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Holland-Walsh characterization for Besov spaces*

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Abstract: Holland and Walsh first gave the derivative-free characterization of Bloch Spaces on the unit disc in $\mathbb{C}$. Later, Nowark got a similar result for the holomorphic Besov Spaces on the unit Ball in $\mathbb{C}^n$. Recently Ren Guangbin extended the result to hyperbolic harmonic Besov spaces on the unit ball in $\mathbb{R}^n$. Based on these results, We obtains the derivative-free characterization of holomorphic Besov spaces on the unit ball in $\mathbb{C}^n$.

Key words: harmonic; holomorphic; Möbius transformation; Besov space; Bloch space

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Besov 空间的 Holland-Walsh 刻画

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摘要：Holland 和 Walsh 首先给出了复单位圆盘上 Bloch 空间的一个不依赖于导函数的刻画. 后来 Nowark 把该结果推广到 n 维复单位球上的全纯函数的 Besov 空间. 最近任广斌把该结果做到了 n 维实单位球上的双曲调和函数 Besov 空间. 我们正是基于这些基础，得到 n 维复单位球上的全纯函数的 Besov 空间的一个不依赖于导函数的刻画.

关键词：调和；全纯；Möbius 变换；Bloch 空间；Besov 空间

0 Introduction

Let $\mathbb{B}_n$ be the unit ball in $\mathbb{C}^n$, $d\nu$ the normalized measure on $\mathbb{B}_n$ and $d\sigma$ the normalized surface measure on the unit sphere $\partial\mathbb{B}_n$.

For any holomorphic function $f \in \mathbb{B}_n$, write $\nabla f(z) = \left(\frac{\partial f}{\partial z_1}(z), \cdots, \frac{\partial f}{\partial z_n}(z)\right)$ and call $\nabla f(z)$ the gradient of f at z. Let φ_a be the Möbius transformation of $\mathbb{B}_n$ (see Ref. [1, p. 25])

$$\varphi_a(z) = \frac{a - P_a z - s_a Q_a z}{1 - \langle z, a\rangle},$$

where $\langle z, a\rangle$ denotes the usual inner product in $\mathbb{C}^n$, $|a|^2 = \langle a, a\rangle$, $P_a z = \frac{\langle z, a\rangle}{|a|^2} a$, $Q_a z = z - P_a z$, and $s_a = (1 - |a|^2)^{1/2}$. We define

$$\widetilde{\nabla} f(z) = \nabla(f \circ \varphi_z)(0)$$

and call $\widetilde{\nabla} f(z)$ the invariant gradient of f at z. Denote

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$$d\tau(z)=(1-|z|^2)^{-n-1}d\nu(z),$$

which is invariant under Möbius transformations φ_a (see Ref. [1, p. 28]), namely

$$\int_{\mathbb{B}_n} f(z)d\tau(z)=\int_{\mathbb{B}_n} f\circ\varphi_a(z)d\tau(z).$$

We consider the Besov space $\mathcal{B}_p$, which consists of holomorphic functions f in $\mathbb{B}_n$ such that $|\widetilde{\nabla}f(z)|\in L^p(\mathbb{B}_n,d\tau)$ (see Ref. [2]). When $p\geqslant 1$, the space $\mathcal{B}_p$ is a Banach space, which is invariant under the action of automorphism group on $\mathbb{B}_n$. When $p\to\infty$, it is exactly the Bloch space $\mathcal{B}$, which consists of holomorphic functions f in $\mathbb{B}_n$ such that

$$\|f\|_{\mathcal{B}}=\sup\{|\widetilde{\nabla}f(z)|: z\in\mathbb{B}_n\}<\infty.$$

We refer to Ref. [1], [2], [3], [4], [5] and [6] for various characterizations of $\mathcal{B}$ and $\mathcal{B}_p$. For example, for any holomorphic function of f on $\mathbb{B}_n$,

(Ⅰ) $f\in\mathcal{B}$ if and only if $\sup\{(1-|z|^2)|\nabla f(z)|: z\in\mathbb{B}_n\}<\infty$;

(Ⅱ) $f\in\mathcal{B}_p$ if and only if $(1-|z|^2)|\nabla f(z)|\in L^p(\mathbb{B}_n,d\tau)$.

The purpose of this paper is to give a new type of Holland-Walsh characterization for the Besov space. We first mention some related known results.

In 1986, Holland and Walsh in Ref. [7] gave the following characterization for the Bloch space $\mathcal{B}(\mathbb{D})$ in the unit disc $\mathbb{D}=\{z\in C: |z|<1\}$.

Theorem 0.1[7] Let f be holomorphic in $\mathbb{D}$. Then $f\in\mathcal{B}(\mathbb{D})$ if and only if

$$\sup\left\{\frac{|f(z)-f(w)|}{|z-w|}(1-|z|^2)^{\frac{1}{2}}(1-|w|^2)^{\frac{1}{2}}: z, w\in\mathbb{D}, z\neq w\right\}<\infty.$$

Stroethoff in Ref. [8] extended the result to the Besov spaces in the unit disc.

Theorem 0.2[8] Let $2<p<\infty$ and f be holomorphic in $\mathbb{D}$. Then $f\in\mathcal{B}_p$ if and only if

$$\int_{\mathbb{D}}\int_{\mathbb{D}}\left[\frac{|f(z)-f(w)|^p}{|w-z|^p}(1-|z|^2)^{\frac{p}{2}}(1-|w|^2)^{\frac{p}{2}}d\tau(z)d\tau(w)\right]<\infty.$$

Nowark Ref. [6] generalized Stroethoff's result to higher dimensions.

Theorem 0.3[6] Let $2n<p<\infty$ and f be holomorphic in $\mathbb{B}_n$. Then $f\in\mathcal{B}_p$ if and only if

$$\int_{\mathbb{B}_n}\int_{\mathbb{B}_n}\left[\frac{|f(z)-f(w)|^p}{|w-P_wz-s_wQ_wz|^p}(1-|z|^2)^{\frac{p}{2}}(1-|w|^2)^{\frac{p}{2}}d\tau(z)d\tau(w)\right]<\infty. \quad (1)$$

Recently, the Holland-Walsh characterization of Bloch space in the unit disc has been extended to the unit ball in Ref. [3].

Theorem 0.4[3] Let f be holomorphic in $\mathbb{B}_n$. Then $f\in\mathcal{B}$ if and only if

$$\sup\left\{\frac{|f(z)-f(w)|}{|z-w|}(1-|z|^2)^{\frac{1}{2}}(1-|w|^2)^{\frac{1}{2}}: z, w\in\mathbb{B}_n, z\neq w\right\}<\infty.$$

Observe that Nowark's characterization for Besov spaces in the unit ball would be natural if the denominator $|w-P_wz-s_wQ_wz|$ in (1) is replaced by $|w-z|$.

Our main result is the following theorem.

Theorem 0.5 Let $4n-2<p<\infty$ and f be holomorphic in $\mathbb{B}_n$. Then $f\in\mathcal{B}_p$ if and only if

$$\int_{\mathbb{B}_n}\int_{\mathbb{B}_n}\left[\frac{|f(z)-f(w)|^p}{|z-w|^p}(1-|z|^2)^{\frac{p}{2}-n-1}(1-|w|^2)^{\frac{p}{2}-2n}d\nu(z)d\nu(w)\right]<\infty.$$

Note that Theorem 0.5 recovers Theorem 0.2 when $n=1$, and recovers Theorem 0.4 when $p\to\infty$.

1 Preliminaries

We shall invoke real techniques to deal with the theory of holomorphic functions. We identify $\mathbb{C}^n$ with $\mathbb{R}^m$, $m=2n$, and denote by Ω the unit ball in $\mathbb{R}^m$.

For any $x\in\mathbb{R}^m$, we write $x=|x|x'$ in polar coordinates, so that $x'\in\partial\Omega$.

We denote the real Möbius transformation in Ω by ψ_a (see Ref. [9, p. 25]). It is an involutionary automorphism of Ω, which is of the form

$$\psi_a(x)=\frac{|x-a|^2a-(1-|a|^2)(x-a)}{||a|x-a'|^2}\quad(a,x\in\Omega).$$

Obviously

$$|\psi_a(x)| = \frac{|x-a|}{||a|x-a'|} \qquad (2)$$

and

$$1-|\psi_a(x)|^2 = \frac{(1-|x|^2)(1-|a|^2)}{||a|x-a'|^2}. \qquad (3)$$

We consider the measure

$$\mathrm{d}\lambda(x) = (1-|x|^2)^{-2n}\mathrm{d}\upsilon(x),$$

which is invariant under the real Möbius transformation ψ_a (see Ref. [4]), namely

$$\int_\Omega f(x)\mathrm{d}\lambda(x) = \int_\Omega f\circ\psi_a(x)\mathrm{d}\lambda(x).$$

Write $x=(z_1,\cdots,z_n)$ with $z_j = x_j + iy_j$, then we can define

$$\nabla f(x) = \left(\frac{\partial f}{\partial z_1},\cdots,\frac{\partial f}{\partial z_n}\right)$$

and the real invariant gradient

$$\Lambda f(x) = \nabla(f\circ\psi_x)(0)$$

and know that

$$|\Lambda f(x)| = (1-|x|^2)|\nabla f(x)|.$$

For any $a\in\Omega$ and $\delta\in(0,1)$ we denote

$$E(a,\delta) = \{x\in\Omega: |\psi_a(x)|<\delta\},$$
$$B(a,\delta) = \{x\in\Omega: |x-a|<\delta\},$$

clearly, $E(a,\delta) = \psi_a(B(0,\delta))$.

From now on, we will use the symbol C to denote a positive constant which may vary at each occurrence and we also use the symbol $M\simeq N$ to denote $C^{-1}M\leqslant N\leqslant CM$.

Lemma 1.1 Let $0\leqslant\alpha\leqslant\lambda$. Then there exists a positive constant $C=C(\delta)$ such that, for any $x\in\Omega$ and $y\in E(x,\delta)$,

$$\frac{(1-|x|^2)^\alpha(1-|y|^2)^{\lambda-\alpha}}{|x-y|^\lambda}\geqslant C.$$

Proof For $x\in\Omega$ and any $y\in E(x,\delta)$, we have (see Ref. [4])

$$1-|x|^2\simeq 1-|y|^2. \qquad (4)$$

So that

$$\frac{(1-|x|^2)^\alpha(1-|y|^2)^{\lambda-\alpha}}{|x-y|^\lambda}\simeq \frac{(1-|x|^2)^{\frac{\lambda}{2}}(1-|y|^2)^{\frac{\lambda}{2}}}{|x-y|^\lambda} = \left(\frac{\sqrt{1-|\psi_x(y)|^2}}{|\psi_x(y)|}\right)^\lambda \geqslant \left(\frac{\sqrt{1-\delta^2}}{\delta}\right)^\lambda$$

as desired.

Let F be the hypergeometric function [10]

$$F(a,b;c;s) = \sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(k)!(c)_k}s^k,$$

for $a,b,c\in\mathbb{R}$ and c neither zero nor a negative integer, where $(a)_k$ denotes the Pochhammer symbol with $(a)_0=1$ and $(a)_k = a(a+1)\cdots(a+k-1)$, $k\in\mathbb{N}$.

Lemma 1.2[4] Let $\alpha>-1$ and $\beta\in\mathbb{R}$. Then, for any $x\in\Omega$, we have

$$\int_\Omega\frac{(1-|y|^2)^\alpha}{||x|y-x'|^{2n+\alpha+\beta}}\mathrm{d}\nu(y)\simeq \begin{cases}(1-|x|^2)^{-\beta}, & \text{if } \beta>0;\\ \ln\dfrac{1}{1-|x|^2}, & \text{if } \beta=0;\\ 1, & \text{if } \beta<0.\end{cases}$$

Recall that the Lemma 4.2 in Ref. [4] is

Lemma 1.3[4] Let $p\geqslant 1$ and $\alpha>-1$, if f is hyperbolic harmonic on Ω. Then

$$\int_\Omega\left(\int_0^1\frac{|\widetilde{\nabla}f(ta)|}{1-t|a|}\mathrm{d}t\right)^p\mathrm{d}\nu_\alpha(a)\leqslant C\int_\Omega|\widetilde{\nabla}f(a)|^p\mathrm{d}\nu_\alpha(a).$$

To prove Theorem 0.5, we need to improve it into the following. The difference is obvious.

Lemma 1.4 Let $p\geqslant 1$ and $\alpha>-1$. If f is harmonic on Ω, then for any real Möbius transformation ψ in Ω

$$\int_\Omega\left(\int_0^1\frac{|\Lambda f\circ\psi(ta)|}{1-t|a|}\mathrm{d}t\right)^p\mathrm{d}\nu_\alpha(a)\leqslant C\int_\Omega|\Lambda f\circ\psi(a)|^p\mathrm{d}\nu_\alpha(a). \qquad (5)$$

Proof Fix $\varepsilon\in(0,1)$. Observe that, for any $t\in[0,1]$ and $a\in\Omega$, if at least one of t and $|a|$ is less than ε, then $|ta| = t|a|<\varepsilon$ such that $\frac{1}{1-t|a|}\leqslant\frac{1}{1-\varepsilon}$. Thus the left side of (5) can be controlled by

$$\int_{\Omega-\varepsilon\Omega}\left(\int_\varepsilon^1\frac{|\Lambda f\circ\psi(ta)|}{1-t|a|}\mathrm{d}t\right)^p\mathrm{d}\nu_\alpha(a) + C\sup_{x\in\varepsilon\Omega}|\Lambda f\circ\psi(x)|^p.$$

Denote the first summand above by I. From the polar coordinate integral formula and Minkowski's inequality we get

$$I = 2n\int_\varepsilon^1\int_{\partial\Omega}\left(\int_\varepsilon^1\frac{|\Lambda f\circ\psi(ts\zeta)|}{1-ts}\mathrm{d}t\right)^p\mathrm{d}\sigma(\zeta)s^{2n-1}(1-s^2)^\alpha\mathrm{d}s\leqslant$$
$$C\int_\varepsilon^1\left(\int_\varepsilon^1\frac{M_p(ts,|\Lambda f\circ\psi|)}{1-ts}\mathrm{d}t\right)^p s^{2n-1}(1-s^2)^\alpha\mathrm{d}s\leqslant$$

$$C\int_{\varepsilon}^{1}\Big(\int_{\varepsilon^2}^{s}h(\rho)\mathrm{d}\rho\Big)^{p}(1-s^2)^{\alpha}\mathrm{d}s,$$

where

$$h(\rho)=\frac{\rho^{\frac{2n-1}{p}}M_p(\rho,|\Lambda f\circ\psi|)}{1-\rho}.$$

Applying Flett's inequality (see Ref. [11, p. 758])

$$\int_0^1\Big(\int_0^s h(\rho)\mathrm{d}\rho\Big)^p(1-s)^{\alpha}\mathrm{d}s\leqslant C\int_0^1 h^p(t)(1-t)^{\alpha+p}\mathrm{d}t,$$

we have

$$I\leqslant C\int_0^1\Big(\int_0^s h(\rho)\mathrm{d}\rho\Big)^p(1-s)^{\alpha}\mathrm{d}s\leqslant$$

$$C\int_0^1 t^{2n-1}(1-t)^{\alpha}M_p{}^p(t,|\Lambda f\circ\psi|)\mathrm{d}t=$$

$$C\int_{\Omega}|\Lambda f\circ\psi(a)|^p\mathrm{d}\nu_{\alpha}(a).$$

It remains to show that

$$\sup_{x\in\varepsilon\Omega}|\Lambda f\circ\psi(x)|^p\leqslant C\int_{\Omega}|\Lambda f\circ\psi(a)|^p\mathrm{d}\nu_{\alpha}(a). \tag{6}$$

It is well known that

$$|g(x)|\leqslant C\int_{E(x,\delta)}|g(w)|\mathrm{d}\lambda(w)$$

for any harmonic function g. Since each partial derivative of harmonic function remains harmonic, we have

$$|\nabla f(x)|\leqslant C\int_{E(x,\delta)}|\nabla f(w)|\mathrm{d}\lambda(w).$$

Recall that $\mathrm{d}\lambda(w)=(1-|w|)^{-2n}\mathrm{d}\nu(w)$ is real Möbius invariant, $|\Lambda f(x)|=(1-|x|^2)|\nabla f(x)|$, $1-|w|\simeq 1-|x|$ for $w\in E(x,\delta)$. Applying Fubini's theorem we obtain

$$|\Lambda f(x)|^p\leqslant C\int_{E(x,\delta)}|\Lambda f(w)|^p\mathrm{d}\lambda(w)\leqslant$$

$$C\int_{E(\psi(x),\delta)}|\Lambda f\circ\psi(y)|^p\mathrm{d}\lambda(y).$$

Because ψ is an involutionary automorphism of Ω, we get

$$|\Lambda f\circ\psi(x)|^p\leqslant C\int_{E(x,\delta)}|\Lambda f\circ\psi(y)|^p\mathrm{d}\lambda(y).$$

Since $1-|y|\simeq 1-|x|\simeq 1$ for $x\in\epsilon\Omega$ and $y\in E(x,\delta)$, the assertion (6) follows.

2 Proof of the Theorem

Theorem 0.5 can be restated as the following result.

Theorem 2.1 Let $p\in(4n-2,\infty)$. Then, for any holomorphic function f on $\mathbb{B}_n$,

$$\int_{\mathbb{B}_n}\int_{\mathbb{B}_n}\Big[\frac{|f(z)-f(w)|^p}{|z-w|^p}(1-|z|^2)^{\frac{p}{2}}\cdot(1-|w|^2)^{\frac{p}{2}-2n}\mathrm{d}\tau(z)\mathrm{d}\nu(w)\Big]<\infty, \tag{7}$$

if and only if

$$\int_{\mathbb{B}_n}(1-|w|^2)^p|\nabla f(w)|^p\mathrm{d}\tau(w)<\infty. \tag{8}$$

Proof of Theorem 2.1 As mentioned above, we identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ and $\mathbb{B}_n$ with Ω, then the holomorphic function f is harmonic on Ω.

If $f\in\mathscr{B}_p$, then for any $a\in\Omega$

$$\frac{|f\circ\psi_z(a)-f\circ\psi_z(0)|}{|a|}=$$

$$\Big|\int_0^1\nabla f\circ\psi_z(ta)\frac{a}{|a|}\mathrm{d}t\Big|\leqslant$$

$$\int_0^1\frac{|\Lambda f\circ\psi_z(ta)|}{1-t|a|}\mathrm{d}t.$$

We take p-th power on both sides and integrate over Ω with respect to $\mathrm{d}\nu_{\alpha}$ for any $\alpha>-1$. Then Lemma 1.3 shows

$$\int_{\Omega}\frac{|f\circ\psi_z(a)-f\circ\psi_z(0)|^p}{|a|^p}\mathrm{d}\nu_{\alpha}(a)\leqslant$$

$$C\int_{\Omega}|\Lambda f\circ\psi_z(a)|^p\mathrm{d}\nu_{\alpha}(a). \tag{9}$$

Because $p>4n-2$, we can set $\alpha=\frac{p}{2}-2n$. Now we take the transform $w=\psi_z(a)$. We integrate both sides in (9) with respect to $\mathrm{d}\tau(z)$ and notice the fact that

$$\mathrm{d}\nu_{\alpha}(a)=(1-|a|^2)^{\alpha}\mathrm{d}\nu(a)=(1-|a|^2)^{\alpha+2n}\mathrm{d}\lambda(a)$$

to yield

$$\int_{\Omega}\int_{\Omega}\frac{|f(w)-f(z)|^p}{|\psi_z(w)|^p}\frac{(1-|\psi_z(w)|^2)^{\frac{p}{2}}}{(1-|w|^2)^{2n}}\mathrm{d}\tau(z)\mathrm{d}\nu(w)\leqslant$$

$$C\int_{\Omega}\int_{\Omega}\Big[|\Lambda f(w)|^p\frac{(1-|\psi_z(w)|^2)^{\frac{p}{2}}(1-|w|^2)^{n+1}}{(1-|w|^2)^{2n}(1-|z|^2)^{n+1}}\mathrm{d}\nu(z)\mathrm{d}\tau(w)\Big]. \tag{10}$$

By (2) and (3), the left side of (10), denoted by I, turns out to be

$$I=\int_{\Omega}\int_{\Omega}\frac{|f(w)-f(z)|^p}{|\psi_z(w)|^p}\frac{(1-|\psi_z(w)|^2)^{\frac{p}{2}}}{(1-|w|^2)^{2n}}\mathrm{d}\tau(z)\mathrm{d}\nu(w)=$$

$$\int_{\Omega}\int_{\Omega}\Big[\frac{|f(z)-f(w)|^p}{|z-w|^p}(1-|z|^2)^{\frac{p}{2}}\cdot(1-|w|^2)^{\frac{p}{2}-2n}\mathrm{d}\tau(z)\mathrm{d}\nu(w)\Big].$$

To estimate the right side of (10), we denote it by J. Because $p>4n-2$, then $\frac{p}{2}-n+1>1$, $\frac{p}{2}-n-1>-1$. By Lemma 1.2 we have

$$\int_\Omega \frac{(1-|\psi_z(w)|^2)^{\frac{p}{2}}(1-|w|^2)^{n+1}}{(1-|w|^2)^{2n}(1-|z|^2)^{n+1}}d\nu(z)=$$
$$\int_\Omega \frac{(1-|w|^2)^{\frac{p}{2}-n+1}(1-|z|^2)^{\frac{p}{2}-n-1}}{||z|w-z'|^p}d\nu(z)\simeq C.$$

Since $|\Lambda f(w)|=(1-|w|^2)|\nabla f(w)|$, we finally obtain

$$J\leqslant C\int_\Omega(1-|w|^2)^p|\nabla f(w)|^p d\tau(w).$$

Combine the above estimates to yield the fact that (8) implies (7).

Conversely, let f be holomorphic in $\mathbb{B}_n$ and satisfy (8), we need to show that (9) holds. Since f is harmonic on Ω, we find that for any fixed $\delta\in(0,1)$

$$(1-|z|^2)^p|\nabla f(z)|^p\leqslant$$
$$C\int_{E(z,\delta)}|f(z)-f(w)|^p d\lambda(w).$$

Since $d\lambda(w)=(1-|w|^2)^{-2n}d\upsilon(w)$, we have

$$\int_\Omega(1-|z|^2)^p|\nabla f(z)|^p d\tau(z)\leqslant$$
$$C\int_\Omega\int_{E(z,\delta)}\frac{|f(z)-f(w)|^p}{(1-|w|^2)^{2n}}d\tau(z)d\nu(w).$$

Lemma 1.1 implies that, for any $w\in E(z,\delta)$,

$$\frac{(1-|z|^2)^{\frac{p}{2}}(1-|w|^2)^{\frac{p}{2}}}{|z-w|^p}\geqslant C,$$

which means

$$\frac{1}{(1-|w|^2)^{2n}}\leqslant C\frac{(1-|w|^2)^{\frac{p}{2}-2n}(1-|z|^2)^{\frac{p}{2}}}{|z-w|^p}.$$

Hence

$$\int_\Omega(1-|z|^2)^p|\nabla f(z)|^p d\tau(z)\leqslant$$
$$C\int_\Omega\int_\Omega\left[\frac{|f(z)-f(w)|^p}{|z-w|^p}(1-|z|^2)^{\frac{p}{2}}\cdot\right.$$
$$\left.(1-|w|^2)^{\frac{p}{2}-2n}d\tau(z)d\nu(w)\right].$$

This completes the proof. □

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