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Some Remarks on Highly Transitive Representations for Free Groups*

SUN Zi-xing[1], ZHU Zuo-tong[2]

(1. *Dept. of Math.*, *Fuyang Normal College*, *Fuyang* 236032, *China*;
2. *College of Mathematics and Computer Science*, *Nanjing Normal University*, *Nanjing* 210097, *China*)

Abstract: The free group $F_\eta (1 < \eta \leqslant \aleph_0)$ has a highly transitive representation in the rational line $\mathbb{Q}$. Let T be any countable dense subset of the irrational, it can be arranged that T be an orbit of $\hat{F}_\eta$ and that every $e \neq \hat{w} \in \hat{F}_\eta$ move every point in T.

Key words: free group; highly transitive representation; countable dense subset

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0 Introduction and main results

Let T be a totally ordered set, $A(T)$ be the group of order-preserving permutations on T. If $\alpha f \leqslant \alpha g$ and $\alpha(f \vee g) = \max\{\alpha f, \alpha g\}$, $\alpha(f \wedge g) = \min\{\alpha f, \alpha g\}$ for $\alpha \in T$ where f, $g \in A(T)$, then $A(T)$ is a l-group. Let G be a l-group if f is an injective from G into $A(T)$, then G has a faithful representation in T. Holland's main result is that every l-group is l-isomorphic to an l-subgroup of the l-group of order-preserving permutation of some totally ordered set[9]. An ordered group is highly transitive if it is n-transitive for every natural number n. A group G of order-preserving permutation of a totally ordered set T is n-transitive if for every $\alpha_1 < \alpha_2 < \cdots < \alpha_n$ and $\beta_1 < \beta_2 < \cdots < \beta_n$ in T, there exists $g \in G$ such that $\alpha_i g = \beta_i$, $i = 1, 2, \cdots, n$.

The free l-group on a countable infinite set of generators was shown to be isomorphic to a doubly transitive sublattice subgroup of the l-permutation group $A(\mathbb{Q})$[1]. Kopytov[2] and McCleary[3] proved independently that the free lattice-ordered group on at least two generators possesses doubly transitive representation (on the rational line if the number of

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Biography: SUN Zi-xing, male, born in 1953, associate professor. Research field: order algebra and combinatorics. E-mail: 11223366808@126.com

generators is finite or countable). Glass and McCleary[4] has showed that the free l-group $F_\eta(1<\eta\leqslant\aleph_0)$ has a pathologically (faithful) 2-transitive representation on $\mathbb{Q}$.

The main results of this paper are

Theorem 1 The free group $F_\eta(2\leqslant\eta\leqslant\aleph_0)$ can be faithfully represented as a highly order-transitive group $\hat{F}_\eta$ of order-preserving permutations of the rational line $\mathbb{Q}$.

Theorem 2 The free group $F_\eta(1<\eta\leqslant\aleph_0)$ has a (faithful) highly transitive representation as a group $\hat{F}_\eta$ of order-preserving permutations of the rational line $\mathbb{Q}$. Furthermore, for every $e\neq w\in F_\eta$, there exists $q_w\in\mathbb{Q}$ such that $\hat{w}$ moves all irrationals above q_w.

Theorem 3 Let T be any countable dense subset of the irrationals, it can be arranged that T be an orbit of $\hat{F}_\eta$ and that every $e\neq\hat{w}\in\hat{F}_\eta$ move every point in T.

1 Farther conclusions of free groups

We know that F_η can be made into a totally ordered group which is dense in itself[8, Chapter Ⅳ, Theorem 8]. Also, every chain which is countable, dense in itself, and lacks end points is order-isomorphic to the rational line $\mathbb{Q}$.

Let X be a fixed set of free generators for F_η. Our fundamental tool will be the notion of a diagram for a reduced group word $w=x_{i_1}^{\pm1}\cdots x_{i_n}^{\pm1}(x_{i_j}\in X)$.

The points of the diagram are the initial subwords $x_{i_1}^{\pm1}\cdots x_{i_k}^{\pm1}(0\leqslant k\leqslant n)(n\in\mathbb{N})$ of w. For each ordered pair (α,β) of points that $\alpha x_{i_j}^{\pm1}=\beta$, the diagram includes an x_{i_j}-arrow, from α to β if the exponent x_{i_j} is $+1$, otherwise from β to α. The remaining aspect of the diagram is a total order on the set Δ of points which is consistent with the arrows in that if there are x-arrows from α_1 to β_1 and α_2 to β_2 (same x for both), then $\alpha_1\leqslant\alpha_2$ iff $\beta_1\leqslant\beta_2$. (An x-arrow from α to β may alternately be described as an x^{-1}-arrow from β to α.) The empty initial subword ($k=0$) is called the base point of diagram.

By a diagram on $\mathbb{Q}$ we mean a diagram which arises from a substitution in $A(\mathbb{Q})$.

Lemma 1 Let $w\in F_\eta$. If $w\neq e$, there exists a diagram on $\mathbb{Q}$ showing this (by making $0\hat{w}\neq 0$). Moreover, given any diagram for w drawn on $\mathbb{Q}$, there is for each $x\in X$ an order-preserving permutation $\hat{x}$ of $\mathbb{Q}$ which acts in accordance with all the x-arrows.

Proof For the first claim, make F_η into a totally ordered group isomorphic as a chain to $\mathbb{Q}$, and use its right regular representation. For the second, the constraints imposed by the collection of x-arrows can be simultaneously satisfied by some $\hat{x}\in A(\mathbb{Q})$ because all open rational intervals are isomorphic as chains. □

Proof of Theorem 1 For each $x\in X$, the action on $\mathbb{Q}$ of its image $\hat{x}$ will be specified at enough points to guarantee the desired results. Each specification will amount to an x-arrow. The proof splits into three distinct phases:

(1) Specifications (essentially within $\mathbb{Q}^+$, the positive rationales) to achieve faithfulness and to link $\mathbb{Q}^+$ with $\mathbb{Q}^-$ by arranging that every n-type in $\mathbb{Q}$ can be sent to an

n-type in $\mathbb{Q}^-$.

(2) Specifications within $\mathbb{Q}^-$ to achieve high order-transitivity.

(3) Synthesis, in which for each $x \in X$ we choose an order-preserving permutation $\hat{x}$ of $\mathbb{Q}$ which meets the specifications for $\hat{x}$.

(Ⅰ)Faithfulness and linkage

F_η is countable, and we enumerate its nonidentity elements as $w_0, w_1, \cdots$. In the rational interval $[0,1]$, we lay out a copy of diagram for $w_0 = x_{i_1}^{\pm 1} \cdots x_{i_n}^{\pm 1}$ showing $e \neq w_0$, with the smallest point λ_0 of the diagram taken to be 0, and the largest point ρ_0 taken to be 1. We specify about the $\hat{x}$'s that the point (corresponding to) $x_{i_1}^{\pm 1} \cdots x_{i_{k-1}}^{\pm 1}$ be sent by $\hat{x}_{i_k}^{\pm 1}$ to the point (corresponding to) $x_{i_1}^{\pm 1} \cdots x_{i_{k-1}}^{\pm 1} x_{i_k}^{\pm 1}$. Similarly, in each interval $[2n, 2n+1]$, we lay out such a diagram for w_n and make such specifications. This is already enough to give faithfulness.

For the sake of the linkage, we want to ensure that all the points in the various diagrams lie in the same orbit of $\hat{F}_\eta$, and this orbit extends down into $\mathbb{Q}^-$. Then because the diagram points are cofinal in $\mathbb{Q}$ and all permutations in $\hat{F}_\eta$ will preserve order, we will have the desired linkage.

To make all diagram points lie in one orbit, it suffices to arrange that for each $n = 0, 1, \cdots$, the points $2n+1$ and $2(n+1)$ lie in the same orbit. For this we construct appropriate "bridges".

We begin with the interval $[1, 2]$. In the original for w_0, $\rho_0 (\leftrightarrow 1)$ must have been moved by at least one free generator, say x_{ρ_0}; and in the diagram for w_1, $x_1 (\leftrightarrow 2)$ must have been moved by some free generator x_{λ_1}. We declare that

(a_1) $1\hat{x}_{\rho_0} = \frac{4}{3}$ if ρ_0 was moved up by x_{ρ_0}; (a_2) $1\hat{x}_{\rho_0}^{-1} = \frac{4}{3}$ if ρ_0 was moved down by x_{ρ_0};

(b_1) $\frac{5}{3}\hat{x}_{\lambda_1} = 2$ if λ_1 was moved up by x_{λ_1}; (b_2) $\frac{5}{3}\hat{x}_{\lambda_1}^{-1} = 2$ if λ_1 was moved down by x_{λ_1}.

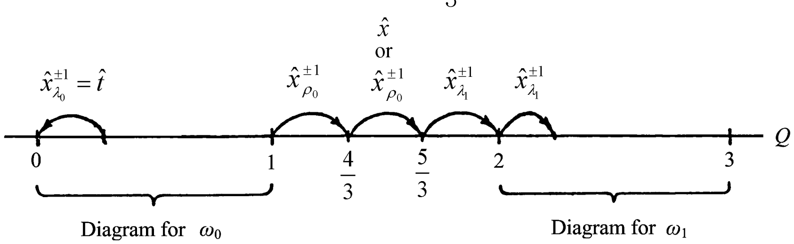

Fig. 1 The diagram of specifications within $\mathbb{Q}^+$ to achieve faithfulness

To connect $\frac{4}{3}$ and $\frac{5}{3}$, we further declare that $\frac{4}{3}\hat{x}_{\rho_0} = \frac{5}{3}$ if (a_1) obtains (or that $\frac{4}{3}\hat{x}_{\rho_0}^{-1} = \frac{5}{3}$ if (a_2) obtains); except that this may conflict with (b_2) or (b_1) if $x_{\rho_0} = x_{\lambda_1}$, so in that case we pick any other $x \in X (\eta > 1)$ and decree that $\frac{4}{3}\hat{x} = \frac{5}{3}$. We build similar bridges in the other intervals $[2n+1, 2n+2]$, $n=1,2,\cdots$.

Replacing x_{λ_0} by $x_{\lambda_0}^{-1}$ if necessary, we may assume $\hat{x}_{\lambda_0}$ moves some positive point to 0. We denote x_{λ_0} by t. Thus the figure contains a leftward t-arrow with head at 0. We specify that $\alpha\hat{t} = \alpha - 1$ for all $\alpha \leqslant 0$. Now the orbit of $\hat{F}_\eta$ containing the diagram points extends down into $\mathbb{Q}^-$, and we have linkage.

(Ⅱ) High order-transitivity

We fix $b \in X$ with $b \neq t$, and specify that $0\hat{b} = 0$. Let S_n be the set of pairs $(\alpha_1, \cdots, \alpha_n)$, $(\beta_1 \cdots, \beta_n)$ of strictly negative rationales with $\alpha_1 < \cdots < \alpha_n$ and $\beta_1 < \cdots < \beta_n$. Enumerate $S = \bigcup_{n=1}^{\infty} S_n$.

For the first pair $(\alpha_1, \cdots, \alpha_{n_1}), (\beta_1, \cdots, \beta_{n_1})$, we specify that $\alpha_i\hat{b} = \beta_i (i=1, \cdots, n_1)$; See figure 2. Next, we pick an integer $m_1 < \min\{\alpha_1, \beta_1\}$, and specify that $m_1\hat{b} = m_1$. For the next pair $(\mu_1, \cdots, \mu_{n_2}), (\nu_1, \cdots, \nu_{n_2})$, we specify that $(\mu_i - m_1)\hat{b} = \nu_i - m_1 (i=1, \cdots, n_2)$, which entails no conflict with any previous specification, thus arranging that $\mu_i\hat{t}^{m_1}\hat{b}\hat{t}^{-m_1} = \nu_i (i=1, \cdots, n_2)$. Next, we pick an integer $m_2 < \min\{\mu_1 - m_1, \nu_1 - m_1\}$, and specify that $m_2\hat{b} = m_2$. For the third pair $(\sigma_1, \cdots, \sigma_{n_3}), (\tau_1, \cdots, \tau_{n_3})$, we specify that $(\sigma_i - m_2)\hat{b} = \tau_i - m_2 (i=1, \cdots, n_3)$, and we pick an integer $m_3 < \min\{\sigma_1 - m_2, \tau_1 - m_2\}$ and specify that $m_3\hat{b} = m_3$.

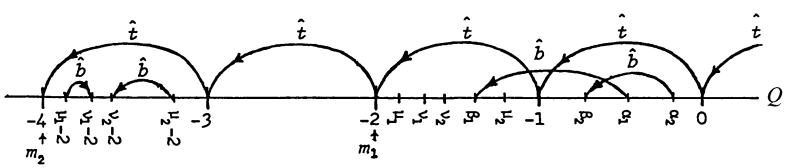

Fig. 2 The diagram of specifications within $\mathbb{Q}^-$ to achieve high order-transitivity

Continuing in this manner, we arrange that every negative n-type can be sent to every other negative n-type (same n) by some $\hat{t}^m\hat{b}\hat{t}^{-m}$.

For each $x \in X$, the set of points which are heads of specified x-arrows is cofinal (coinitial) in $\mathbb{Q}$ if and only if the same holds for tails. For each x other than t, the set of points which are ends of specified x-arrows has no limit points in $\mathbb{R}$, and for t only the non-positive real are limit points. Hence we can extend each $\hat{x}$ to an order-preserving permutation of $\mathbb{Q}$. □

Proof of Theorem 2 Faithfulness and linkage can be achieved directly by White's Theorem[5] that the maps $\alpha f = \alpha + 1$ and $\alpha g = \alpha^3$ freely generate a copy $\hat{F}_2$ of F_2 within $A(\mathbb{R})$, and that indeed every $e \neq \hat{w} \in \hat{F}_2$ moves every transcendental number.

We have $\hat{F}_2 \leqslant A(\mathbf{A})$, where $\mathbf{A}$ denotes the real algebraic. Let $f_i = g^{-i}fg^i$, $i=0,1,\cdots$. Let $\hat{k}_i = \hat{f}_i$, but change $\hat{k}_2$ to the left of 0 to make $(\hat{k}_1, \hat{k}_2)$ highly order-transitive. Now use the copy of $\hat{F}_\eta$ freely generated by $\{\hat{k}_i \mid 0 \leqslant i \leqslant \eta\}$.

This gives Theorem 2 with "irrationals" changed to "transcendental" and of course the chains $\mathbf{A}$ and $\mathbb{Q}$ are isomorphic. □

Proof of Theorem 3 This time we change the proof of Theorem 1 by changing the first part, with an argument modeled after the proof of [7, Theorem 3].

For any T_1 and T_2 satisfying our hypotheses, there is an order-preserving permutation of $\mathbb{Q}$ sending T_1 onto T_2[6, Lemma 21]. Thus we may assume that T is a coset of $\mathbb{Q}$ in $(\mathbb{R}, +)$, so that T is mapped onto itself by integer translations.

As before, let $\alpha\hat{t} = \alpha - 1$, specify that $0\hat{b} = 0$, and make specifications for $\hat{b}$ in $\mathbb{Q}^-$ to arrange high order-transitivity. These specifications amount to a collection of $\hat{b}$-arrows, and more $\hat{b}$-arrows, and more $\hat{b}$-arrows will be added later. Let $B \subseteq \mathbb{Q}^-$ be the set of points that are ends of $\hat{b}$-arrows. B has no limit points in $\mathbb{R}$.

For faithfulness, we use a brute force argument quite different from that of Theorem 1. Let Y be X with t deleted. Let P be the set of pairs (α, z), where $\alpha \in \mathbb{Q} \cup T$ and $z = y^{\pm 1}$ for some $y \in Y$. We proceed inductively through the enumeration, defining $\hat{y}$-arrows ($y \in Y$) as we go. There are two cases: $z = y$ and $z = y^{\pm 1}$.

When we reach (α, y), we specify a $\hat{y}$-arrows with tail at α (unless there is already such an arrow). Now we explain how to choose the head β of this arrow. Let λ be the

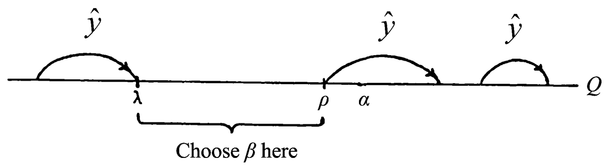

Fig. 3 The diagram specify $\hat{y}$-arrows about faithfulness

largest point in $\mathbb{Q} \cup T$ which is the head of an already existing $\hat{y}$-arrow whose tail lies below α; or if there aren't any, let $\lambda = -\infty$. (Only finitely many $\hat{y}$-arrows have been added so far during the induction, and by the construction of $\hat{b}$-arrows prior to the induction there must be the largest such point, ever for $y = b$.) Let ρ be the smallest of the points that are heads of already existing $\hat{y}$-arrows whose tails lie above α; or if there aren't any, let $\rho = +\infty$. We choose the head β of the new $\hat{y}$-arrows from (λ, ρ), and from the same set $\mathbb{Q}$ or T as α.

If $\alpha \in T$, we impose one more constraint on β, namely that β not differ by an integer from α or from either end of any previously defined $\hat{u}$-arrow ($u \in y$). (The $\hat{b}$-arrows specified prior to the induction have their ends in $\mathbb{Q}$, so there are only finitely many ends of $\hat{u}$-arrows in T.) The reason for the constraint is this: A loop is a path which starts at some $\delta \in \mathbb{Q} \cup T$ and follows a sequence of arrows ($\hat{t}$-arrows are included here) and eventually returns to δ. (No arrow has its two ends the same, removing any ambiguity from this definition.) The constraint makes sure that no sequence of $\hat{t}$-arrows leads from β to α or to any point which is either end of an already existing $\hat{u}$-arrow ($u \in y$). This in turn guarantees that as we proceed through the induction, there can never arise for the first time a loop involving a point T. And this guarantees that $w \neq e$ does not fix any points in

T. So faithfulness is obtained.

We treat the case y^{-1} similarly. This time we define $\hat{y}^{-1}$-arrow from α to β, i. e. , a $\hat{y}$-arrow from β to α. $\hat{y}^{-1}$-arrow has no limit point in $\mathbb{R}$, and for y only the non-positive real number are limit points. Hence we can extend each $\hat{z}$ to an order-automorphism of $\mathbb{Q}$ (onto $\mathbb{Q}$ because of the inclusion z^{-1} in the induction) which maps T onto itself. As before, we can arrange the faithfulness of $\hat{F}_\eta$ on $\mathbb{Q}$. □

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自由群的高可迁表示的注

孙自行[1],朱作桐[2]

(1. 阜阳师范学院数学系,安徽阜阳 236032;2. 南京师范大学数学与计算机科学学院,江苏南京 210097)

摘要: 自由群 $F_\eta(1<\eta\leqslant\aleph_0)$ 在有理数集$\mathbb{Q}$上有一个高可迁表示,若 T 是无理数集上的一个任意可数稠密子集,则可使得 T 是 $\hat{F}_\eta$ 的一个轨道且对任意 $e\neq\hat{w}\in\hat{F}_\eta$ 变换 T 中的每一个点.

关键词: 自由群;高可迁表示;可数稠密子集