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Relation $\mathcal{H}^*$ on Completely Archimedean Semigroups*

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Abstract: Relation $\mathcal{H}^*$ on a completely archimedean semigroup is investigated, and some equivalent conditions for $\mathcal{H}^*$ to be a congruence are given.

Key words: completely archimedean semigroup; π-group; relation $\mathcal{H}^*$; congruence

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1 Introduction and preliminaries

Bogdannovic defined a kind of generalized Green's relations $\mathcal{L}^*$, $\mathcal{R}^*$, $\mathcal{H}^*$ and $\mathcal{J}^*$ on the π-regular semigroups[1]. It is well known that relation $\mathcal{H}$ is a congruence on a completely simple semigroup. However, it is not necessary that relation $\mathcal{H}^*$ is a congruence on a completely archimedean semigroup. It is natural to ask what is the case relation $\mathcal{H}^*$ is a congruence. The aim of this paper is to give an answer to this question.

An element a of semigroup S is called π-regular if there exists a positive integer n such that $a^n \in a^n S a^n$. A semigroup S is called π-regular if all of its elements are π-regular. A semigroup S is a completely archimedean semigroup if it is a nil-extension of a completely simple semigroup, i. e., S is a π-regular semigroup and all of its idempotents are primitive. A completely regular semigroup S is cryptic if $\mathcal{H}$ is a congruence on S, further, if $S/\mathcal{H}$ is a regular band, cryptogroup S is called a regular cryptogroup.

Throughout this paper, denote the set of all idempotents of the semigroup S by $E(S)$ and the set of all regular elements of S by Reg S. If a is an element of π-regular semigroup S, we denote $\min\{n \in N \mid a^n \in \text{Reg } S\}$ by $r(a)$. A partial multiplicative set Q is called partial semigroup if $a(bc) = (ab)c$ whenever $a(bc)$ or $(ab)c$ in Q for any $a, b, c \in Q$. Moreover, call partial semigroup Q a power partial semigroup if for each $a \in Q$, there exists a natural

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number n such that $a^n \notin Q$ (see ref. [2] for details). Also, we use the symbol $\langle\varphi\rangle$ to denote the value of a constant maping φ.

Generalized Green's equivalences on π-regular semigroup S were defined by ref. [1] as follows:

$$a\mathcal{L}^* b \Longleftrightarrow Sa^{r(a)} = Sb^{r(b)} \quad a\mathcal{R}^* b \Longleftrightarrow a^{r(a)}S = b^{r(b)}S,$$
$$\mathcal{H}^* = \mathcal{L}^* \cap \mathcal{R}^*, \quad a\mathcal{J}^* b \Longleftrightarrow Sa^{r(a)}S = Sb^{r(b)}S.$$

Denote the $\mathcal{K}^*$-class containing e by $K_e{}^*$, where $\mathcal{K}^* \in \{\mathcal{L}^*, \mathcal{R}^*, \mathcal{H}^*\}$. Other notations and terminologies not given here can be found in the texts of ref. [3] and ref. [1].

In ref. [1], Bogdannovic gave a construction of a semigroup S which has a completely simple ideal with Rees matrix semigroup $\mathcal{M}(I,G,\Lambda;P)$ and a partial semigroup Q.

Specifically, if Q is a power partial semigroup, we can easily get

Lemma 1.1 A semigroup S is a completely archimedean semigroup if and only if it is isomorphic to some $\Sigma = \mathcal{M}(I,G,\Lambda;P) \bigcup Q$ with a multiplication defined as follows:

(1) $(i,a,\lambda)(j,b,\mu) = (i,ap_{\lambda j}b,\mu)$; (2) $p(i,a,\lambda) = (i\xi_p,\varphi(p,i)a,\lambda)$;

(3) $(i,a,\lambda)p = (i,a\psi(p,\lambda),\lambda\eta_p)$; (4) $pq = r \in Q \Rightarrow pq = r \in \Sigma$;

(5) $pq \notin Q \Rightarrow pq = (i\xi_q\xi_p, \varphi(p,i\xi_p)\varphi(q,i)p^{-1}_{\lambda\eta_p\eta_q,i}, \lambda\eta_p\eta_q)$.

for all $p,q \in Q, a,b \in G$, $i,j \in I$, $\lambda,\mu \in \Lambda$, where $\xi: p \mapsto \xi_p$ ($\eta: p \mapsto \eta_p$) is a mapping from Q into the semigroup $T(I)$ ($T(\Lambda)$) of all the mappings of $I(\Lambda)$ into itself. □

2 Main results

For a completely archimedean semigroup S, the next lemma is obvious:

Lemma 2.1 Let $S = Q \bigcup T$ be a completely archimedean semigroup, where T is the completely simple kernel of S and $Q = S \backslash T$. Then

(1) $\mathcal{L}^S = \mathcal{L}^T \bigcup 1_Q$; (2) $\mathcal{R}^S = \mathcal{R}^T \bigcup 1_Q$; (3) $\mathcal{H}^S = \mathcal{H}^T \bigcup 1_Q$;

(4) $\mathcal{D}^S = \mathcal{J}^S$ and $\mathcal{D}^S = (T \times T) \bigcup 1_Q$. □

Lemma 2.2 Let S be a completely archimedean semigroup. Denote

$$P(e) = \{x \in S: ex = xe \in G_e \text{ for some } e \in E(S)\}$$

in which G_e is the maximal subgroup of S containing e as its identity. Then we have $P(e) = H_e^*$ and

(1) $P(e)$ is a π-group; (2) $P(e) \bigcap P(f) = \varnothing$ if $e,f \in E(S)$, and $e \neq f$;

(3) $S = \bigcup_{e \in E(S)} P(e)$.

Proof Let $e = (i,p^{-1}_{\lambda i},\lambda) \in E(S)$ and $x \in S$. Suppose that $ex = xe \in G_e$. Then by Lemma 1.1 $xe = x(i,p^{-1}_{\lambda i},\lambda) = (i\xi_x, *, \lambda)$, and $ex = (i,p^{-1}_{\lambda i},\lambda)x = (i, *, \lambda\eta_x)$. So we have $i\xi_x = i$, $\lambda\eta_x = \lambda$. Further $x^{r(x)}e = xx^{r(x)-1}(i,p^{-1}_{\lambda i},\lambda) = (i, *, \lambda)$, $ex^{r(x)} = (i,p^{-1}_{\lambda i},\lambda)x^{r(x)-1}x = (i, *, \lambda)$. It means that $x^{r(x)}\mathcal{H}e$, i.e., $x \in H_e^*$. On the other hand, if $x\mathcal{H}^*e$ for some $x \in S$ and $e \in E(S)$, we have $x^{r(x)} \in G_e$. By Theorem I. 4.3 of ref. [1], we know that $ex =$

$xe \in G_e$. The remainder is obvious. □

The above statement is not true for the nil-extension of general completely regular semigroups. For example, suppose that $S = \bigcup_{\alpha \in Y} S_\alpha$ is a nil-extension of a completely regular semigroup and $\alpha, \beta \in Y$ with $\alpha > \beta$, where S_α is a completely archimedean semigroup for any $\alpha \in Y$ (see Theorem Ⅸ.1.1 of ref.[1] for details). For any $e \in E(S_\alpha), f \in E(S_\beta)$, then $e(efe)^0 = (efe)^0 e = (efe)^0 \in G_{(efe)^0} \subseteq S_\beta$, but $e \notin S_\beta$.

At the same time it is not necessary that $\mathcal{H}^*$ is a congruence on a completely archimedean semigroup S, i.e., it is not necessary that S is a band of π-groups. The reader can find an example in ref.[3].

In the following, we give some equivalent conditions for $\mathcal{H}^*$ to be a congruence on completely archimedean semigroup S.

Lemma 2.3 Let S be a completely archimedean semigroup. Then $\mathcal{H}^*$ is a congruence on S if and only if $xyz\mathcal{H}^* xy'z$, for any $x, y, y', z \in S$.

Proof Suppose that S is a completely archimedean semigroup and satisfies $xyz\mathcal{H}^* xy'z$, for any $x, y, y', z \in S$. By Lemma 2.2, $S = \bigcup_{e \in E(S)} H_e^*$, in which $H_e^* = \{x \in S : ex = xe \in G_e$ for some $e \in E(S)\}$ and H_e^* is a π-group. We will claim that $H_e^* H_f^* \subseteq H_{ef}^*$. Let $a \in H_e^*, b \in H_f^*$. Then

$$(ab)(ef) = a(be)f \,\mathcal{H}^*\, a(eb)f = (ae)(bf) \in G_e G_f \subseteq G_{(ef)^0} \subseteq H_{ef}^*$$
$$(ef)(ab) = e(fa)b \,\mathcal{H}^*\, e(af)b = (ea)(fb) \in G_e G_f \subseteq G_{(ef)^0} \subseteq H_{ef}^*.$$

By Lemma 1.1 and the proof of lemma 2.2, we know that $ab \in H_{ef}^*$ and $H_e^* H_f^* \subseteq H_{ef}^*$, which means that $\mathcal{H}^*$ is a congruence. And hence $S/\mathcal{H}^*$ is a band. Further, $H_e^* H_f^* H_g^* = H_e^* H_{f'}^* H_g^*$ for any $e, f, f', g \in E(S)$. Let $f' = e$. Then $H_e^* H_f^* H_g^* = H_e^* H_g^*$. So, $S/\mathcal{H}^*$ is a rectangular band.

Conversely, if S is a completely archimedean semigroup and relation $\mathcal{H}^*$ is a congruence, then S is a rectangular band of π-groups. For any $x, y, y', z \in S$, suppose that $x \in H_e^*$, $y \in H_f^*$, $y' \in H_{f'}^*$, $z \in H_g^*$, we have $xyz \in H_e^* H_f^* H_g^* = H_e^* H_g^* = H_{eg}^*$, $xy'z \in H_e^* H_{f'}^* H_g^* = H_e^* H_g^* = H_{eg}^*$. So, $xyz\mathcal{H}^* xy'z$, for any $x, y, y', z \in S$. □

Lemma 2.4 Let S be a nil-extension of a cryptogroup. Then $\mathcal{H}^*$ is a congruence on S if and only if $(xy)^{r(xy)} \mathcal{H}^* x^{r(x)} y^{r(y)}$ for any $x, y \in S$.

Proof By the definition of $\mathcal{H}^*$, for any $x, y \in S$, $x\mathcal{H}^* x^{r(x)}$, $y\mathcal{H}^* y^{r(y)}$ and $xy\mathcal{H}^* (xy)^{r(xy)}$, since $\mathcal{H}^*$ is a congruence, $(xy)^{r(xy)} \mathcal{H}^* x^{r(x)} y^{r(y)}$. Conversely, let $x, y \in S$ and $x\mathcal{H}^* y$. Then $xz\mathcal{H}^* (xz)^{r(xz)} \mathcal{H}^* x^{r(x)} z^{r(z)}$, $yz\mathcal{H}^* (yz)^{r(yz)} \mathcal{H}^* y^{r(y)} z^{r(z)}$ for any $z \in S$. Because S is a nil-extension of a cryptogroup, we have $x^{r(x)} z^{r(z)} \mathcal{H}^* y^{r(y)} z^{r(z)}$. And hence $xz\mathcal{H}^* yz$. Similarly, $zx\mathcal{H}^* zy$. □

Lemma 2.5 Let S be a nil-extension of a regular cryptogroup. Then $\mathcal{H}^*$ is a congruence on S if and only if $\mathcal{L}^*$ and $\mathcal{R}^*$ are congruences on S.

Proof Let S be a nil-extension of a regular cryptogroup and $\mathcal{H}^*$ a congruence on S.

Suppose that $a, b \in S$ and $a\mathcal{L}^* b$. For any $c \in S$, by the regular cryptic of Reg S and Lemma 2.4, we have

$$ac\mathcal{L}^*(ac)^{r(ac)}\mathcal{L}^* a^{r(a)}c^{r(c)}\mathcal{L}^* b^{r(b)}c^{r(c)}\mathcal{L}^*(bc)^{r(bc)}\mathcal{L}^* bc,$$
$$ca\mathcal{L}^*(ca)^{r(ca)}\mathcal{L}^* c^{r(c)}a^{r(a)}\mathcal{L}^* c^{r(c)}b^{r(b)}\mathcal{L}^*(cb)^{r(cb)}\mathcal{L}^* cb.$$

So $\mathcal{L}^*$ is a congruence. Similarly, $\mathcal{R}^*$ can be proved to be a congruence. The other part is obvious. □

Lemma 2.6 Let S be a completely archimedean semigroup. Then $\mathcal{H}^*$ is a congruence on S if and only if $\langle\xi_x\rangle = i, \langle\eta_x\rangle = \lambda$, where $(i, *, \lambda) = x^{r(x)}$.

Proof Suppose that S is a completely archimedean semigroup, and relation $\mathcal{H}^*$ is a congruence on S. Let $x \in S$, and $x^{r(x)} = (i, *, \lambda)$. For any $y = (j, b, \mu) \in$ Reg S, $xy\mathcal{H}^* x^{r(x)}y = (i, *, \lambda)(j, b, \mu) = (i, *, \mu)$. On the other hand $xy = (j\xi_x, *, \mu)$ by Lemma 1.1. So for any $j \in I$, $j\xi_x = i$ which means $\langle\xi_x\rangle = i$. Similarly, $\langle\eta_x\rangle = \lambda$.

Conversely, suppose that $x, y \in S$ and $x\mathcal{H}^* y$. Then $x^{r(x)}\mathcal{H} y^{r(y)}$. By hypotheses, $\langle\xi_x\rangle = \langle\xi_y\rangle$, $\langle\eta_x\rangle = \langle\eta_y\rangle$. Then for any $z \in S$, by Lemma 1.1.

Case 1: when $z \in$ Reg S, by the hypotheses, it is obvious that $xz\mathcal{H}^* yz$.

Case 2: when $z \notin$ Reg S,

(1) and $x, y \in$ Reg S, $xz\mathcal{H}^* yz$ holds obviously.

(2) and $x, y \in S\backslash$Reg S, if $xz, yz \in S\backslash$Reg S, then $\langle\xi_{xz}\rangle = \langle\xi_z\rangle\langle\xi_x\rangle = \langle\xi_x\rangle$ and $\langle\xi_{yz}\rangle = \langle\xi_z\rangle\langle\xi_y\rangle = \langle\xi_y\rangle$ by Lemma 1.1, so $\langle\xi_{xz}\rangle = \langle\xi_{yz}\rangle$, and hence $xz\mathcal{R}^* yz$ holds. Similarly, we can prove that $xz\mathcal{L}^* yz$. If $xz, yz \in$ Reg S, suppose that $xz \in G_e$, $yz \in G_f$, then $xz = xze = x(ze)$, $yz = yzf = y(zf)$. By the hypotheses and $\langle\xi_x\rangle = \langle\xi_y\rangle$, we have $xz\mathcal{R}^* yz$. Similarly $xz\mathcal{L}^* yz$. If $xz \in G_e$ for some $e \in E(S)$ and $yz \in S\backslash$Reg S, then $\langle\xi_{yz}\rangle = \langle\xi_z\rangle\langle\xi_y\rangle = \langle\xi_y\rangle$, $xz = x(ze)$. That $xz\mathcal{R}^* yz$ also follows from that $\langle\xi_x\rangle = \langle\xi_y\rangle$. Similarly, we can get $xz\mathcal{L}^* yz$. In the same way, $zx\mathcal{H}^* zy$. So $\mathcal{H}^*$ is a congruence on S. □

Theorem 2.7 Let S be a completely archimedean semigroup. Then the following conditions are equivalent:

(1) Relation $\mathcal{H}^*$ is a congruence on S; (2) S is a rectangular band of π-groups; (3) $xyz\mathcal{H}^* xy'z$, for any $x, y, y', z \in S$; (4) $(xy)^{r(xy)}\mathcal{H}^* x^{r(x)}y^{r(y)}$ for any $x, y \in S$; (5) $\mathcal{L}^*$ and $\mathcal{R}^*$ are congruences on S; (6) $\langle\xi_x\rangle = i$, $\langle\eta_x\rangle = \lambda$, if $x^{r(x)} = (i, *, \lambda)$.

Proof Obviously, (1) and (2) are equivalent, and by Lemma 2.3, Lemma 2.4, Lemma 2.5 and Lemma 2.6, the proof is completed. □

Corollary 2.8 When relation $\mathcal{H}^*$ is a congruence on completely archimedean semigroup S, it is the maximum idempotent-separating congruence. □

Corollary 2.9 Let S be a completely archimedean semigroup. Then the following conditions are equivalent:

(1) Relation $\mathcal{R}^*$ is a congruence on S; (2) $(xy)^{r(xy)}\mathcal{R}^* x^{r(x)}y^{r(y)}$ for any $x, y \in S$; (3) $\langle\xi_p\rangle = i$, for any $p \in S\backslash$Reg S, if $p^{r(p)} = (i, *, *)$. □

Corollary 2.10 Let S be a completely archimedean semigroup. Then the following conditions are equivalent: (1) Relation $\mathcal{L}^*$ is a congruence on S; (2) $(xy)^{r(xy)}\ \mathcal{L}^*\ x^{r(x)}y^{r(y)}$ for any $x,y\in S$; (3) $\langle\eta_p\rangle=\lambda$, for any $p\in S\backslash \mathrm{Reg}\, S$, if $p^{r(p)}=(*,*,\lambda)$. □

Corollary 2.11 Let S be a completely archimedean semigroup. Then relation $\mathcal{R}^*$ ($\mathcal{L}^*$) is a congruence if and only if S is a left (right) zero band of a nil-extension of right (left) groups. □

Corollary 2.12 Let S be a completely archimedean semigroup. If relation $\mathcal{R}^*$ ($\mathcal{L}^*$) is a congruence, then S is the disjoint union of its some right (left) ideals R_e^* (L_e^*), in which $e\in E(S)$. □

At last, we give a description of a nil-extension of a rectangular band with $\mathcal{H}^*$ being a congruence.

Let $E=I\times J$ be a rectangular band and Q be a power partial semigroup, where I and λ are left and right zero band respectively, $\xi:p\mapsto\xi_p$ ($\eta:p\mapsto\eta_p$) be a maping from Q into the semigroup $T(I)(T(J))$ of all the transformations of $I(J)$ into itself, and

(1) for any $p\in Q, \xi_p=\mathrm{const}$, $\eta_p=\mathrm{const}$;

(2) for any $p,q\in Q, pq\in Q\Rightarrow\xi_{pq}=\xi_p$, $\eta_{pq}=\eta_q$.

Define a multiplication on $\Sigma=(I\times J)\bigcup Q$ as follows:

(i) $(i,j)(i',j')=(i,j')$ for any $i,i'\in I$, $j,j'\in J$;

(ii) $p(i,j)=(\langle\xi_p\rangle,j)$, $(i,j)p=(i,\langle\eta_p\rangle)$, for any $i\in I$, $j\in J$, $p\in Q$;

(iii) $pq=\begin{cases} r & \text{if } p,q\in Q \text{ and } pq=r\in Q\\ (\langle\xi_p\rangle,\langle\eta_q\rangle) & \text{if } pq\notin Q\end{cases}$

Theorem 2.13 Σ with the multiplication defined above is a semigroup. A semigroup S is a nil-extension of a rectangular band with $\mathcal{H}^*$ being a congruence if and only if it is isomorphic to some Σ. □

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完全阿基米德半群上关系 $\mathcal{H}^*$

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摘要: 给出了完全阿基米德半群上关系 $\mathcal{H}^*$ 是同余的若干等价条件.

关键词: 完全阿基米德半群;π-群;$\mathcal{H}^*$ 关系;同余