag{3.10b}$$

# 4 Analytical representation

Equations (3.5) have been solved in [15], where integrals (3.8) are also calculated. This would clearly lead to full-explicit expressions for the contact stresses $\tau(x)$ or $\sigma(x)$ by means of Eqs.(3.6, 3.7). Looking at Eqs.(3.9, 3.10) for the wave field (see also (3.7)), we only need to report the following values of integrals $G_0^m$, $G_1^m$ $(m = 0, 1, 2, ...)$ :

$$G_0^m = -\frac{2}{\ln\sin\frac{\pi b}{2a}}\cos\frac{\pi m b}{a} + \frac{m}{\pi}\left(\frac{2}{\pi} + \frac{1}{\ln\sin\frac{\pi b}{2a}}\right)\left[\left(\sum_{h=0}^{m-2}\right)' I_h\, J_{m-2-h} - \pi J_m + \pi\cos\frac{\pi b}{a} J_{m-1}\right] + \\ +\frac{m/\pi^2}{\ln\sin\frac{\pi b}{2a}}\left[\left(\sum_{h=0}^{m-2}\right)' I_h\left(P_{m-2-h} + P_{h-m+1}\right) - \pi(P_m + P_{-m-1}) + \pi\cos\frac{\pi b}{a}(P_{m-1} + P_{-m})\right], \tag{4.1}$$

$$G_1^m = -\frac{2\cos^2\frac{\pi b}{2a}}{\ln\sin\frac{\pi b}{2a}}\cos\frac{\pi m b}{a} + \frac{m}{\pi}\frac{\cos^2\frac{\pi b}{2a}}{\ln\sin\frac{\pi b}{2a}}\left[\left(\sum_{h=0}^{m-2}\right)' I_h\, J_{m-2-h} - \pi J_m + \pi\cos\frac{\pi b}{a} J_{m-1}\right] + \\ +\frac{m}{\pi^2}\frac{\cos^2\frac{\pi b}{2a}}{\ln\sin\frac{\pi b}{2a}}\left[\left(\sum_{h=0}^{m-2}\right)' I_h\left(P_{m-2-h} + P_{h-m+1}\right) - \pi(P_m + P_{-m-1}) + \pi\cos\frac{\pi b}{a}(P_{m-1} + P_{-m})\right] - \\ -\frac{m}{\pi^2}\left[\left(\sum_{h=0}^{m-2}\right)' I_h\left(J_{m-1-h} + J_{h-m+2}\right) - \pi(J_{m+1} + J_{-m}) + \pi\cos\frac{\pi b}{a}(J_m + J_{1-m})\right], \tag{4.2}$$

where the prime near summations over $h$ means that the term under summation is absent when $m < 2$. The symbols $I_k$, $J_{\pm k}$, $P_{\pm k}$ $(k = 0, 1, 2, ...)$ stand for the following integrals

$$I_k = \int_\alpha^\beta z^k\sqrt{(z-\alpha)(\beta-z)}\,dz, \quad J_{\pm k} = \int_\alpha^\beta z^{\pm k}\frac{dz}{\sqrt{(z-\alpha)(\beta-z)}}, \tag{4.3}$$

$$P_{\pm k} = \int_\alpha^\beta z^{\pm k}\frac{\ln(\beta - z)}{\sqrt{(z-\alpha)(\beta-z)}}\,dz \quad (\alpha = e^{-i\pi b/a},\ \beta = e^{i\pi b/a} = \alpha^{-1}),$$

which are explicitly calculated in [15, Appendix], as follows ($k = 0, 1, 2, ...$) :

$$I_k = \frac{\pi^2(-1)^k\,(\beta-\alpha)^2}{(k+1)(k+2)}\sum_{\ell=0}^{k}\frac{\alpha^\ell\beta^{k-\ell}}{\ell!(k-\ell)!\,\Gamma\left(-\frac{1}{2}-\ell\right)\Gamma\left(-\frac{1}{2}+\ell-k\right)}; \qquad (4.4)$$

$$J_k = \pi^2(-1)^k\sum_{\ell=0}^{k}\frac{\alpha^\ell\beta^{k-\ell}}{\ell!(k-\ell)!\,\Gamma\left(\frac{1}{2}-\ell\right)\Gamma\left(\frac{1}{2}+\ell-k\right)}, \quad J_{-k} = J_{k-1}; \qquad (4.5)$$

$$P_k = \sum_{j=0}^{k}\binom{k}{j}\alpha^j(\beta-\alpha)^{k-j}\times\left\{\frac{\Gamma(1/2+k-j)\Gamma(1/2)}{\Gamma(1+k-j)}[\ln(\beta-\alpha)+\psi(1/2)-\psi(1+k-j)]\right\}; \qquad (4.6)$$

$$P_{-k} = \sum_{j=0}^{k-1}\binom{k-1}{j}(\alpha+\beta)^{k-1-j}(-1)^j\,P_j - (\ln\alpha)J_{k-1} - Q_{k-1}\ , \quad k = 1, 2, ...; \qquad (4.7a)$$

$$2kQ_k = (2k-1)(\alpha+\beta)Q_{k-1} - (2k-2)Q_{k-2} + 2\,I_{k-2},\ , \quad k = 2, 3, ...; \qquad (4.7b)$$

$$Q_0 = 2\pi\ln\frac{\sqrt{\alpha}+\sqrt{\beta}}{2}, \quad Q_1 = \pi(\alpha+\beta)\left(\ln\frac{\sqrt{\alpha}+\sqrt{\beta}}{2}+\frac{1}{2}\right)-\pi. \qquad (4.7c)$$

In the above formulas, besides to $\alpha, \beta$ which are given in Eq.(4.3), we have denoted by $\Gamma(z) \equiv \int_0^\infty e^{-t}\,t^{z-1}dt$ the (Euler's) *Gamma function* and by $\psi(z) \equiv \Gamma'(z)/\Gamma(z)$ the so-called *psi-function*; useful properties of these functions are

$$\Gamma(1) = 1, \quad \Gamma(z+1) = z\,\Gamma(z), \quad \Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z), \quad \psi(z) = \psi(z+1) - 1/z;$$

$-\psi(1) = 0.577216$ is Euler's constant. Such special functions can be evaluated by means of their well-known analytical approximations [12]

# 5 Physical remarks

The developed analytical method permits rapid implementation to derive important physical conclusions, since dependence of the principal physical quantities is expressed in corresponding formulas explicitly. We have performed such an implementation for numerous combinations of the input data, which contain the information about basic geometrical and physical parameters of the structure. From the results presented in the included figures, as well as from many other tables obtained, the following remarks of physical nature can be extracted **(see Figs.2 – 6)**.

1) If the amplitude of the punch vibrations is kept as a constant versus the basic frequency parameter (i.e., versus $ak$ in the anti-plane case and $ak_1$ or $ak_2$ in the in-plane case), then it is very interesting to trace the dependence of the integral of the contact stress, taken over the single punch base, as a function of the frequency parameter. Note that if we assume the mass of the punch to be negligibly small, the quoted integral equals the applied force.

Thus, we can observe that in both anti-plane and in-plane cases, the graph for the applied force versus frequency in the *one-mode range* (3.1), always has a local maximum inside this frequency interval. See lines no.1 in Figs.2, 3. The smaller is the punch (i.e., parameter $b/a$), the more left is the position of such a maximum in the interval. For very wide punch, when its size approaches its possible maximum value (equal to the period of the structure), the position of the local maximum moves so right that it falls out of the (one-mode) interval. Moreover, the value of the maximum grows with increasing of the punch size, that is quite natural from the physical point of view.

2) If we compare the value of the applied force required to support a certain constant amplitude of vibration, then it is easily seen from our graphs that in the in-plane case its value is always higher than in the anti-plane case, under the same values of all other parameters.

3) Another interesting question is the behavior of the *principal* displacement versus frequency and size of the punch. Reducing the analysis to the one-mode range, we mean that: in the anti-plane problem, which studies the case of horizontal (out-of-plane) vibrations of punches, the (out-of-plane only) displacement $w$ tends asymptotically to a constant value as $y \to \infty$, as directly follows from Eq. (3.9); in the in-plane problem, which studies the case of vertical vibrations, the horizontal displacement $u_x$ vanishes as $y \to \infty$ but the vertical one $u_y$ tends again asymptotically to a certain constant value, thus representing itself the principal displacement at infinity - as follows from Eqs. (3.10). Examples of these asymptotic behaviors are shown in Figs. $4-6$ for various punch sizes; as physically expected, for a given amplitude of vibration in the contact zone, the smaller is the punch the smaller is the principal displacement at infinity. For the dependence of the principal displacements (at infinity) versus frequency, see lines no.2 in Figs. 2, 3.

4) It is very interesting to investigate the behavior of the *energy flux* (or energy intensity) in its dependence upon the main geometrical and physical parameters. This represents the energy produced by the elastic stresses on their work with the particles displacements, calculated over the period of the harmonic oscillations with respect to time. It is well known (see, for example,

[10]) that such a quantity is given by the following expression

$$E = \frac{T}{2} \operatorname{Re}(\Sigma^* v) , \qquad T = 2\pi/\omega , \tag{5.1}$$

where $v$ is a pertinent component of the velocity vector and $\Sigma$ the corresponding component of the stress, while $T$ is the period of oscillations ($^*$ means complex conjugate). In our problem, it is involved the principal displacement vector as defined above: so, here we have $v = \dot{u} = -i\omega u$, where $u = w$ or $u_y$ according to anti-plane or in-plane case. As a consequence, Eq.(5.1) reduces to

$$E = \pi \operatorname{Im}(\Sigma^* u) = -\pi \operatorname{Im}(\Sigma u^*) . \tag{5.2}$$

This formula can be applied to control the law of energy conservation from $y = 0$ to $y \to \infty$. In the anti-plane problem, $u = w(x, y) \to w_\infty$ and $\Sigma = \tau_{yz}(x, y) \to \tau_\infty$, as $y \to \infty$; hence, by integration over a period $(-a, a)$, this law should imply the following equality:

$$w_0 \operatorname{Im}(H_0^\tau) = 2a \operatorname{Im}(\tau_\infty w_\infty^*) \tag{5.3}$$

In the in-plane problem, the tangential component of the displacement vector and corresponding component of the stress vanish at infinity. Moreover, as follows from the pertinent formulas, the periodic application of outer forces does not produce a surface Rayleigh wave (as it could happen, for example, in the case of a single point force applied to the boundary). Thus, in the in-plane problem the law of energy conservation should imply the following equality:

$$u_0 \operatorname{Im}(H_0^\sigma) = 2a \operatorname{Im}(\sigma_\infty u_\infty^*) , \tag{5.4}$$

where we put - from Eq.(5.2) - $u = u_y(x, y) \to u_\infty$ and $\Sigma = \sigma_{yy}(x, y) \to \sigma_\infty$, as $y \to \infty$.

By deducing $H_0^{\tau,\sigma}$ from system (3.7) and calculating the fields at infinity from Eqs.(3.9, 3.10), it can be verified that Eqs.(5.3) and (5.4) hold (identically) whatever be the frequency in the range (3.1).

Some examples reflecting a non-monotonic behavior of the energy flux versus frequency are shown as lines no. 3 in figures 2 and 3 for anti-plane and in-plane problems, respectively.

5) Finally, we have compared our analytical results with those from a direct numerical treatment of the basic integral equations. One of the typical outcomes of such a treatment is reflected in Fig.2 as (dashed) line no. 4. Further verifications of this type confirm in all cases a very good precision of the obtained analytical formulas almost throughout the one-mode range. In this connection, we note that even in the last part of this range the error is *at*

*worse* less than 4%, as one can see in lines 2 of Figs.5, 6 (for $|u_y(y)|/u_0$ at $ak_2 = 2.5$), which of course should start from the unit value.

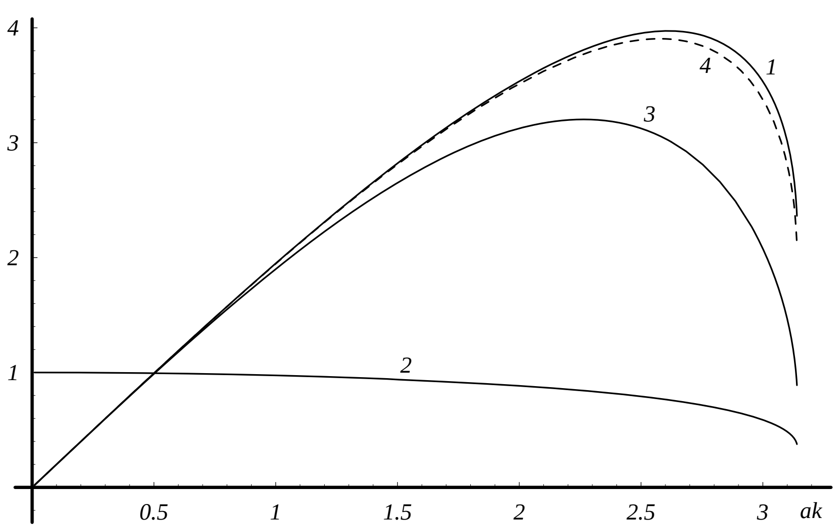


Figure 2: For anti-plane problem, with $b/a = 0.5$. Total force applied $|H_0^\tau|/(\mu w_0)$ (line 1), principal (far-field) displacement $|w_\infty|/w_0$ (line 2) and (integrated) energy flux $E/(\pi\mu w_0^2)$ (line 3) vs. frequency parameter $ak$ in the one-mode range $(0, \pi)$. In (dashed) line 4: total force by a direct numerical method.

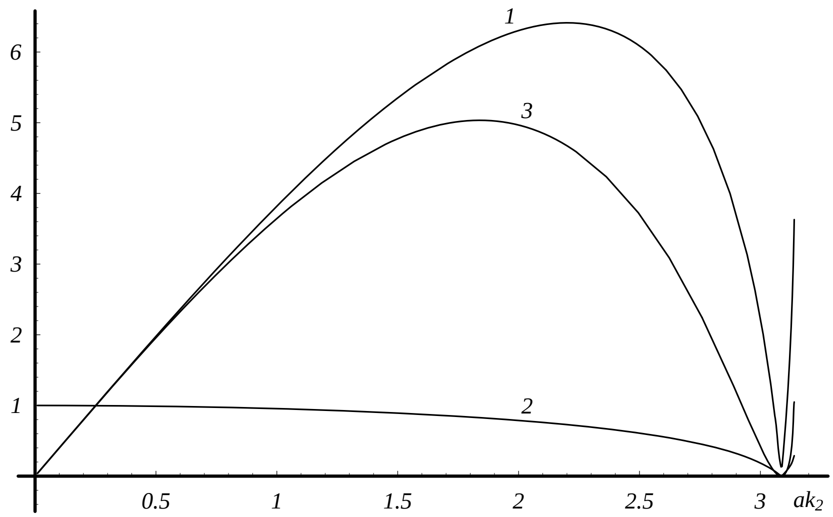


Figure 3: For in-plane problem, with $b/a = 0.5$ and $k_2/k_1 = 2$. Total force applied $|H_0^\sigma|/(\mu u_0)$ (line 1), principal (far-field) displacement $|u_\infty|/u_0$ (line 2) and (integrated) energy flux $E/(\pi \mu u_0^2)$ (line 3) vs. frequency parameter $ak_2$ in the one-mode range $(0, \pi)$.

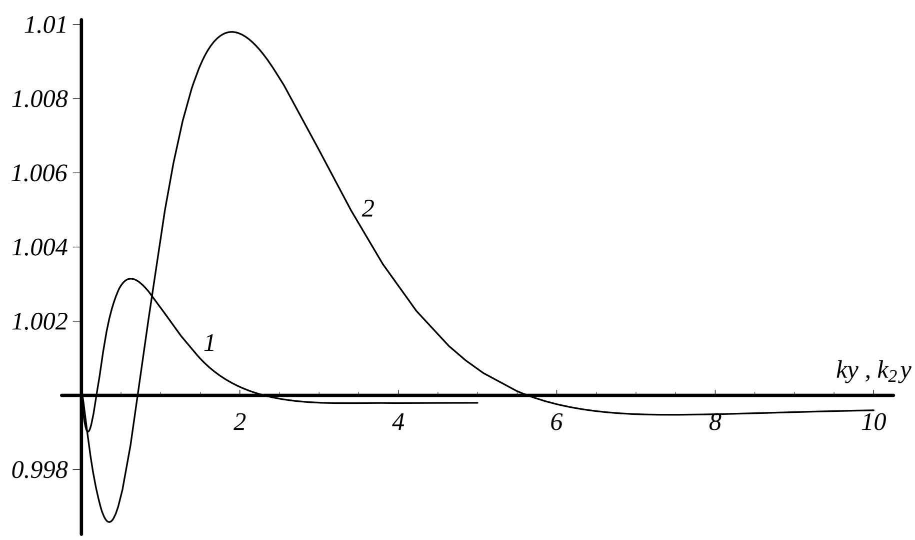


Figure 4: For both anti-plane and in-plane problems, with $x/b = 0.75$ , $b/a = 0.9$ and $ak = ak_2 = 2.5$ $(k_2/k_1 = 2)$. Displacement fields $|w(y)|/w_0$ and $|u_y(y)|/u_0$ vs. depth parameters $ky$ and $k_2y$ (lines 1 and 2, respectively).

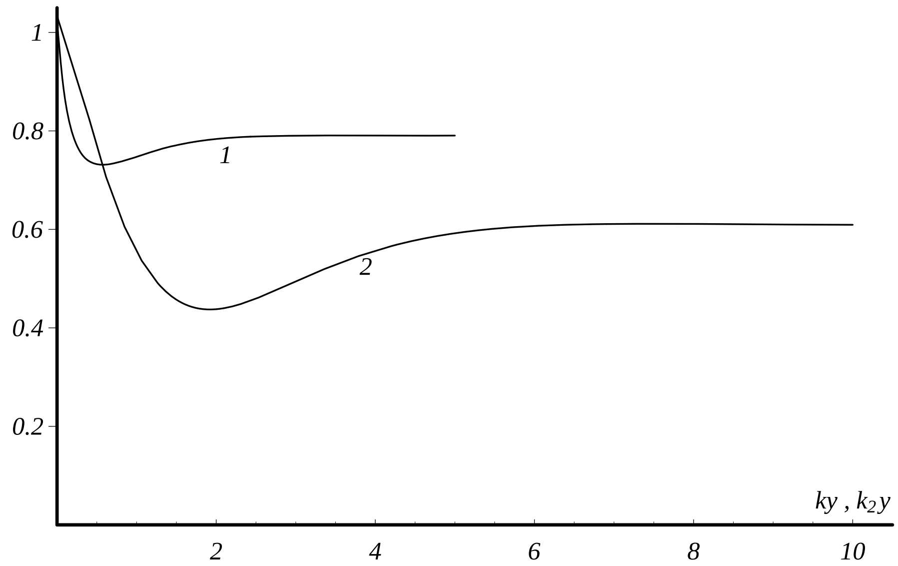


Figure 5: For both anti-plane and in-plane problems, with $x/b = 0.75$ , $b/a = 0.5$ and $ak = ak_2 = 2.5$ ($k_2/k_1 = 2$). Displacement fields $|w(y)|/w_0$ and $|u_y(y)|/u_0$ vs. depth parameters $ky$ and $k_2y$ (lines 1 and 2, respectively).

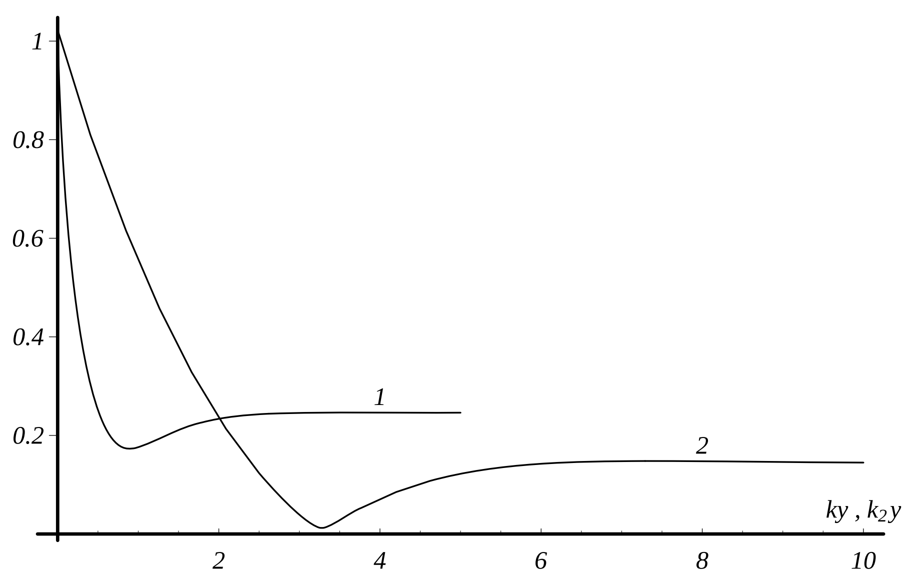


Figure 6: For both anti-plane and in-plane problems, with $x/b = 0.75$ , $b/a = 0.1$ and $ak = ak_2 = 2.5$ ($k_2/k_1 = 2$). Displacement fields $|w(y)|/w_0$ and $|u_y(y)|/u_0$ vs. depth parameters $ky$ and $k_2y$ (lines 1 and 2, respectively).

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## References

[1] J.D.Achenbach, *Wave Propagation in Elastic Solids*, North-Holland: Amsterdam, 1973.

[2] K.Aki, P.G.Richards, *Quantitative Seismology. Theory and Methods (1,2)*, Freeman: San Francisco, 1980.

[3] J. Brouwer, K. Helbig, *Shallow High-Resolution Reflection Seismics* (Handbook of Geophysical Exploration: Seismic Exploration), Pergamon: New York, 1998.

[4] D. Guerguiev et al., Analysis of Floquet wave generation and propagation in a plate with multiple arrays of line attachments, *J. Sound Vibr.* **234** (no.5, 2000), 819-840.

[5] K. J. Hsiaa et al., Collapse of stamps for soft lithography due to interfacial adhesion, *Appl. Phys. Letters* **86** (2005), 154106.

[6] C.Y. Hui et al., Constraints on Microcontact Printing Imposed by Stamp Deformation, *Langmuir* **18** (2002), 1394-1407.

[7] K.L. Johnson, *Contact Mechanics*, Cambridge University Press: Cambridge, 2004.

[8] N.I. Muskhelishvili, *Some Basic Problems of the Mathematical Theory of Elasticity,* P. Noordhoff: Groningen, 1953.

[9] A. Papoulis, *The Fourier integral and its applications,* McGraw-Hill: New York, 1962.

[10] A.D.Pierce, *Acoustics*, ASA Publ.: New York, 1991.

[11] J.J. Postel, E. Gillot, M. Larroque, *Review of specific parameters in high-resolution seismic,* EAGE 66th Conf., Paris, 2004, Z99. (http://www.freepatentsonline.com/6714867.html)

[12] A.P. Prudnikov, Y.A.Brychkov, O.I.Marichev. *Integrals and Series* (vol.2), Gordon and Breach Science Publishers: Amsterdam, 1986.

[13] E. Scarpetta, M.A. Sumbatyan, Explicit analytical results for one-mode normal reflection and transmission by a periodic array of screens, *J. Math. Anal. Appl.* **195** (1995), 736-749.

[14] E. Scarpetta, In-plane problem for wave propagation through elastic solids with a periodic array of cracks, *Acta Mechanica* **154** (2002), 179-187.

[15] E. Scarpetta, M.A. Sumbatyan, Some analytical results for acoustic scattering through a periodic array of elastic membranes *Quart. Appl. Math.* **65** (2007), 737-755.

[16] M. Shen, W. Cao, Acoustic band-gap engineering using finite-size layered structures of multiple periodicity, *Appl. Phis. Letters* **75** (no.23, 1999), 3713-3715.

[17] I.N. Sneddon, *Mixed Boundary Value Problems in Potential Theory,* North-Holland: Amsterdam; John Wiley: New York, 1966.

[18] United States Patent 6028818 : Method and apparatus for multiple seismic vibratory surveys. (http://www.freepatentsonline.com/6028818.html)

[19] United States Patent 6807508 : Seismic prospecting method and device using simultaneous emission of seismic signals based on pseudo-random sequences. (http://www.freepatentsonline.com/6807508.html)

[20] United States Patent 6714867 : Method for seismic monitoring of an underground zone by simultaneous use of several vibroseismic sources. (http://www.freepatentsonline.com/6714867.html)

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