Quantitative and Qualitative Analysis in Social Sciences
Volume 1, Issue 2, 2007, 25-32
ISSN: 1752-8925

Linear Portfolio Weights

Giovanni Barone-Adesi[a]
Swiss Finance Institute
University of Lugano

Abstract

Simple heuristics for large portfolio choice in small samples are proposed. The loss of efficiency from true optimum is observed by simulation. The performance of chosen portfolios is reasonable when true arbitrage opportunities and good deals are absent.

JEL Classifications: G11.

Keywords: Portfolio selection, heuristic methods.

Acknowledgements: The financial support of FinRisk, a network of the Swiss National Fund, is gratefully acknowledged.

[a] Swiss Finance Institute, Facoltà di Economia, University of Lugano, via Guiseppe Buffi 13, Lugano CH-6904, Switzerland; tel.: +41 58 666 4753, email: baroneg@lu.unisi.ch

1 Introduction

Forming optimal portfolios in a mean-variance framework still relies on Markowitz's (1952) pioneering efforts to a large extent. Over the last fifty years, the increase in computer power has solved the computational difficulties that originally limited the application of Markowitz's approach. However, computer improvements cannot solve the statistical difficulty stemming from the need to estimate very large covariance matrices from limited time series of data. This problem often forces practitioners to rely on index models to simplify the optimization procedures. Even index models, however, may become unreliable when the number of observations from which their inputs are estimated is small. This is the case, as an example, if we attempt to form a portfolio of hedge funds from a large universe of assets with a limited history.

To bypass the statistical problems associated with the inversion of large covariance matrices we propose and test a very simple heuristic optimization method, based on linear expressions for the 'optimal' portfolio weights. Our simulations rely on sample average returns and covariances with the equally weighted portfolio of funds in our universe. Simulations show that these methods achieve Sharpe ratios not very far from the optimal in most cases that are likely to occur in reality, though they overestimate the achievable performance. The global minimum variance portfolio appears to be more elusive, especially if the equally weighted portfolio is far from the efficient frontier.

2 Portfolio Weights

It is well known that each asset in an optimal mean-variance portfolio has a weight that equates the ratio of its contribution to portfolio return to its contribution to portfolio risk. These contributions, however, cannot be measured precisely from a short series of data, making the identification of optimal portfolios difficult. To simplify the problem we will focus on the two minimum variance portfolios, the ones that minimize variance or maximize the ratio of excess return over risk, henceforth the global minimum variance portfolio and the Sharpe portfolio.

Intuitively, portfolio weights are inversely related to each asset's contribution to risk in the minimum variance portfolio. Therefore, assets with higher covariance should have smaller weights on average. A very high correlation between a couple of assets with different expected return may override this feature, offering a 'good deal' (Cochrane and Saá-Requejo, 2000), or even an arbitrage opportunity. If the number of available degrees of freedom is small, relative to the number of assets, it is impossible to identify 'good deals' and arbitrage opportunities reliably. Linear combinations of assets appear to offer profit opportunities, but estimation noise accounts for most of them. Out-of-sample estimates show that those profits are illusory.

If profit opportunities related to 'good deals' and arbitrage cannot be identified reliably within the sample on the basis of purely statistical analysis, it is more reasonable to ignore them. The equally weighted portfolio is then a reasonable starting point for our heuristic optimization. Assets with higher expected return receive higher weights, assets with higher covariance lower weights. The weight changes in the Sharpe portfolio are chosen to reflect the contributions to portfolio risk and excess return, above the risk-free rate. Only contributions to risk, that is covariances, are considered in the weights of the global minimum variance portfolio.

The portfolio weights for the Sharpe portfolio are therefore:

$$X_i = \frac{1}{N}\left(1 + \frac{1}{N}\left(\frac{R_i}{R_E} - \frac{cov\left(R_i,\ R_E\right)}{var\left(R_E\right)}\right)\right), \tag{1}$$

where R_i and R_E are the expected returns on stock i, $i = 1, 2, ..., N$ and the equally weighted portfolio, respectively. We proxy them with sample averages in our estimates. Similarly, cov and var refer to covariance and variance or their estimates. The global minimum variance portfolio weights are:

$$X_i = \frac{1 - \frac{1}{N} cov\left(R_i,\ R_E\right)/var\left(R_E\right)}{N-1}. \tag{2}$$

The above weights define heuristic proxies to the two portfolios. Although it would be possible in principle to iterate our procedure, computing new covariances of asset returns with the portfolios defined in equations (1) and (2), we will not do that in this paper, because our focus will be on very short time series. Iterating would then introduce a serious risk of overfitting. Therefore, we will consider some simple covariance structures and verify the performance of our estimators in the next section through a simulation experiment.

3 Simulation

To test the performance of our heuristic portfolio weights we select three sets of securities. The first set includes 50 stocks with equal mean returns and volatilities. The second set includes two groups of 25 stocks with different means and volatilities. The third set includes three groups of 25 stocks with different means and volatilities. The last of the three groups adds only noise. In every case we consider 12 observations, drawn from the normal distribution, calibrated to simulate yearly returns. The simple covariance structures we consider allow for easy determination of the correct optimal portfolios. They are compared with our heuristic portfolios and the loss of efficiency is estimated. Without loss of generality we set the risk-free rate at zero in all of our simulations.

3.1 The First Set

The first set of stocks includes 50 uncorrelated stocks with expected return equal to 10% and standard deviation of 30%. Because all the stocks have the same expected return, the efficient frontier degenerates to one point, the minimum variance portfolio. Its expected return is 10% and its standard deviation is $0.30/\sqrt{50} = 0.04243$. All the stocks have equal weight, 0.02, in the optimal portfolio.

Sample average returns for our 50 stocks range between $+30\%$ and -6% over the 12 time periods. Covariances with the equally weighted portfolio returns range between 1% and -0.5%. The equally weighted portfolio has sample mean 0.1197 and standard deviation 0.051. Our estimated weights range between 0.0184 and 0.0212 for the global minimum variance portfolio, and between 0.0184 and 0.0216 for the Sharpe portfolio. The two sets of weights have sample correlation 0.932. Therefore they appear to be stable and extremely similar to each other, with small deviations from their true optimal value, 0.02. Our method therefore seems to be reasonably efficient in this simple experiment.

Our estimated weights produce estimates of 0.12 and 0.121 for the expected returns of the global minimum variance and the Sharpe portfolio, with corresponding standard deviations of 0.0477 and 0.0478. They therefore overstate the correct values. However, if we apply our estimated weights to the true stock parameters we find expected returns of 0.10 and volatilities 0.04245 and 0.04246 respectively, against a true volatility of 0.04243.

3.2 The Second Set

Our second set includes 50 stocks. 25 stocks have expected return 0.08 and volatility (sigma) 0.24, 25 stocks have expected return 0.12 and volatility 0.27. All the correlations between the 50 stocks are initially set to zero. We then repeat the experiment setting the correlations to 0.50. Table 1 reports the main results of this experiment. For zero correlation, the true minimum variance frontier is very elongated (see Figure 1). The minimum variance portfolio is therefore very close to the portfolio that optimizes the Sharpe ratio. Estimated expected returns and volatilities are about 15% or 20% higher than true values in the upper lines of the table. This is due to sample fluctuation, not to loss of efficiency. That is shown in the last two columns, where the estimated portfolio returns are computed applying the same estimated weights ($\widehat{w}$) to the true parameter values. Therefore, we may conclude that in this instance our method provides good estimates for portfolio weights, though not precise estimates of portfolio parameters.

Table 1: The second set (50 stocks with equal correlations)

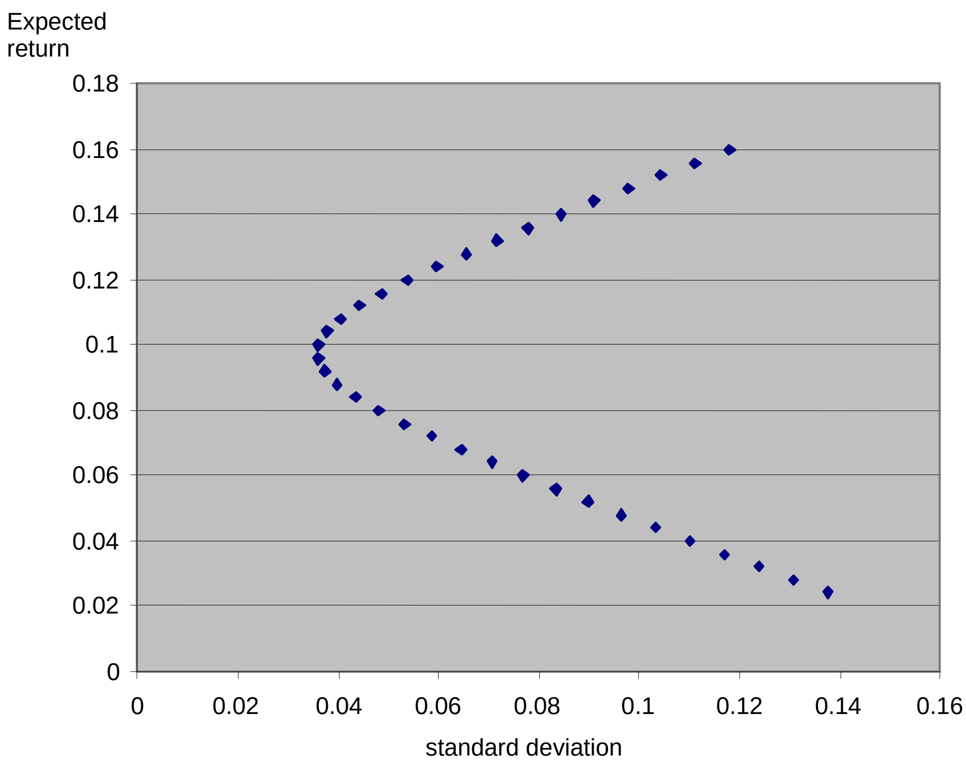
	true exp. return	true sigma	est. exp. return	est. sigma	$\widehat{w}\times$ true return	est. sigma with $\widehat{w}$, true cov.
correlation=0.0						
equally weighted	0.100	0.0361	0.1168	0.0435		
minimum variance	0.096	0.0360	0.1169	0.0410	0.0999	0.0361
Sharpe	0.100	0.0361	0.1179	0.0410	0.1000	0.0362
correlation=0.5						
equally weighted	0.10	0.1821	0.0945	0.1835		
minimum variance	0.06	0.1706	0.0946	0.1829	0.1000	0.1821
Sharpe	0.24	0.3346	0.0955	0.1828	0.1001	0.1821

Notes: The expected returns and sigmas (standard deviations) of the equally weighted portfolio, the minimum variance portfolio and the portfolio with maximum Sharpe ratio are shown for the zero correlation and the 0.5 correlation cases. True values are compared to estimated values and to true values of portfolios based on estimated weights.

Estimates for pairwise correlations set to be equal to 0.5 are reported in the lower part of Table 1. It is apparent that our method now misses completely both the minimum variance and the Sharpe portfolio. This result stems from the fact that the positive correlations introduce a 'good deal', the opportunity of gaining an almost riskless profit investing in the stocks with higher expected return and shorting the ones with lower expected return. The efficient frontier is almost linear, leading to an optimal Sharpe portfolio far away from the global minimum variance portfolio. Sample fluctuations appear to be minor in this instance, because corresponding values in the third and fourth columns are now similar.

Figure 1

Mean-variance frontier for the second set, zero correlations.



Notes: The expected returns and sigmas (standard deviations) of the equally weighted portfolio, the minimum variance portfolio and the portfolio with maximum Sharpe ratio are shown for the zero correlation and the 0.5 correlation cases. True values are compared to estimated values and to true values of portfolios based on estimated weights.

3.3 The Third Set

Our third set comprises 75 stocks, divided into three groups of 25 stocks each. The expected return of any stock in each group is 0.06, 0.09 and 0.0, respectively. Correspondingly, volatilities are 0.2, 0.3 and 0.3. Pairwise correlations are 0.5 between the first 50 stocks, zero otherwise. These characteristics are summarized at the top of Table 2. Estimates that refer to the whole set of 75 stocks are in lines containing the number 75 in their first cell. The introduction of 25 stocks with purely noisy returns causes a clear deterioration in the quality of the estimated performance of our portfolios. The use of estimated or true parameter values leads to similar conclusions. The only parameter for which a somewhat useful estimate is obtained is the optimal Sharpe ratio, which is about 30% off using in sample values, or 20% off applying estimated weights to the true parameters. These results are confirmed by the values in the row 'Sharpe 50', which excludes the 25 noisy stocks. Similar results are obtained for the global minimum variance portfolio excluding the noisy stocks (row 'min. variance50'), showing that the estimates of the global minimum portfolio are particularly sensitive to the introduction of noisy stocks.

Table 2: The third set (75 stocks, including 25 'pure noise')

Stocks	Exp. return	Sigma	Correlations		
25	0.06	0.2	1	0.5	0
25	0.09	0.3	0.5	1	0
25	0	0.3	0	0	1

	True values		Estimated values		Values using est. weights	
	exp. return	sigma	average	sigma	exp. est. w	sigma est. w
eq. weighted75	0.05	0.12044	0.06708	0.10013		
min. variance50	0.063	0.14186	0.10179	0.17414	0.07495	0.17849
min. variance75	0.013	0.05576	0.06642	0.10233	0.04953	0.11970
Sharpe50	0.072	0.14988	0.10039	0.13759	0.07253	0.17308
Sharpe75	0.072	0.14988	0.13312	0.04746	0.04375	0.11350
eq. weighted50	0.075	0.15716	0.10182	0.17490		

Notes: Following the notation in Table 1, portfolio parameters with (75) and without (50) noisy stocks are compared.

4 Conclusion

The proposed method provides weights for building the global minimum variance portfolio and the optimal Sharpe portfolio that differ little from the equally weighted portfolio. In all of our experiments weights differed from 1/N by less than 10%. Moreover, the two sets of estimated weights, for the global minimum variance and the Sharpe portfolio, were highly correlated (>0.9). The advantage of our method is the ease of computation for any number of assets. However, its reliability suffers if the universe of assets allows for good deals or it is noisy. In both of these cases our starting point, the equally weighted portfolio, is not close to the two sought portfolios. The impossibility of identifying reliably noisy stocks and good deals in small samples on purely statistical grounds motivated our heuristic strategy choices. Not surprisingly, our strategy performs poorly when its assumptions are false. Otherwise, it provides useful rules of thumb to approximate the global minimum variance and the optimal Sharpe portfolios.

References

[1] Cochrane J. H., and J. Saá-Requejo (2000). Beyond Arbitrage: Good Deal Asset Price Bounds in Incomplete Markets, Journal of Political Economy, 108, 79-119.

[2] Markowitz, H. M. (1952). Portfolio Selection, Journal of Finance 7, 1, 77-91.