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Dynamical Systems With Enveloping Semigroup Containing Finitely Many Idempotents*

SHAO Song

(*Department of Mathematics*, *USTC*, *Hefei* 230026, *China*)

Abstract: Let (X, T) be a topological dynamical system and its enveloping semigroup $E(X, T)$ the closure of $\{T^n : n \in \mathbb{Z}_+\}$ in X^X. An $u \in E(X, T)$ with $u^2 = u$ is called an idempotent. The systems with enveloping semigroup containing finitely many idempotents are studied in this paper. It is shown that they are semi-distal. And comparing it with other dynamical properties we try to point out that the property we defined is not complex. Examples are given in the final section.

Key words: idempotent; scrambled set; substitution

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0 Introduction

We use $\mathbb{Z}$ to denote the integers, $\mathbb{Z}_+$ the non-negative integers, $\mathbb{Z}_-$ the non-positive integers and $\mathbb{N}$ the natural numbers. A topological dynamical system is a pair (X, T), where X is a compact metric space with a metric d and T is a surjective continuous map from X to itself. A pair $(x, y) \in X \times X$ is said to be proximal if $\liminf_{n \to +\infty} d(T^n x, T^n y) = 0$ and the one such that $\lim_{n \to +\infty} d(T^n x, T^n y) = 0$ is said to be asymptotic. If in addition $x \neq y$, then the pair (x, y) is said to be proper. The sets of proximal pairs and asymptotic pairs of (X, T) are denoted by $\mathrm{prox}(X, T)$ and $\mathrm{asym}(X, T)$ respectively. $x \in X$ is a recurrent point if there is $\{n_i\} \subset \mathbb{N}$ such that $T^{n_i} x \to x$, when $n_i \to +\infty$. A pair $(x, y) \in X^2$ which is not proximal is said to be distal. A pair is said to be a Li-Yorke pair if it is proximal but not asymptotic. A pair $(x, y) \in X^2 \setminus \Delta_X$ is said to be a strong Li-Yorke pair if it is proximal and is a recurrent point of X^2. It is easy to check that a strong Li-Yorke pair is a Li-Yorke pair. A system without proper proximal pairs (Li-Yorke pairs, strong Li-Yorke pairs) is called distal (al-

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Biography: SHAO Song, male, born in 1976, PhD. E-mail: songshao@ustc.edu.cn.

most distal, semi-distal respectively). Following the definitions a distal system is almost distal and an almost distal system is semi-distal. (X,T) is equicontinuous if for any $\varepsilon > 0$, there is $\delta > 0$ such that $d(x,y) < \delta$ implies $d(T^n x, T^n y) < \varepsilon$, for every $n \in \mathbb{Z}_+$. An equicontinuous system is distal.

A factor map $\pi: (X,T) \longrightarrow (Y,S)$ is a continuous onto map from X to Y such that $S \circ \pi = \pi \circ T$; in this situation (X,T) is said to be an extension of (Y,S) and (Y,S) the factor of (X,T). If $\pi: (X,T) \to (Y,T)$ is an extension, then set $R_\pi = \{(x_1, x_2) \in X^2 : \pi x_1 = \pi x_2\} = (\pi \times \pi)^{-1} \Delta Y \subset X \times X$. An extension $\pi: (X,T) \to (Y,T)$ is called asymptotic if $R_\pi \subset \mathrm{Asmp}(X,T)$. Similarly we define proximal, distal, equicontinuous extensions. π is almost one to one if there exists a dense G_δ set $Y_0 \subset Y$ such that $\pi^{-1}(y)$ is a singleton for any $y \in Y_0$.

Let $\mathrm{Trans}T = \{x : \omega(T,x) = X\}$, where $\omega(T,x)$ is ω limit set of x. Say (X,T) is transitive if $\mathrm{Trans}T \neq \emptyset$. In fact $\mathrm{Trans}T$ is a dense G_δ set when it is not empty. Say (X,T) is minimal if $\mathrm{Trans}T = X$ and $x \in X$ is minimal point if it belongs to some minimal subsystem of X. If $(X \times X, T \times T)$ is transitive then (X,T) is said to be weakly mixing.

In order to study the asymptotic behavior of a topological system (X,T) Ellis introduced in 1960 the enveloping semigroup $E(X,T)$ which has been proved to be a very powerful tool in the theory of topological dynamical system[10]. It is defined as the closure of the set $\{T^n : n \in \mathbb{Z}_+\}$ in X^X (with its compact, usually non-metrizable, pointwise convergence topology). Ellis pointed out that for any system the idempotents in $E(X,T)$ always exist. He also showed that a system (X,T) is equicontinuous iff $E(X,T)$ is a group of homeomorphism and (X,T) is distal iff $E(X,T)$ is a group. Hence in some sense a system whose enveloping semigroup containing only one idempotent is not complex. It is well known that in the minimal case a PI system is relatively simple. Let (X,T) be a minimal dynamical system. (X,T) is said to be strictly proximal isometric or strictly PI if it can be obtained from the trivial system by a (countable) transfinite succession of proximal and equicontinuous extensions. And (X,T) is said to be proximal isometric or PI if it is the factor of a strictly PI system by a proximal extension. It is well known that X is PI if and only if it satisfies the following property: whenever W is a closed invariant subset of $X \times X$ which is topologically transitive and has a dense subset of minimal points, then W is minimal[2]. On the other hand, if (X,T) is not a PI system then for every $x \in X$ and every minimal left ideal I in $E(X,T)$, $Id(I)x$ is uncountable, where $Id(I)$ is the set of idempotents of I[8]. And (X,T) is a minimal weakly mixing system iff for any $x \in X$ the set $Id(E(X,T))x$ is dense in X[11]. Hence it seems that as the cardinal of the set of idempotents of the enveloping semigroup becomes bigger the systems become more complex. So we focus our attention on the simpler case: the systems whose enveloping semigroup containing only finitely many idempotents (we call them FID systems for short). Glasner showed that a minimal system whose enveloping semigroup contains only finitely many minimal left ideals is PI[13]. Obviously the enveloping semigroup of an FID system has only finitely many minimal ideals. Hence an FID minimal system is PI. This is another reason invoking us to study FID systems.

Recently Akin studied distality concepts for Ellis actions [6]. He defined a system without strong Li-Yorke pairs to be semi-distal, i. e. every $(x, y) \in (X\times X, T\times T)$ which is proximal and recurrent is in the diagonal. Akin gave an elegant characterization to the semi-distal system in viewpoint of the enveloping semigroup. Let the adherence semigroup $\mathcal{H}(X,T)$ be $\limsup\{T^n\} = \bigcap_k \overline{\{T^n \mid n=k,k+1,\cdots\}} \subset E(X,T)$. He pointed out that a system is semi-distal iff every idempotent in $\mathcal{H}(X,T)$ is minimal. We show that any FID system is semi-distal but the converse does not hold. Hence we also have that a transitive FID system is minimal.

Chaos is a complex dynamical behavior[1]. There are lots of definitions on chaos. Among them chaos in the sense of Li-Yorke is a basic one[7]. A set $S\subset X$ is called a scrambled set if any pair of distinct points $(x, y)\in S\times S$ is a Li-Yorke pair. (X,T) is called chaos in the sense of Li-Yorke if it admits an uncountable scrambled set. Also we are interested in the system whose scrambled set is finite because it is simple in the viewpoint of chaos. We show that the transitive FID systems are this kind of systems. But the converse is not true. An example will be given to show this. F. Blandchard etc. call a system without Li-Yorke pairs almost distal[7]. They pointed out that a system (X,T) is almost distal iff $(\mathcal{H}(X,T),T)$ is a minimal system iff the only minimal left ideal of $\mathcal{H}(X,T)$ is itself. We conjecture that every transitive FID system is almost distal.

The paper is organized as follows. In section 1 we discuss the properties of the systems with finitely many idempotents. In the last section we give the examples to show that for any $n\in\mathbb{N}$ we have a system with $\max\{\#(C): C$ is the scrambled set of $X\} = n$, where $\# C$ is the cardinal of set C. And for any $n\in\mathbb{N}$, we give the systems whose cardinal of idempotents of $\mathcal{H}(X,T)$ is n. We also show that some substitution minimal systems are semi-distal and some are not.

1 FID systems

First we introduce some concepts on semigroup.

Definition 1 A set E is an Ellis semigroup if it satisfies the following three conditions:

ⅰ) E is a semigroup.

ⅱ) E has a compact Hausdorff topology.

ⅲ) The left translation map $R_p: E\longrightarrow E, q\longmapsto qp$ is continuous for every $p\in E$.

It is easy to see that for a system (X,T) the enveloping semigroup $E(X,T)$ and the adherence semigroup $\mathcal{H}(X,T)$ are Ellis semigroups. Ellis-Namakura Theorem says that for any Ellis semigroup E the set $Id(E)$, the set of idempotents of E, is not empty. We can introduce a quasi-order (a reflexive, transitive relation) $<_R$ on the set $Id(E)$ by defining $v <_R u$ *iff* $uv = v$. If $v <_R u$ and $u <_R v$ we say that u and v are equivalent and write $u \sim_R v$. Similarly we define $<_L$ and $\sim_L$.

An idempotent $u \in Id(E)$ is maximal if $v \in Id(E)$ and $u <_R v$ implies $v <_R u$. And the minimal idempotent is defined similarly. A non-empty subset $I \subset E$ is a left ideal if it is closed and $EI \subset I$. A minimal left ideal is left ideal that does not contain any proper left ideal of E. Obviously every left ideal is semigroup and every left ideal contains some minimal ideal. By the theory of semitopological semigroup every idempotent in the minimal left ideal is minimal and an idempotent is minimal iff it is contained in some minimal left ideal[16].

The following theorem is a basic result in the theory of semitopological semigroup. For completeness we give a proof[9,10,16].

Theorem 1 Let E be Ellis semigroup and $c \in Id(E)$. Then there are minimal and maximal idempotents u and v respectively, such that $u <_R c <_R v$.

Proof Let I be any minimal left ideal of E. Then Ic is also a minimal left ideal and by Ellis-Namakura theorem let w be an idempotent of Ic. Let $w = kc$ with $k \in I$. Set $u = cw = ckc \in Ic$. Then

$$u^2 = (ckc)(ckc) = ckckc = c(kc)(kc) = cww = cw = u.$$

Hence $u \in Id(E)$ and $cu = ccw = cw = u$. As $u \in Ic$ and Ic is minimal left ideal, u is minimal.

Now show the existence of a maximal idempotent v. Let $\{c_i\}$ be a totally ordered family in $Id(E)$ with $c <_R c_i$. Regard $\{c_i\}$ as a net, and let (a subnet of) $c_i \to r \in E$. Then for fixed i, if $c_i <_R c_j$, $c_j c_i = c_i$, so $rc_i = c_i$. Let $H = \{q \in E : qc_i = c_i$, for all $c_i\}$. Then H is a nonempty closed semigroup, and hence by Ellis-Namakura theorem, H contains an idempotent s. Clearly $c <_R c_i <_R s$. Thus by Zorn's Lemma there is a maximal idempotent v such that $c <_R v$.

Now we begin to show that any FID system (X, T) is semi-distal, i. e. every idempotent of $\mathscr{E}(X, T)$ is minimal.

Proposition 1 Let (X, T) be a system. Then the following conditions are equivalent:

(1) every idempotent of $\mathscr{E}(X, T)$ is maximal.

(2) every idempotent of $\mathscr{E}(X, T)$ is minimal.

(3) (X, T) is semi-distal.

Proof (1)⇒(2) We show that if every idempotent is maximal then any idempotent u is minimal. By Theorem 1 there is a minimal idempotent v such that $v <_R u$. Since every idempotent is maximal, $u <_R v$. But v is minimal, thus u is minimal.

(2)⇒(1) is similar to (1)⇒(2).

(2)⇔(3) [6].

Following Proposition 5.33. of [10] (also see [9]) we have:

Theorem 2 For a system (X,T) the maximal idempotents in $\mathscr{H}(X,T)$ are dense in $Id(\mathscr{H}(X,T))$.

Corollary 1 For system (X,T) if every minimal idempotent of $\mathscr{H}(X,T)$ is an isolated point in $Id(\mathscr{H}(X,T))$ then (X,T) is semi-distal.

Proof We show that every idempotent is minimal. For any $u \in Id(\mathscr{H}(X,T))$ by Theorem 1 there is some minimal idempotent $v \in \mathscr{H}(X,T)$ such that $v <_R u$. As v is an isolated point in $Id(\mathscr{H}(X,T))$, there is a neighborhood U of v in $\mathscr{H}(X,T)$ such that $U \cap Id(\mathscr{H}(X,T)) = \{v\}$. Since the set of maximal idempotents in $\mathscr{H}(X,T)$ are dense in $Id(\mathscr{H}(X,T))$, there is a maximal idempotent in U. As there is only one idempotent in U, v is maximal. Thus as $v <_R u$ we have $u <_R v$. Since v is minimal, by the definition u is minimal too. So (X,T) is semi-distal.

Theorem 3 Let (X,T) be a TDS. Then

(1) An FID system is semi-distal.

(2) If (X,T) has a non-minimal idempotent, then $Id(E(X,T))$ is infinite.

(3) If (X,T) is a transitive FID system, then it is minimal.

(4) Any transitive but not minimal system has infinitely many idempotents.

Proof By Corollary 1. we have (1). (1)⇔(2), (3)⇔(4) are obvious. So we only need to show (3). Let $x \in \mathrm{Trans}T$. Then by $x \in \omega(x,T)$ there is some $p \in E(X,T)$ such that $px = x$. Thus $\{p \in E(X,T): px = x\}$ is a nonempty closed semigroup and contains some idempotent u. As (X,T) is FID, u is minimal. So $x = ux$ is minimal point. That is, (X,T) is minimal.

Remark 1) In fact we can show that (1) ~ (4) are equivalent.

2) The Floyd-Auslander system is an example which is semi-distal with infinitely many minimal idempotents (see, for example, [2]). So semi-distality does not imply FID. It is easy to check that Floyd-Auslander system is also chaotic in the sense of Li-Yorke. Hence a semi-distal system may also be complex in the viewpoint of chaos.

3) Two systems (X,T), (Y,S) are said to be disjoint if their product system $(X \times Y, T \times S)$ is the only closed invariant set of $X \times Y$ projecting onto both X and Y. As any minimal semi-distal system is disjoint from all weakly mixing systems[6], any minimal FID system is also disjoint from all weakly mixing systems. Also from this we can see FID is not a complex dynamical property.

As for minimal system (X,T) it is weakly mixing iff its maximal equicontinuous factor is trivial and any equicontinuous system is FID. From these facts we can also have that any minimal weakly mixing system is disjoint from all minmal FID systems.

Definition 2 (1) A system (X,T) is said to be a CID system if $Id(E(X,T))$ is a countable set; to be a ζ- ID system if the cardinal of $Id(E(X,T))$ is ζ.

(2) A system (X,T) is said to be an FS system if $s = \max\{\# C: C$ is the scrambled set of $X\}$ is finite; to be a CS system if s is countable; to be a ζ-S system if the cardinal of s is ζ.

Remark 1) Since $Id(E(X,T)) = Id(\mathcal{H}(X,T)) \cup \{id\}$, where $id = T^0$ is the identity map of X, we can replace $E(X,T)$ by $\mathcal{H}(X,T)$ in the definition. In the sequel we will show that id is a very special idempotent.

2) Any almost distal system is FS. When ζ is uncountable, a ζ-S system is Li-Yorke chaotic.

Proposition 2 Any transitive FID system is FS.

Proof Let (X,T) be a transitive FID system. By Theorem 3 it is minimal. Let $S = \{x_i : i \in I\}$ be any scrambled set in X, where I is an index set. Now we show that $\# I$ is finite. Let x be a point of S. For any $i \in I$, (x, x_i) is proximal and hence there is a minimal idempotent u_i such that $x_i = u_i x$ [2]. If $i \neq j$, then $u_i \neq u_j$ by $x_i \neq x_j$. So $\# S = \# I \leqslant \# (Id(E(X,T))) < \infty$, i. e. $(X. T)$ is FS.

Remark (1) In fact the same proof shows that any minimal ζ-ID system is ζ-S system.

(2) The converse of Proposition 2 does not hold. We will give an minimal system which is FS but not FID in the next section.

(3) For the general case an FID system need not be FS. In [12] Huang-Ye constructed an example which is completely Li-Yorke chaotic, i. e. the whole space is a scrambled set. This example is Li-Yorke chaotic but has only two idempotents in $E(X,T)$ (see [12] for details).

Glasner showed that a minimal system whose enveloping semigroup contains finitely many minimal left ideals is PI system [13]. In [18] McMahon showed that if X is a minimal system whose $E(X,T)$ has less than 2^{Ω} minimal left ideals, where Ω is the first uncountable order number, then X is PI. Moreover if a minimal system X is not PI, then for every x in X and every minimal left ideal I in $E(X,T)$ the set $Id(I)x$ is uncountable. Hence in the minimal case

$$\text{Distality} \subset \text{FID} \subset \text{CID} \subset \text{PI} \subset \zeta\text{-ID},$$

where $\zeta \geqslant 2^{\Omega}$. We also have

$$\text{FID} \subset \text{semi-distality} \subset \text{PI}.$$

We don't know whether a CID system is semi-distal?

Let $\pi : (X,T) \to (Y,S)$ be a factor map. Then there is a unique continuous semigroup homomorphism $\phi : E(X,T) \to E(Y,S)$ such that $\pi(px) = \phi(p)\pi(x)$, $x \in X$, $p \in E(X,T)$. We can get ϕ as follows. Let $\phi : \{T^n : n \in \mathbb{Z}_+\} \to \{S^n : n \in \mathbb{Z}_+\}$, $T^n \mapsto S^n$, where $\{T^n : n \in \mathbb{Z}_+\}$, $\{S^n : n \in \mathbb{Z}_+\}$ with the topology inherited from X^X and Y^Y. Then ϕ is uniformly continuous, and hence has a continuous extension, still called ϕ, to a continuous map of $E(X,T)$ to $E(Y,S)$. ϕ has the required properties.

Proposition 3 1) $\pi : (X,T) \to (Y,S)$ is a homomorphism. If (X,T) is FID, then (Y,S) is FID.

2) Any subsystem of an FID system is still FID.

3) The product of finitely many FID systems is FID.

Proof (1) Let $\phi: E(X) \to E(Y)$ be the corresponding semigroup homomorphism. For any $v \in Id(E(Y))$, $\phi^{-1}(v)$ is the closed subsemigroup of E(X). Hence by Ellis-Namakura theorem $Id(\phi^{-1}(v)) \neq \emptyset$. Let $u \in Id(\phi^{-1}(v))$, then $\phi(u) = v$. Thus $\phi(Id(E(X))) = Id(E(Y))$. Especially, if (X,T) is FID, then (Y,S) is FID.

(2) Let (X,T) be a system and (Y,T') be its subsystem, i. e. $Y \subset X$ be an invariant closed subset and $T|_Y = T'$. We show that for any $u' \in Id(E(Y,T'))$ there is some $u \in Id(E(X,T))$ such that $u|_Y = u'$. Hence $\# Id(E(Y,T')) \leqslant \# Id(E(X,T))$. Especially, if (X,T) is FID, then (Y,T') is FID too.

Let $u' \in Id(E(Y,T'))$ and $A = \{p \in E(X,T): p|_Y = u'\}$. Set $\{T'^{\alpha}\}$ be a net with $T'^{\alpha} \to u'$. Let (a subnet of) $T^{\alpha} \to p$. As $(T|_Y)^{\alpha} = T'^{\alpha}$, $p|_Y = u'$, i. e. A is nonempty. Now we show that A is a semigroup. Let $p, q \in A$. For any $y \in Y$, $pq(y) = p(q(y)) = p(u'(y)) = u'(u'(y)) = u'^2(y) = u'(y)$. So $(pq)|_Y = u'$, i. e. $pq \in A$. Thus A is a nonempty closed semigroup. By Ellis-Namakura Theorem there is an $u \in Id(A) \subset Id(E(X,T))$. That is, for any $u' \in Id(E(Y,T'))$ there is some $u \in Id(E(X,T))$ such that $u|_Y = u'$.

(3) Let $\{(X_i, T_i)\}_{i=1}^n$ be topological dynamical systems. Then $E(X_1 \times X_2 \times \cdots \times X_n, T_1 \times T_2 \times \cdots \times T_n) \subset E(X_1, T_1) \times E(X_2, T_2) \times \cdots \times E(X_n, T_n)$. By this fact it is easy to see that the product of finitely many FID systems is FID.

Remark Let $\pi: (X,T) \to (Y,S)$ be an extension and (Y,S) be FID. For most π, (X,T) will be not FID. We will, in the next section, give an example to show that even if π is almost one to one (X,T) can not be FID.

Proposition 4 Let (X,T) be a topological dynamical system. Then the following conditions are equivalent:

(1) id is a minimal idempotent of $E(X,T)$;

(2) id is a minimal idempotent of $\mathscr{H}(X,T)$;

(3) (X,T) is distal.

If in addition (X,T) is transitive then (1)～(3) are equivalent to(4) $\# Id(\mathscr{H}(X,T)) = 1$;

Proof (1) ⇒(2) is clear.

(2) ⇒(3) Let id be a minimal idempotent of $\mathscr{H}(X,T)$ hence there is minimal left ideal I such that $id \in I$. Then $\mathscr{H}(X,T) = \mathscr{H}(X,T)\, id \subset \mathscr{H}(X,T)\, I \subset I$, i. e. $\mathscr{H}(X,T) = I$ is a minimal left ideal. For any $(x,y) \in Prox$ we have $px = py$ for some $p \in \mathscr{H}(X,T)$. Hence $A = \{p \in \mathscr{H}(X,T): px = py\}$ is a left ideal. As $\mathscr{H}(X,T)$ itself is a minimal left ideal, $A = \mathscr{H}(X,T)$. Especially $id \in A$ and we have $x = y$. Hence by definition of distality (X,T) is distal.

(3)⇒(1) by the fact that (X,T) is distal iff $E(X,T)$ is a group.

Now assume that (X,T) is transitive. (3) ⇒(4) is easy. (4) ⇒(2) is similar to the proof above. If $Id(\mathscr{H}(X,T)) = \{u\}$, then $\mathscr{H}(X,T) = u\mathscr{H}(X,T)$ is a group. As (X,T) is transitive, let $x \in \mathrm{Trans}T$. Then $ux = x$ and it is easy to see $uy = y$ for any $y \in X$. Hence $u = id$.

Call a system (X, T) weakly rigid if for any $n \in \mathbb{N}$ (X^n, T) is pointwise recurrent, i. e. every point of X^n is recurrent. By the definition of the adherence semigroup we have that (X, T) is weakly rigid iff $id \in \mathcal{A}(X, T)$.

Corollary 2 If (X, T) is semi-distal but not distal, then $id \notin \mathcal{A}(X, T)$.

Corollary 3 Let (X, T) be semi-distal. Then (X, T) is distal iff (X, T) is weakly rigid.

3 Examples

In this section we will give examples to show that for any $n \in \mathbb{N}$ we have a system with $\max\{\#(C): C$ is the scrambled set of $X\} = n$. And for any $n \in \mathbb{N}$, we give systems whose cardinal of idempotents of $\mathcal{A}(X, T)$ is n. We show that some substitution minimal systems are semi-distal and some are not.

First we introduce some basic concepts on the substitution. Let S be a finite alphabet with m symbols, $m \geqslant 2$. We usually suppose that $S = \{0, 1, \cdots, m-1\}$. Let $\Omega = S^{\mathbb{Z}}$ be the set of all bisequences $x = \cdots x_{-1} x_0 x_1 \cdots, x_i \in S$, $i \in \mathbb{Z}$, with the product topology. A metric compatible is given by $d(x, y) = \dfrac{1}{1+k}$, where $k = \min\{|n| : x_n \neq y_n\}, x, y \in \Omega$. The shift map $\sigma: \Omega \longrightarrow \Omega$ is defined by $(\sigma x)_n = x_{n+1}$ for all $n \in \mathbb{Z}$. The pair (Ω, σ) is called a shift dynamical system. The points of Ω are called bisequences. Similarly we can replace $\mathbb{Z}$ by $\mathbb{Z}_+$, and σ will not be a homeomorphism but a surjective map. The points of $S^{\mathbb{Z}_+}$ are called sequences. The elements of $S^* = \bigcup_{k \geqslant 1} S^k$ are called blocks (over S). For $\alpha \in S^*$ denote the length of α by $N(\alpha)$. If $\omega \in \Omega$ and $a \leqslant b \in \mathbb{Z}$, then $\omega[a, b]$ will denote the $(b-a+1)-$block occurring in ω starting at place a and ending at place b. That is, if $\omega = (\cdots \omega_{-1}\ \omega_0\ \omega_1 \cdots)$, then $\omega[a, b] = \omega_a \omega_{a+1} \cdots \omega_b$. Sometimes we also write $\omega[a; b-a+1] = \omega[a, b]$. Similarly we define $\omega[a, b]$ when ω is a sequence or a block.

A substitution θ is a map $\theta: S \longrightarrow S^*$. If $\theta: S \longrightarrow S^n, n \geqslant 2$, then θ is called a substitution of constant length n. Let λ or $\lambda(\theta)$ denote the map of S^2 into S^2 such that $pq \in S^2$ implies $\lambda(pq) = (\theta p)_{n_p - 1}(\theta q)_0$, where n_p is the length of $\theta(p)$, $(\theta p)_{n_p - 1}$ is the last symbol in θp and $(\theta q)_0$ is the first symbol in θq. Let L or $L(\theta)$ denote the set of all $pq \in S^2$ such that there exists $i \in \mathbb{N}$ with $\lambda^i(pq) = pq$. Let μ be the period of $\lambda|_L$, that is, μ is the least $i \in \mathbb{N}$ such that $pq \in L$ implies $\lambda^i(pq) = pq$. Let $pq \in L$ and we define a sequence ω_{pq} as follows: $\omega_{pq}[-N(\theta^{\mu k} p), N(\theta^{\mu k} q) - 1] = \theta^{\mu k} p\ \theta^{\mu k} q$, for $k = 1, 2, \cdots$.

let L_0 or $L_0(\theta)$ denote the set of all $pq \in L$ such that for each $r \in \mathit{Range}\ (\omega_{pq})$ there exists $i \in \mathbb{N}$ such that pq occurs in $\theta^i r$. Let W or $W(\theta)$ denote $\{\omega_{pq} : pq \in L\}$ and W_0 or $W_0(\theta)$ denote $\{\omega_{pq} : pq \in L_0\}$. We have the following results: L_0 and so W_0 is not empty; W_0 coincides with the set of all almost periodic points under σ in W. Let $X_\theta = \overline{\text{Orb}\ (w_{pq},\sigma)}$ be the closed σ — orbit of w_{pq} in Ω. We call (X_θ,σ) a substitution minimal system (for details, see [4]).

By $Z(n)$, $(n \geqslant 2)$ we denote the additive group of n-adic integers: $Z(n) = \left\{ z = \sum_{i=0}^{\infty} z_i n^i \mid z_i = 0,1,\cdots,n-1 \right\}$. A metric in $Z(n)$ is given by $\rho(z,z') = \frac{1}{\min\{i: z_i \neq z'_i\}+1}$, where $z = \sum_{i=0}^{\infty} z_i n^i, z' = \sum_{i=0}^{\infty} z'_i n^i \in Z(n)$. $(Z(n),\rho)$ is a compact metric group. A map $\tau: Z(n) \longrightarrow Z(n)$ is defined by $\tau(z) = z+1$. We call $(Z(n),\tau)$ an n-adic system. Clearly, τ is an isometric homeomorphism, that is, $\rho(\tau(z),\tau(z')) = \rho(z,z')$. Every element z of the orbit of 0 in $Z(n)$ is of the form $z = \sum_{i=0}^{\infty} z_i n^i$, where either $z_i = 0$ for all but finitely many i, or $z_i = n-1$ for all but finitely many i. We shall regard such elements both as ordinary integers and as elements of $Z(n)$.

Definition 2 Let $S = \{0,1,\cdots,m-1\}$ and θ be a substitution of length r over S. We call θ an admissible substitution if θ is one to one, primitive and every point in W_0 is not periodic. By primitivity we mean that there is $k \in \mathbb{N}$ such that $a,b \in S$ implies a occurs in $\theta^k(b)$.

Remark The reason why we assume θ to be primitive is that if θ is primitive then all elements in W_0 generate the same flow, which we denote by (X_θ,σ). And in this case we have $X_\theta = X_{\theta^k}$ for any $k \geqslant 1$, so that we may replace θ by a power of θ without changing the flow.

In the sequel, θ will denote a fixed admissible substitution of length r over S.

For a fixed substitution θ of length r, we use the term basic r^k — block to denote any one of the r^k — blocks $\theta^k(p)$ $(p \in S)$.

Theorem 4 There is a homomorphism $\pi: (X_\theta,\sigma) \longrightarrow (Z(r),\tau)$. Moreover, if $x \in X_\theta$ and $z = \sum_{i=0}^{\infty} z_i r^i \in Z(r)$, then $\pi(x) = z$ iff $x[-z^{(k+1)}; r^{k+1}]$ is a basic r^{k+1} — block for each k, where $z^{(k)} = \sum_{i=0}^{k-1} z_i r^i$.

Proof Let $\omega_{pq} \in W_0$. We define a map $\pi: (X_\theta,\sigma) \longrightarrow (Z(n),\tau)$ by $\pi(\omega) = \lim_{i\to\infty} \tau^{k_i}(0)$ where $\{k_i\}$ satisfies $\lim_{i\to\infty} \sigma^{k_i}(\omega_{pq}) = \omega$. We can easily verify that π is well defined and has the property as mentioned (see [3] for details).

Theorem 5 If $z \in Z(r)$, then $\pi^{-1}(z)$ consists of at most m^2 points.

Proof Let x be an arbitrary element in $\pi^{-1}(z)$. By the definition of π we can choose $\{k_i\}$ such that $\lim_{i\to\infty} \sigma^{k_i}(\omega_{pq}) = x$ and $\lim_{i\to\infty} \tau^{k_i}(0) = z$.

Assume $z = \sum_{i=0}^{\infty} z_i r^i$. If $z \in \mathbb{Z}$ then $\#\ \pi^{-1}(z) = \#\ \pi^{-1}(0)$. By Theorem 4 $\pi^{-1}(0) = W_0$

and $\# \pi^{-1}(z) = \# W_0 \leqslant \# S^2 = m^2$.

Now assume $z \in Z(n)\setminus \mathbb{Z}$. Then $z^{(k)} \to +\infty$ and $r^k - z^{(k)} - 1 \to +\infty (k \to \infty)$. Let $A_k = \{\theta^k(a): a \in S\}$, then $\# A_k = m$ for each $k \in \mathbb{N}$. Define $f_k: A_{k+1} \longrightarrow A_k$ by $f_k(x) = x[z^{(k+1)} - z^{(k)}, z^{(k+1)} + r^k - z^{(k)} - 1] = x[z_k r^k, z_k r^k + r^k - 1]$. Let $A = \prod_{k=1}^{\infty} A_k$ be the direct product and $P_k: A \longrightarrow A_k$ the project map for each k.

Set $T = \{x \in A: P_k(x) = f_k(P_{k+1}(x)), \forall k \geqslant 1\}$. The map which maps $x \in \pi^{-1}(z)$ to $\{x[-z^{(k)}, r^k - z^{(k)} - 1]\}_{k=1}^{\infty} \in T$ is an injective map. Thus $\# \pi^{-1}(z) \leqslant \# T \leqslant m$.

Lemma 1 Let $\pi: (X,T) \to (Y,S)$ be a proximal extension and (Y,S) be a distal system. Then for any proximal pair (x,x') of X, $\pi(x) = \pi(x')$.

Proof Since $(\pi(x),\pi(x'))$ is a proximal pair of Y and Y is distal, we have $\pi(x) = \pi(x')$.

Theorem 6 Every scrambled set S contained in X_θ consists of at most m points.

Proof By Lemma 1 S is contained in some fibre of π, i. e. there is some $z \in Z(r)$ such that $S \subset \pi^{-1}(z)$. From the proof of Theorem 5 it is easy to see $\# S \leqslant m$.

Remark For substitution over two symbols, we can get the result above in a slightly different manner[5,14].

The following example shows that for any $m \in \mathbb{N}$ there is a system whose maximum cardinal of scrambled set is m.

Example 1 Let $S = \{0,1,\cdots,m-1\}$ and $\theta: S \longrightarrow S^3$ with $0 \longmapsto 001, 1 \longmapsto 102, 2 \longmapsto 203, \cdots, m-1 \longmapsto (m-1)\,00$. Then $L_0 = L = S^2$, $\# L_0 = m^2$ and $\# \pi^{-1}(0) = \# L_0 = m^2$.

For every $\omega_{pq}, \omega_{rs} \in S^2$, ω_{pq} and ω_{rs} are Li-Yorke pairs iff $q \neq s$. And in this example the maximum cardinal of the scrambled set is m.

The following example shows that for any $m \in \mathbb{N}$ there is a system (X,T) whose number of idempotents is m.

Example 2 Let $S = \{0,1,\cdots,m-1\}$ and $\theta: S \longrightarrow S^3$ with $0 \longrightarrow 010, 1 \longrightarrow 020, 2 \longrightarrow 030, \cdots, m-1 \longrightarrow 000$. Then the number of idempotents of $\mathcal{H}(X_\theta,\sigma)$ is m.

Proof Let $\pi: (X_\theta,\sigma) \longrightarrow (Z(3),\tau)$ be the homomorphism above. Let $z' = \sum_{i=0}^{\infty} 3^i = (\cdots,1,1,1,\cdots)$ and $E = \{\tau^k z': k \in \mathbb{Z}\}$. We claim that $Z(3)\setminus E = \{z \in Z(3): \pi^{-1}(z)$ is a single point$\}$ and when $z \in E \# (\pi^{-1}(z)) = m$.

As $W_0 = \{\omega_{00}\}$, $\# (\pi^{-1}(z)) = 1$ when $z \in \mathbb{Z}$. Now suppose $z \in Z(3)\setminus E$. If $x \in \pi^{-1}(z)$, then $x[-z^{(k)}; 3^k]$ is a basic 3^k-block. As $z \notin E$, x_0 is uniquely determined. Since $\mathbb{Z}$ and E are τ-invariant, $Z(3)\setminus(\mathbb{Z} \cup E)$ is also τ-invariant. And $\sigma^n x \in \pi^{-1}(\tau^n z)$, $x_n = (\sigma^n x)_0$ is uniquely determined for each n. Hence $\pi^{-1}(z) = \{x\}$ is a single point set.

Suppose next that $z \in E$. As E is τ-invariant, we may assume $z = z' = \sum_{i=0}^{\infty} 3^i$. Both $z^{(k)}$

and $3^k - z^{(k)}$ increase to infinity. By Theorem 4 there are points $x^{(1)}, x^{(2)}, \cdots, x^{(m)} \in X_\theta$ such that for each $k \in \mathbb{N}$,

$$\{x^{(i)}[-z^{(k)}; 3^k]: i = 1, 2, \cdots, m\} = \{\theta^k(i): i \in S\}.$$

Hence $\pi^{-1}(z) = \{x^{(1)}, x^{(2)}, \cdots, x^{(m)}\}$ and π is an asymptotic extension.

Since X_θ is minimal and π is asymptotic, for any $i \in \{1, 2, \cdots, m\}$, $\{p \in E(X): p\pi^{-1}(z') = x^{(i)}\}$ is a nonempty semigroup and hence contains idempotents. For each i, we take an idempotent u_i from the above semigroup. We show $Id(\mathcal{H}(X)) = \{u_i: i = 1, 2, \cdots, m\}$.

For any $u \in Id(\mathcal{H}(X))$, since $(Z(3), \tau)$ is equicontinuous, u is identity of $E(Z(3))$. Hence $u\pi^{-1}(z') = \{x^{(i)}\} = u_i\pi^{-1}(z')$ for some $i \in \{1, 2, \cdots, m\}$. So for any $n \in \mathbb{Z}$, $u\pi^{-1}(\tau^n z') = \{\sigma^n x^{(i)}\} = u_i\pi^{-1}(\tau^n z')$. Since for any $z \in Z(3) \backslash \mathrm{Orb}(z', \tau)$, $\pi^{-1}(z)$ is a single point. And $u|_{\pi^{-1}(Z(3)\backslash \mathrm{Orb}(z', \tau))} = id_{X_\theta} = u_i|_{\pi^{-1}(Z(3)\backslash \mathrm{Orb}(z', \tau))}$. Thus $ux = u_i x$ for any $x \in X_\theta$, i. e. $u = u_i$. Hence we have $Id(\mathcal{H}(X)) = \{u_i: i = 1, 2, \cdots, m\}$.

Now we show that for some substitutions they are semi-distal and some not. For convenience we study the substitution over two symbols.

Recall that there is a homomorphism $\pi: (X_\theta, \sigma) \longrightarrow (Z(n), \tau)$ mapping w_{pq} to 0. Furthermore, if $\pi(w) = \sum_{i=0}^{\infty} z_i n^i$, then for each $k \geqslant 1$, $w\left[-\sum_{i=0}^{k-1} z_i n^i, n^k - \sum_{i=0}^{k-1} z_i n^i - 1\right]$ is either $\theta^k(0)$ or $\theta^k(1)$. Set $J_k = \{m: 0 \leqslant m \leqslant n^k - 1, \theta^k(0)$ and $\theta^k(1)$ disagree at place $m\}$, $J_\infty = \left\{\sum_{i=0}^{\infty} z_i n^i: z_i \in J_1 \text{ for all } i\right\}$, $E = \bigcup \{\tau^k J_\infty: k \in Z\}$, $Z^* = Z(n) - E$. Then we have $Z^* = \{\pi^{-1}(z)$ is a single point$\}$. If $z \in E$, then $\pi^{-1}(z)$ consists of exactly two points unless $z \in Z$ and a and b disagree at both endpoints. In this case, $\pi^{-1}(z)$ consists of four points (see [14]).

Proposition 5 Let (X_θ, σ) be a substitution minimal flow, where θ is a substitution of constant length $n (n \geqslant 3)$ on the symbols 0 and 1. Let $\theta(0) = a_0 a_1 \cdots a_{n-1}$ and $\theta(1) = b_0 b_1 \cdots b_{n-1}$.

1) If there is some $k \in \{0, 1, \cdots, n-1\}$ such that $a_k \neq b_k$ and $a_i = b_i, i \neq k$, then $\pi: (X_\theta, \sigma) \longrightarrow (Z(n), \tau)$ is an asymptotic extension, and (X_θ, σ) is semi-distal.

2) If $a_{n-1} \neq b_{n-1}$ and there are $i, j \in \{0, 1, \cdots, n-2\}$ with $a_i \neq b_i, a_j = b_j$, then $\pi: (X_\theta, \sigma) \longrightarrow (Z(n), \tau)$ is a proximal but not an asymptotic extension, and (X_θ, σ) is not semi-distal.

Proof (1) As above $J_\infty = \left\{\sum_{i=0}^{\infty} k n^i\right\}$, $E = \left\{\sum_{i=0}^{\infty} k n^i + t: t \in Z\right\}$, $Z^* = Z(n) - E$. If $k = 0$ or $n-1$, then $E = Z$. Suppose that $z \in E$. We may assume that $z = 0$. Then $\pi^{-1}(z) = \{w, w': w(0) \neq w'(0), w(i) = w'(i), i \neq 0\}$. If $k \neq 0, n-1$, then $Z \subset Z^*$. Suppose that $z \in E$. We may assume that $z = \sum_{i=0}^{\infty} k n^i$. Then $\pi^{-1}(z) = \{w, w': w(0) \neq w'(0), w(i) = w'(i), i \neq 0\}$. So we have that $\pi: (X_\theta, \sigma) \longrightarrow (Z(n), \tau)$ is an asymptotic extension.

(2) We may assume $a_{n-1}=0$ and $b_{n-1}=1$ (otherwise we use θ^2 instead of θ). Let $\pi:(X_\theta,\sigma)\longrightarrow(Z(n),\tau)$ be the homomorphism above. Then there is $z\in Z(n)-Z$ such that $\#(\pi^{-1}(z))=2$. Let $\pi^{-1}(z)=\{w,w'\}$. It is easy to verify that (w,w') is proximal and not asymptotic. Now we are going to show that $(w,w')\in R(T\times T)$.

If $z=\sum_{i=0}^{\infty} z_i n^i$, let $z^{(k)}$ denote the $k-$th partial sum of z, that is, $z^{(k)}=\sum_{i=0}^{k-1} z_i n^i$. As $z\in Z(n)-Z$, both $z^{(k)}$ and $n^k-z^{(k)}$ increase with k. By the construction of π, we have for each $k\in\mathbb{N}$

$$\{w[-z^{(k)},n^k-z^{(k)}-1],w'[-z^{(k)},n^k-z^{(k)}-1]\}=\{\theta^k(0),\theta^k(1)\}. \qquad (*)$$

We may assume that there are infinitely many zeros in $\{w(n^k-z^{(k)}-1):k\in\mathbb{N}\}$, i. e., there are $k_1<k_2<\cdots$ such that $w(n^{k_i}-z^{(k_i)}-1)=0$ for each $i\in\mathbb{N}$. Since $a_{n-1}=0$ we have $w[-z^{(k_i)},n^{k_i}-z^{(k_i)}-1]=\theta^{k_i}(0)$ for each $i\in\mathbb{N}$. It follows by $(*)$ that $w'[-z^{(k_i)},n^{k_i}-z^{(k_i)}-1]=\theta^{k_i}(1)$ for each $i\in\mathbb{N}$.

Let $j>i$. Then

$$(w[-z^{(k_j)},n^{k_j}-z^{(k_j)}-1])[n^{k_j}-n^{k_i},n^{k_j}-1]=\theta^{k_i}(0)=w[-z^{(k_i)},n^{k_i}-z^{(k_i)}-1]$$

$$(w'[-z^{(k_j)},n^{k_j}-z^{(k_j)}-1])[n^{k_j}-n^{k_i},n^{k_j}-1]=\theta^{k_i}(1)=w'[-z^{(k_i)},n^{k_i}-z^{(k_i)}-1]$$

Hence $d((w,w'),\sigma^{t_{ij}}(w,w'))<S_{ij}$, where $t_{ij}=n^{k_j}-z^{(k_j)}-1-(n^{k_i}-z^{(k_i)}-1)$ and $S_{ij}=\dfrac{1}{\min\{z^{(k_i)},n^{k_i}-z^{(k_i)}-1\}}$.

Since $\lim S_{ij}=0$, (w,w') is a recurrent point of $(X_\theta\times X_\theta,\sigma\times\sigma)$.

Remark Similar to Example 2, the number of idempotents of $\mathscr{E}(X_\theta)$ of the system in (1) is two. And by the fact an FID system is semi-distal, the system in (2) has infinitely many idempotents. Also from this example we can see the almost one to one extension of an FID system need not be FID.

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包络半群仅含有限个幂等元的系统

邵　松

（中国科学技术大学数学系，安徽合肥 230026）

摘要：设 (X, T) 为拓扑动力系统，它的包络半群 $E(X, T)$ 定义为 $\{T^n : n \in \mathbb{Z}_+\}$ 在 X^X 中乘积拓扑下的闭包，如果元素 $u \in E(X, T)$ 满足 $u^2 = u$，则称之为幂等元. 本文研究包络半群仅含有限个幂等元的系统的性质，证明这类系统为 semi-distal 的，并且对照其他动力学性质，指出这类系统的动力学性状相对简单，并提供了许多例子进行论证.

关键词：幂等元；混乱集；子替换