Klein-Gordon Equation in Hydrodynamical Form

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We follow and modify the Feshbach-Villars formalism by separating the Klein-Gordon equation into two coupled time-dependent Schrödinger equations for particle and antiparticle wave function components with positive probability densities. We find that the equation of motion for the probability densities is in the form of relativistic hydrodynamics where various forces have their classical counterparts, with the additional element of the quantum stress tensor that depends on the derivatives of the amplitude of the wave function. We derive the equation of motion for the Wigner function and we find that its approximate classical weak-field limit coincides with the equation of motion for the distribution function in the collisionless kinetic theory.

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I. INTRODUCTION

In relativistic wave mechanics, it is well known that the naive probability density $\rho = 2\text{Im}(\psi^* \partial_t \psi)$, as constructed from the wave function ψ of the Klein-Gordon equation, is not necessarily a positive quantity. The presence of a negative probability density may appear to preclude its description in relativistic hydrodynamics. The origin of such pathology arises because the wave equation is second order in the time-derivative, and the corresponding probability density involves a time-derivative of the wave function. Dirac's resolution of the problem led to the introduction of another wave equation, the Dirac equation, that is first order in the time-derivative [1, 2]. A parallel resolution of the pathology was provided by Pauli and Weisskopf in quantum field theory with the introduction of particles and antiparticles, and the wave field ψ is interpreted not as the probability amplitude but as comprising of operators that create and destroy particles in various modes [3].

Wave mechanical interpretation of the wave field ψ was however reintroduced by Feshbach and Villars [4] by noting that the Klein-Gordon equation actually contains both particle and antiparticle degrees of freedom. The particle-antiparticle separation can be achieved by writing the Klein-Gordon and the Dirac equations as a set of coupled time-dependent Schrödinger equations for the particle and antiparticle wave function components with positive probability densities [4].

In many practical problems, as for example in the evolution of dense matter with relativistic constituents in relativistic hydrodynamics [5–10] or in the quarkonium two-body problem [11, 12], particles and antiparticles can be approximately treated as two distinct types of interacting particles, and they possess positive probability densities. For these problems, we are motivated to follow the Feshbach-Villars formalism where the particle and antiparticle probability densities can be positive definite.

We wish however to modify the formulation of Feshbach and Villars [4]. In their formulation, the relativistic properties of the momentum variables in the coupled Schrödinger equations are not readily apparent and the Schrödinger equation for the antiparticle wave function contains the operator $-(\mathbf{p}-e\mathbf{A})^2/2m$ that differs from the standard kinetic energy operator by a sign. It will be desirable to reformulate the problem so that the relativistic properties of the momentum variables become more apparent and the Schrödinger equations for the particle and antiparticle wave functions contain kinetic energy operators with the same sign. As a result, the connection to relativistic hydrodynamics can be more readily worked out.

Using the new set of coupled Schrödinger equations, we wish to search for a hydrodynamical description for the evolution of relativistic probability densities. Hydrodynamics and quantum mechanics have many elements in common, since the density fields and the velocity fields are important dynamical variables in both descriptions. It is therefore well-known that the Schrödinger equation can be cast into a hydrodynamical form for the evolution of the probability density [13, 14]. Such a correspondence has been utilized [15] to form the foundation for an "*a posteriori*" theoretical support for the validity of treating a nucleus as a liquid drop, as in Bohr and Wheeler [16], and treating the fission of a nucleus in liquid-drop hydrodynamics, as in Hill and Wheeler [17]. There are however important differences associated with quantum effects that are absent in classical hydrodynamics [14, 15, 18, 19]. The deviation of the classical hydrodynamical description from the quantum treatment is embodied in the presence of the quantum stress

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tensor $p_{ij}^{(q)}$ in quantum fluids [14, 15]. Quantum shell effects manifest themselves as nuclear shell effects superimposed on a smooth hydrodynamical liquid-drop background and they lead to the intrinsic deformation in many nuclei [19].

In the related area of hadron and nuclear collisions, a relativistic hydrodynamical description of the collision process is a reasonable concept, as pioneered by the work of Landau [5] and supported by recent experimental findings [20–22]. Relativistic hydrodynamics has been applied to study the evolution of a quark-gluon matter in the work of Bjorken [6], Baym *et al.* [7], Ollitraut [8], and many others [9]. They led to successful investigations on the dynamics of matter with relativistic constituents in extreme conditions, as occurs in relativistic heavy-ion collisions [20, 23]. Wave mechanical description of quantum systems in terms of probability densities and probability currents contains the proper theoretical ingredients appropriate for continuum hydrodynamics. It will therefore be of interest to generalize previous formulation of hydrodynamics in the Schrödinger equation in Ref. [14] to Klein-Gordon and Dirac equations, in order to investigate how the wave mechanics of Klein-Gordon and Dirac particles may be used to provide the foundation for relativistic hydrodynamics of dense and compressed systems with relativistic constituents.

As the foundation of hydrodynamics is usually presented within the framework of the kinetic theory [24–27], it is instructive to investigate how the present quantum probability density approach and the kinetic theory approach are connected. We shall study how the equation of motion for the Wigner function, derived from the time-dependent Schrödinger equations representing the Klein-Gordon equation, can be related to the equation of motion for the phase space distribution function in the kinetic theory, in the classical weak-field and collisionless limit.

This paper is organized as follows. In Section II, we introduce the particle and antiparticle wave functions to represent the Klein-Gordon wave function and its time derivative. The Klein-Gordon equation is then separated into two coupled time-dependent Schrödinger equations. The particle and antiparticle probability densities obey equations of continuity containing additional terms, but the total net particle number remains a conserved quantity. In Section III, we examine the Euler equation for the motion of the probability fluid of a particle or an antiparticle. To provide insight into the dynamics, we specialize to the simplified case when the particle-antiparticle pair production can be suppressed. The motion of the probability fluids of particle and antiparticle obeys relativistic fluid dynamics equations, and the quantum stress tensor provides part of the source of the equation of state of relativistic matter. In Section IV, we examine the generalization to relativistic hydrodynamics for a simple many-body system in the mean-field description. In Section V, we derive the equation of motion for the Wigner function and compare it with the equation of motion for the distribution function in the kinetic theory. In Section VI, we present our summary and discussions.

II. SEPARATION OF KLEIN-GORDON EQUATION INTO PARTICLE AND ANTIPARTICLE COMPONENTS

In the present investigation, we are interested in the temporal evolution of a single-particle boson state ν with an initial wave function $\psi(\mathbf{r},t)$ at $t=0$, in external fields that consist of a scalar field $\mathcal{S}$ and a vector gauge field $(A^0,\mathbf{A})$ with a coupling constant e. The single-particle state is characterized by a (net) particle number n_ν that is a conserved quantity (see Eqs. (15) and (16) below). The particle number of a state is quantized; it has the value of $n_\nu=1$ when ν is a particle state and $n_\nu=-1$ when ν is an antiparticle state. The evolution of the state ν is described by the Klein-Gordon equation

$$(i\hbar\partial_t - eA^0)^2\psi = [(\frac{\hbar}{i}\nabla - e\mathbf{A})^2 + (m+\mathcal{S})^2]\psi. \tag{1}$$

To separate out the particle and antiparticle degrees of freedom, we introduce the energy function E and the auxiliary wave function ψ_4. For a given wave function ψ for the state ν with a charge $e_\nu = n_\nu e$, the energy function E is the positive root of the following quadratic equation of E,

$$\int d\mathbf{r}\ \ \psi^*\{(E-e_\nu A^0)^2 + [i\hbar\partial_t(E-e_\nu A^0)] - [(\frac{\hbar}{i}\nabla - e_\nu\mathbf{A})^2 + (m+\mathcal{S})^2]\}\psi = 0, \tag{2}$$

which is obtained by taking the scalar product of the wave function with the Klein-Gordon equation (1). We may not know $\partial_t E$ initially at $t=0$, but E and $\partial_t E$ can presumably be evaluated self-consistently and iteratively, when we succeed in getting the equations (Eq. (7) below) that allow us to advance the wave function to the next time step. Following Feshbach and Villars [4], we introduce the auxiliary wave function ψ_4 to represent the time-derivative of the wave function ψ,

$$(i\hbar\partial_t - eA^0)\psi = (E - e_\nu A^0)\psi_4. \tag{3}$$

Our definition of ψ_4 in the above equation differs from that of Feshbach and Villars [4], where the right-hand side is given as $m\psi_4$. The present definition exhibits better the relativistic properties of the momentum variables and facilitates the representation of the Klein-Gordon equation in relativistic hydrodynamics. The Klein-Gordon equation (1) becomes

$$(i\hbar\partial_t - eA^0)\psi_4 = \frac{1}{E - e_\nu A^0}\left\{(\frac{\hbar}{i}\nabla - e\mathbf{A})^2 + (m+\mathcal{S})^2]\psi - [i\hbar\partial_t(E - eA^0)]\psi_4\right\}. \tag{4}$$

We define the particle and antiparticle components $\chi_\pm$ of the state ν as linear combinations of ψ and ψ_4,

$$\chi_+ = \frac{1}{\sqrt{2}}[\psi + \psi_4] \tag{5}$$

$$\chi_- = \frac{1}{\sqrt{2}}[\psi^* - \psi_4^*] \quad \text{or} \quad \chi_-^* = \frac{1}{\sqrt{2}}[\psi - \psi_4]. \tag{6}$$

Our definition of χ_- in Eq. (6) contains a complex conjugation that is different from the definition of χ_- of Feshbach and Villars [4]. Such a modification allows one to obtain a Schrödinger equation for χ_- that contains the kinetic energy operator $(\mathbf{p} - e\mathbf{A})^2/2(E - e_\nu A^0)$, instead of a kinetic energy operator with the opposite sign in Feshbach and Villars [4]. In terms of the wave functions χ_+ and χ_- as defined in Eqs. (5) and (6), equations (3) and (4) become

$$\begin{aligned}(i\hbar\partial_t - e_\pm A^0)\chi_\pm &= \frac{1}{2(E - e_\nu A^0)}\left\{(\frac{\hbar}{i}\nabla - e_\pm\mathbf{A})^2 + (m+\mathcal{S})^2 + [(E - e_\nu A^0)^2 - i\hbar\partial_t(E - e_\nu A^0)]\right\}\chi_\pm \\ &+ \frac{1}{2(E - e_\nu A^0)}\left\{(\frac{\hbar}{i}\nabla - e_\pm\mathbf{A})^2 + (m+\mathcal{S})^2 - [(E - e_\nu A^0)^2 - i\hbar\partial_t(E - e_\nu A^0)]\right\}\chi_\mp^*,\end{aligned} \tag{7}$$

where $e_\pm = \pm e$. Thus we obtain the central result that the Klein-Gordon equation can be reduced to a set of two time-dependent Schrödinger equations in which the particle and the antiparticle appear as distinct types of particles with opposite charges $e_\pm$ with the particle wave function χ_+ and the antiparticle wave function χ_-. A general solution of the Klein-Gordon equation contains two components that can be represented by a column vector,

$$\Psi = \begin{pmatrix}\chi_+ \\ \chi_-\end{pmatrix}. \tag{8}$$

Both χ_+ and χ_- have positive norms, $|\chi_\pm|^2$, which can be interpreted as the probability densities of particles and antiparticles respectively, as in hydrodynamics.

Note that our denominator in Eqs. (7) differs from the denominator of m in the formulation of Feshbach and Villars [4]. The relativistic properties of the momentum variable are more apparent in Eq. (7). These relativistic properties will facilitate the subsequent representation of the Klein-Gordon equation in relativistic hydrodynamical form.

The second term of the new equation (7) represents the particle-antiparticle coupling and pair production. It is proportional to $((\hbar/i)\nabla - e_\pm\mathbf{A})^2 + (m+\mathcal{S})^2 - [(E - e_\nu A^0)^2 - i\hbar\partial_t(E - e_\nu A^0)]$ and involves essentially the deviation of E^2 from $\mathbf{p}^2 + m^2$ that increases with the strength of the interaction. The coupling is inversely proportional to the energy (or mass) of the particle. The particle-antiparticle coupling and the rate of pair production are large, when the strength of the interaction relative to the rest mass of the particle is large [10, 28, 29]. The particle-antiparticle coupling is small when the strength of the interaction relative to the energy (or mass) of the particle is small.

For stationary states $\chi_\pm$ with $e_\nu = n_\nu e = \pm e$ in static external fields, Eq. (7) gives

$$\left\{(E - e_\pm A^0)^2 - (\frac{\hbar}{i}\nabla - e_\pm\mathbf{A})^2 - (m+\mathcal{S})^2\right\}\chi_\pm = \left\{-(E - e_\pm A^0)^2 + (\frac{\hbar}{i}\nabla - e_\pm\mathbf{A})^2 + (m+\mathcal{S})^2\right\}\chi_\mp^*. \tag{9}$$

If the right-hand side representing the pair production coupling can be neglected, the above equation is just the time-independent Klein-Gordon equation for a stationary state of a particle (or an antiparticle) in external fields. Equation (7) has the correct limit for stationary states in static external fields in the absence of pair productions.

In a static case with time-independent external fields, the energy function E is a constant of motion, although the wave functions $\chi_\pm$ and the associated probability fluids in the static fields can still evolve dynamically in space-time. With time-dependent external fields, the energy of the single-particle state changes because energy may be supplied or removed by the external fields. We envisage conceptually that a two-component wave function $\Psi(\mathbf{r},t) = (\chi_+(\mathbf{r},t), \chi_-(\mathbf{r},t))$ for the state ν with $e_\nu = n_\nu e$ is initially known as $t = 0$. The knowledge of the initial wave functions allows us to obtain E from Eq. (2) and to construct χ_+ and χ_- at the next time step with the help of Eq. (5) and (6). In the initial evaluation of E and in the subsequent stepwise increment of the wave function in time, we need the quantity $\partial_t E(t)$, which may be self-consistently determined by an iterative procedure.

After the wave function Ψ has been advanced to the next time step, we can evaluate $E(t)$ using Eq. (2) that is a quadratic equation in E. It contains both a positive-E and a negative-E solution. We shall limit our attention only on the positive-E solution so that we can speak of both particle and antiparticles states of positive energies. This stepwise increment allows the determination of the evolution of the state $\Psi(\mathbf{r},t)$ and $E(t)$ as a function of time.

To see how the probability fluids behave in space and time, we write the wave functions $\chi_\pm$ in terms of the amplitude and phase functions,

$$\chi_\pm(\mathbf{r},t)=\phi_\pm(\mathbf{r},t)e^{iS_\pm(\mathbf{r},t)/\hbar-i\Omega(t)}. \tag{10}$$

We construct $\chi_\pm^*\times(7)$-$\chi_\pm\times(7)^*$. After some manipulations, we find

$$\partial_t[(E-e_\nu A^0)\phi_\pm^2]+\nabla\cdot[\phi_\pm^2(\nabla S_\pm-e_\pm\mathbf{A})]=X_\pm, \tag{11}$$

where

$$\begin{aligned}2X_\pm &= \{\chi_\pm^*(\frac{\hbar}{i}\nabla-e_\pm\mathbf{A})^2\chi_\mp^*-\chi_\pm(\frac{\hbar}{-i}\nabla-e_\pm\mathbf{A})^2\chi_\mp\}\\ &\quad+[(m+\mathcal{S})^2+(E-e_\nu A)^2](\chi_\pm^*\chi_\mp^*-\chi_\pm\chi_\mp)\\ &\quad+[i\hbar\partial_t(E-e_\nu A^0)](\chi_\pm^*\chi_\mp^*+\chi_\pm\chi_\mp).\end{aligned} \tag{12}$$

As the quantities X_+ and X_- are not generally a full divergence, the total number of particles and antiparticles in the two components are not conserved. However, the difference of the particle number and antiparticle numbers of the two components satisfies the equation

$$\partial_t[(E-e_\nu A^0)(\phi_+^2-\phi_-^2)]+\nabla\cdot[\phi_+^2(\nabla S_+-e_+\mathbf{A})]-\nabla\cdot[\phi_-^2(\nabla S_--e_-\mathbf{A})]=X_+-X_-. \tag{13}$$

But, X_+-X_- is a complete divergence,

$$\begin{aligned}X_+-X_- = &-\nabla\cdot(\chi_+^*\nabla\chi_-^*-\chi_-^*\nabla\chi_+^*)/2+\nabla\cdot(\chi_+\nabla\chi_--\chi_-\nabla\chi_+)/2\\ &-\nabla\cdot[e_+A^0(\chi_+^*\chi_-^*+\chi_+\chi_-)/i].\end{aligned} \tag{14}$$

Therefore, the quantity

$$n_{\rm particle}=\int d\mathbf{r}\frac{E-e_\nu A^0}{m}(\phi_+^2-\phi_-^2)=\int d\mathbf{r}\frac{E-e_\nu A^0}{m}(|\chi_+|^2-|\chi_-|^2) \tag{15}$$

is a conserved quantity because the right-hand side of the equation (14) is a complete divergence. The additional number of particles produced is equal to the additional number of antiparticles produced. The equal increase in particle and antiparticle numbers associated with X_+ and X_- represents the occurrence of particle-antiparticle pair production.

A single-particle solution with a $n_{\rm particle}=n_\nu=\pm1$ is one in which $|\chi_\pm|^2\gg|\chi_\mp|^2$ and can be normalized to be

$$\int d^3r\frac{E-e_\nu A^0}{m}[|\chi_\pm|^2-|\chi_\mp|^2]=1 \text{ for a particle state with } n_{\rm particle}=\pm1. \tag{16}$$

III. KLEIN-GORDON EQUATION IN HYDRODYNAMICAL FORM

To obtain the Euler equation, we construct $\chi_\pm^*\times(7)$+$\chi_\pm\times(7)^*$. Putting all terms together, we get

$$\begin{aligned}&\phi_\pm^2(-2\partial_tS_\pm-2e_\pm A^0)\\ &=\frac{1}{2(E-e_\nu A^0)}\Big\{-[2\phi_\pm\nabla^2\phi_\pm-2\phi_\pm^2(\nabla S_\pm-e_\pm\mathbf{A})^2]\\ &\quad+[(m+\mathcal{S})^2+(E-e_\nu A^0)^2]2\phi_\pm^2\\ &\quad+\chi_\pm^*(\frac{\hbar}{i}\nabla-e_\pm\mathbf{A})^2\chi_\mp^*+\chi_\pm(\frac{\hbar}{-i}\nabla-e_\pm\mathbf{A})^2\chi_\mp\\ &\quad+[(m+\mathcal{S})^2-(E-e_\nu A)^2](\chi_\pm^*\chi_\mp^*+\chi_\pm\chi_\mp)\\ &\quad+[i\hbar\partial_t(E-e_\nu A^0)](\chi_\pm^*\chi_\mp^*-\chi_\pm\chi_\mp)\Big\}.\end{aligned} \tag{17}$$

In the present hydrodynamical description, we investigate the evolution of the probability densities of particles and antiparticles in a single-particle system with a definite particle number n_ν in external fields. The rate of the particle-antiparticle pair production depends on the strength of the external interaction relative to the particle rest mass. In the case of a strong interaction and large pair production probabilities, a hydrodynamical description will be complicated as it will involve a component that describes the pair production process.

A hydrodynamical description will be appropriate after the active pair production stage has passed and the expansion of the system is driven by a slowly varying external field or a mean field. It is this type of motion for which we wish to provide a hydrodynamical description. This is the case when the interaction is present but not large compared to the rest mass, so that the probability of pair production is small, and can be suppressed in the lowest-order approximation. This is equivalent to the case of a "simple fluid" in relativistic hydrodynamics, in which the chemical composition of the fluid ceases to change [30]. In such circumstances, one can speak of a single-particle system with a definite particle number n_ν, which can take on $n_\nu = 1, -1$ values. We shall now consider this simplifying case with suppressed pair production by discarding the last four terms in the curly bracket on the right hand-side of Eq. (17). After dividing by $-2\phi_\pm^2$, the equation for the phase function $S_\pm$ for this simplified case is

$$\begin{aligned}(\partial_t S_\pm + e_\pm A^0) &= \frac{1}{2(E - e_\nu A^0)}\Big\{[(\nabla^2\phi_\pm)/\phi_\pm - (\nabla S_\pm - e_\pm \mathbf{A})^2] \\ &\quad -(m+\mathcal{S})^2 - (E - e_\nu A^0)^2\Big\}. \end{aligned} \tag{18}$$

For this case with suppressed pair production, $e_\nu = n_\nu e = e_\pm$. We take the gradient ∇_i of the above for $i = 1, 2, 3$, and multiply by $\phi_\pm^2(E - e_\pm A^0)$. We obtain

$$\begin{aligned}&(E - e_\pm A^0)\phi_\pm^2 \partial_t(\nabla_i S_\pm - e_\pm A^i) \\ &= \Big\{\phi_\pm^2 \nabla_i[(\nabla^2\phi_\pm)/2\phi_\pm] - \sum_{j=1}^3 \phi_\pm^2(\nabla_j S_\pm - e_\pm A^j)\nabla_j(\nabla_i S_\pm - e_\pm A^i) \\ &\quad -(m+\mathcal{S})\nabla_i\mathcal{S} - \sum_{j=1}^3 \phi_\pm^2(\nabla_j S_\pm - e_\pm A^j)e_\pm F^{ij}\Big\} - (E - e_\pm A^0)\phi_\pm^2 e_\pm F^{0i} \\ &\quad +\frac{e_\pm \nabla_i A^0 \phi_\pm^2}{2(E - e_\pm A^0)}\Big\{(\nabla^2\phi_\pm)/\phi_\pm - (\nabla S_\pm - e_\pm \mathbf{A})^2] - (m+\mathcal{S})^2 + (E - e_\pm A^0)^2\Big\}. \end{aligned} \tag{19}$$

Using the equation of continuity for this simplified case without pair production, we obtain

$$\begin{aligned}&\partial_t\left[\frac{(E - e_\pm A^0)\phi_\pm^2(\nabla_i S_\pm - e_\pm A^i)}{m+\mathcal{S}}\right] + \sum_{j=1}^3 \nabla_j\left[\frac{\phi_\pm^2(\nabla_j S_\pm - e_\pm A^j)(\nabla_i S_\pm - e_\pm A^i)}{m+\mathcal{S}}\right] \\ &= -\frac{m}{m+\mathcal{S}}\sum_{j=1}^3 \nabla_j p_{ij}^{(q)} - \phi_\pm^2\nabla_i\mathcal{S} + \frac{1}{m+\mathcal{S}}\Big\{-(E - e_\pm A^0)\phi_\pm^2 e_\pm F^{0i} - \sum_{j=1}^3 \phi_\pm^2(\nabla_j S_\pm - e_\pm A^j)e_\pm F^{ij}\Big\} \\ &\quad +\frac{e_\pm \nabla_i A^0 \phi_\pm^2}{2(E - e_\pm A^0)(m+\mathcal{S})}\Big\{(\nabla^2\phi_\pm)/\phi_\pm - (\nabla S_\pm - e_\pm \mathbf{A})^2] - (m+\mathcal{S})^2 + (E - e_\pm A^0)^2\Big\} \\ &\quad -(E - e_\pm A^0)\phi_\pm^2(\nabla_i S_\pm - e_\pm A^i)\frac{\partial_t\mathcal{S}}{(m+\mathcal{S})^2} - \sum_{j=1}^3\left[\frac{\phi_\pm^2(\nabla_j S_\pm - e_\pm A^j)(\nabla_i S_\pm - e_\pm A^i)}{m+\mathcal{S}}\right]\frac{\nabla_j\mathcal{S}}{(m+\mathcal{S})^2}. \end{aligned} \tag{20}$$

We can identity the fluid energy density $\epsilon_\pm$ as

$$\epsilon_\pm = (m+\mathcal{S})\phi_\pm^2, \tag{21}$$

as it corresponds to the fluid energy density for the fluid element at rest. The fluid element is characterized by a relativistic 4-velocity u^μ. We can identify

$$\begin{aligned} u_\pm^0 &= \frac{E - e_\pm A^0}{m+\mathcal{S}} \\ u_\pm^i &= \frac{\nabla_i S_\pm - e_\pm A^i}{m+\mathcal{S}}, \qquad \text{for } i = 1, 2, 3, \end{aligned} \tag{22}$$

which obeys $(u^0_\pm)^2 - (\mathbf{u}_\pm)^2 = 1$ in the absence of the quantum effects. We can then write an equation of motion for the probability densities in terms of the hydrodynamical equation

$$
\begin{aligned}
&\partial_t(\epsilon_\pm u^0_\pm u^i_\pm) + \sum_{j=1}^{3} \nabla_j \epsilon_\pm u^i_\pm u^j_\pm + \frac{m}{m+\mathcal{S}} \sum_{j=1}^{3} \nabla_j p^{(q)}_{\pm ij} \\
= &-\phi^2_\pm \nabla_i \mathcal{S} + \frac{1}{m+\mathcal{S}} \left\{ -(E - e_\pm A^0)\phi^2_\pm e_\pm F^{0i} - \sum_{j=1}^{3} \phi^2_\pm (\nabla_j S_\pm - e_\pm A^j) e_\pm F^{ij} \right\} \\
&+ \frac{e_\pm \nabla_i A^0 \phi^2_\pm}{2(E - e_\pm A^0)(m+\mathcal{S})} \left\{ (\nabla^2 \phi_\pm)/\phi_\pm - (\nabla S_\pm - e_\pm \mathbf{A})^2] - (m+\mathcal{S})^2 + (E - e_\pm A^0)^2 \right\} \\
&-(E - e_\pm A^0)\phi^2_\pm (\nabla_i S_\pm - e_\pm A^i) \frac{\partial_t \mathcal{S}}{(m+\mathcal{S})^2} - \sum_{j=1}^{3} \left[\frac{\phi^2_\pm (\nabla_j S_\pm - e_\pm A^j)(\nabla_i S_\pm - e_\pm A^i)}{m+\mathcal{S}} \right] \frac{\nabla_j \mathcal{S}}{(m+\mathcal{S})^2},
\end{aligned}
\tag{23}
$$

where $i, j = 1, 2, 3$. This is the Klein-Gordon equation for the particle and antiparticle probability densities in hydrodynamical form. The first two terms on the left-hand side correspond to $\partial_\mu T^{\mu i}_\pm$, with the energy momentum tensor of the probability fluid $T^{\mu i}_\pm = \epsilon_\pm u^\mu_\pm u^i_\pm$, for $\mu = 0, 1, 2, 3$. The third term on the left-hand side is the quantum stress tensor arising from the spatial variation of the amplitude of the single-particle wave function [14],

$$
p^{(q)}_{\pm\, ij} = -\frac{\hbar^2}{4m} \nabla^2 \phi^2_\pm \delta_{ij} + \frac{\hbar^2}{m} \nabla_i \phi_\pm \nabla_j \phi_\pm . \tag{24}
$$

In Eq. (23) there is no thermal stress tensor contribution in Eq. (23) for a single-particle. As indicated in [14], a thermal stress tensor $p^{(t)}_{ij}$ will however arise from different velocity fields when there are many particles in a many-body system. The first two terms on the right-hand side contain forces coming from the scalar interaction, the electric field F^{0i} and the magnetic field F^{ij}, as in classical formulations. The third term on the right-hand side is the relativistic correction to the time-like part of the vector interaction, and the last two terms represent relativistic corrections associated with the spatial and temporal variation of the scalar interaction. Thus the dynamics of the probability fluid obeys an equation similar to the hydrodynamical equation, with forces on fluid elements arising from what one expects in classical considerations. The additional element is the presence of the quantum stress tensor $p^{(q)}_{ij}$ that is proportional to $\hbar^2$ and arises from the quantum nature of the fluid.

While we have examined the hydrodynamical form when the particle-antiparticle pair production has been suppressed, the particle-antiparticle pair production can be included in future studies, at the expenses of increasing greatly the complexity of the simple picture we have obtained.

IV. APPLICATION TO A MANY-BODY SYSTEM IN THE MEAN-FIELD DESCRIPTION

The external mean field we have been studying can come internally from the single-particle state probability density as in the Gross-Pitaevskii equation [31–34] or from the self-consistent mean-field in a many-body system [14, 15, 35]. A many-body system in the time-dependent mean-field description consists of a collection of independent particles moving in self-consistent mean-fields generated by all other particles [14, 15, 35]. Each single-particle state $\psi_{a\nu}$ is characterized by a state label a, particle type ν, energy e_ν, and occupation number $n_{a\nu}$. For simplicity, we consider the case in which the mean-field potential arises from a scalar two-body interaction $v_s(\mathbf{r}_1, \mathbf{r}_2)$ and a time-like vector interaction $v_0(\mathbf{r}_1, \mathbf{r}_2)$. We further neglect the last three terms on the right-hand side of Eq. (23) which represent higher-order relativistic corrections. The equation of motion for the energy density $\epsilon_{a\nu}$ and velocity fields $u^i_{a\nu}$ for $i = 1, 2, 3$ and $\nu = \pm$, in the single particle state a and particle type ν, is then

$$
\begin{aligned}
&\partial_t(\epsilon_{a\nu} u^0_{a\nu} u^i_{a\nu}) + \sum_{j=1}^{3} \nabla_j \epsilon_{a\nu} u^i_{a\nu} u^j_{a\nu} + \frac{m}{m+\mathcal{S}} \sum_{j=1}^{3} \nabla_j p^{(q)}_{(a\nu)ij} \\
= &-\phi^2_{a\nu} \nabla_i \mathcal{S} + \frac{E - e_{a\nu} A^0_\pm}{m+\mathcal{S}} \phi^2_{a\nu} e_{a\nu} \frac{\partial A^0}{\partial x^i},
\end{aligned}
\tag{25}
$$

where, in the frame with the fluid element at rest,

$$
\mathcal{S}(\mathbf{r}, t) = \int d^3\mathbf{r}_2 \; n(\mathbf{r_2}, t) v_s(\mathbf{r}, \mathbf{r}_2), \tag{26}
$$

$$A^0(\mathbf{r},t) = \int d^3\mathbf{r}_2 \left\{ n_+(\mathbf{r_2},t)e_+ + n_-(\mathbf{r_2},t)e_- \right\} v_0(\mathbf{r},\mathbf{r}_2), \tag{27}$$

$$n_\nu = \sum_a n_{a\nu}\phi_{a\nu}^2, \quad \text{and} \quad n = n_+ + n_-. \tag{28}$$

We consider a strongly interacting system in which the number of particles and antiparticles are equal so that $n_+(\mathbf{r_2}) = n_-(\mathbf{r_2})$ and $n_+(\mathbf{r_2})e_+ + n_-(\mathbf{r_2})e_-$ is zero. Then the contribution from the second term on the right-hand side of Eq. (25) is zero. Multiplying Eq. (25) by $n_{a\nu}$ and summing over $\{a,\nu\}$, we get

$$\begin{aligned}&\partial_t(\sum_{a\nu} n_{a\nu}\epsilon_{a\nu}u^0_{a\nu}u^i_{a\nu}) + \sum_{j=1}^3 \nabla_j(\sum_{a\nu} n_{a\nu}\epsilon_{a\nu}u^i_{a\nu}u^j_{a\nu}) \\ &+\frac{m}{m+\mathcal{S}}\sum_{j=1}^3 \nabla_j(\sum_{a\nu} n_{a\nu}p^{(q)}_{(a\nu)ij}) + (\sum_{a\nu} n_{a\nu}\phi^2_{a\nu})\nabla_i\mathcal{S} = 0.\end{aligned} \tag{29}$$

We define the total energy density ϵ by

$$\sum_{a\nu} n_{a\nu}\epsilon_{a\nu} = \epsilon, \tag{30}$$

and the average 4-velocity u by

$$u = \sum_{a\nu} n_{a\nu}\epsilon_{a\nu}u_{a\nu}/\epsilon. \tag{31}$$

We can introduce the thermal stress tensor $p_{ij}^{(t)}$ for $\{i,j\} = 1,2,3$ as the correlation of the deviations of the single-particle velocity fields from the average

$$\sum_{a\nu} n_{a\nu}\epsilon_{a\nu}(u^i_{a\nu} - u^i)(u^j_{a\nu} - u^j) \equiv p_{ij}^{(t)}. \tag{32}$$

For the case with the suppression of pair production, we obtained the equation of hydrodynamics

$$\partial_t(\epsilon u^0 u^i) + \sum_{j=1}^3 \left\{ \nabla_j \left(\epsilon u^i u^j + p_{ij}^{(t)} + p_{ij}^{(v)} \right) + \frac{m}{m+\mathcal{S}}\nabla_j p_{ij}^{(q)} \right\} = 0, \tag{33}$$

where the total quantum stress tensor is

$$p_{ij}^{(q)} = -\frac{\hbar^2}{4m}\nabla^2 \sum_{a\nu} n_{a\nu}\phi^2_{a\nu}\delta_{ij} + \frac{\hbar^2}{m}\sum_{a\nu} n_{a\nu}\nabla_i\phi_{a\nu}\nabla_j\phi_{a\nu}, \tag{34}$$

and the pressure due to the interaction $p_{ij}^{(v)}$ is

$$\frac{\partial}{\partial x^j}p_{ij}^{(v)}(\mathbf{r},t) = n(\mathbf{r},t)\nabla_i\mathcal{S}(\mathbf{r},t) = n(\mathbf{r},t)\frac{\partial}{\partial x^j}\int d^3\mathbf{r}_2 n(\mathbf{r}_2,t)v_s(\mathbf{r},\mathbf{r}_2). \tag{35}$$

The mean-field stress tensor $p_{ij}^{(v)}$ can also be given as

$$p_{ij}^{(v)} = \left\{ n\frac{\partial(W^{(v)}n)}{\partial n} - W^{(v)}n \right\}\delta_{ij}, \tag{36}$$

where $W^{(v)}$ is the energy per particle arising from the mean-field interaction. As an illustrative example, we can consider a density-dependent two-body interaction

$$v_s(\mathbf{r},\mathbf{r}_2) = [a_2 + a_3 n((\mathbf{r}+\mathbf{r}_2)/2)]\delta(\mathbf{r}-\mathbf{r}_2). \tag{37}$$

The contribution of the mean-field interaction to the stress tensor is then

$$p_{ij}^{(v)} = \frac{1}{2}[a_2 + 2a_3 n(\mathbf{r})]n^2\delta_{ij}, \tag{38}$$

whose magnitude increases with the density and the strengths of the interaction.

The quantum stress tensor $p_{ij}^{(q)}$ and the thermal stress tensor $p_{ij}^{(t)}$ can take on different values, depending on the occupation numbers $n_{a\nu}$ of the single-particle states that determine the degree of thermal equilibrium of the system. The quantum stress tensor depends on the amplitudes of the wave functions while the thermal stress tensor depends on the phases of the wave functions and the deviation of the velocity fields from the mean velocities. The quantum stress tensor is less sensitive to the degree of thermalization as compared to the thermal stress tensor. In the time-dependent mean-field description, the motion of each particle state can be individually followed [35]. The occupation numbers $n_{a\nu}$ of the single-particle states will remain unchanged, if there are no additional residual interaction between the single particles due to residual interactions. When particle residual interactions are allowed as in the extended time-dependent mean-field approximation [36], the occupation numbers will change and will approach an equilibrium distribution as time proceeds.

We note that the total pressure arises from many sources. We come to the observation that in situations when $|p_{ij}^{(q)} + p_{ij}^{(v)}| \gg p_{ij}^{(t)}$ for a strongly-coupled system at low and moderate temperatures, there can be situations when the system behaves quasi-hydrodynamically, even though the state of the system has not yet reached thermal equilibrium. In this case, the hydrodynamical state is maintained mainly by the quantum stress tensor and the strong mean fields.

What we have discussed in this Section is only a theoretical framework that exhibits clearly the different sources of stress tensors. To study specifically the dynamics of the quark-gluon plasma, for example, it will be necessary to investigation the specific nature of different constituents and their interactions. Nevertheless, the general roles played by the different components of stress tensors can still be a useful reminder on the importance of the quantum and mean-field stress tensors in the strongly-coupled regime, at temperature just above the transition temperature T_c.

V. CONNECTION TO THE KINETIC THEORY

The foundation of hydrodynamics is usually presented within the framework of kinetic theory, in which particles and antiparticles are described as distinct constituents and their interactions are weak [24–27]. In such a description, particles are considered to be approximately on-the-mass-shell, and their inter-particle collisions lead to thermalization. The state of the system for particles or antiparticles of type ν is described by a distribution function $f_\nu(\mathbf{r},\mathbf{p};t)$ in phase space. Successive gradient expansions of small deviations from the equilibrium distribution lead to various approximations of the transport coefficients. The time-dependencies of the moments of various kinematic operators lead to hydrodynamical equations. The near-mass-shell condition restricts its application to systems with weak interactions that can be taken as perturbations in quantum field theory.

For a dense and strongly interacting system, such as a nucleus or a strongly-coupled quark-gluon plasma, a reasonable description of a non-equilibrium system can be formulated in a different approach, the quantum probability density approach considered here. In this approach, constituents of the quantum system move in the strong mean fields generated by all other particles. Each particle executes its single-particle motion in the time-dependent mean field, and the residue interaction between particles lead to "collisions" that change the single-particle state occupation numbers $n_{a\nu}$. These residual-interaction collisions occur in such a way that the single particle occupation numbers approaches a thermal distribution as a function of time [36].

In the quantum probability density approach, as the particles resides in a strong field, they are off-the-mass shell and their energies depend on their local potential. The dynamics of each single particle state is described by a wave function with a positive probability density and a probability current. The equations for the total probability density and probability current are analogous to the relativistic hydrodynamical equations. The stress tensors in such a description arises from many components: the quantum stress tensor, the thermal stress tensor, and the mean-field interaction stress tensor. The behavior of the dynamics in such a quantum description need not be the same as classical hydrodynamics because the constitutive equations relating various stress components with densities and other attributes may respond differently to the evolving dynamics [38]. We mention earlier that in situations in which the quantum stress tensor and interaction stress tensor dominate over the thermal stress tensor, the degree of thermalization may not be important in the hydrodynamical behavior of the system. In the other extreme when the stress tensor arises predominantly from the thermal stress tensor, the dynamics will then depend on the degree of thermal equilibrium.

Although the quantum probability density approach is particularly appropriate for cases of strongly interacting quantum systems, it also has a well-defined classical and weak-coupling limit that should coincide with the kinetic theory approach. It is therefore instructive to investigate how the two approaches are connected by studying how the equation of motion for the Wigner function, derived from the time-dependent Schrödinger equations representing the Klein-Gordon equation, can be related to the equation of motion for the phase space distribution function in the kinetic theory, in the classical weak-field and collisionless limit.

Both in the present quantum probability density approach and the kinetic theory approach, the application to relativistic hydrodynamics are confronted with the problem of pair production in which the interactions will lead to spontaneous production of particle-antiparticle pairs [28, 29]. The standard hydrodynamics is one in which pair-production probability is suppressed. The equation of motion of the single-particle state of particle type ν is then obtained from Eq. (7) by neglecting the second term on the right-hand side. We further consider the motion to be sufficiently slow so that we can neglect the term $\partial_t(E-e_\nu A^0)$. The equation of motion for the single-particle state is then

$$(i\hbar\partial_t - e_\nu A^0)\chi_\nu \;=\; \frac{1}{2(E-e_\nu A^0)}\left\{(\frac{\hbar}{i}\nabla - e_\nu \mathbf{A})^2 + (m+\mathcal{S})^2 + [(E-e_\nu A^0)^2]\right\}\chi_\nu, \tag{39}$$

which can be written as

$$i\hbar\partial_t\chi_\nu \;=\; \frac{1}{2(E-e_\nu A^0)}\left\{(\frac{\hbar}{i}\nabla - e_\nu \mathbf{A})^2\right\}\chi_\nu + V(r)\chi_\nu, \tag{40}$$

where

$$V(r) \;=\; \left\{\frac{(m+\mathcal{S})^2}{2(E-e_\nu A^0)} + \frac{E+e_\nu A^0}{2}\right\}. \tag{41}$$

From this time-dependent Schrödinger equation, we wish to obtain the corresponding equation of motion for the single-particle Wigner function. The construction of a gauge invariant Wigner function from the density matrix in the presence of an external gauge field $(A^0, \mathbf{A})$ has been examined by many authors [39–48], and we can follow similar procedures. We construct $\chi_\nu^\dagger(\mathbf{r}_1,t)\chi_\nu(\mathbf{r}_2,t)$ and introduce $(\mathbf{r}_1+\mathbf{r}_2)/2=\mathbf{r}$, and $\mathbf{r}_1-\mathbf{r}_2=\mathbf{s}$. We define the three-dimensional gauge-invariant single-particle Wigner function for particle type ν as

$$f_\nu(\mathbf{r},\mathbf{p};t) \;=\; \int d\mathbf{s}e^{i\mathbf{p}\cdot\mathbf{s}}\{\mathcal{F}_I\}^{-1}\chi_\nu^\dagger(\mathbf{r}+\mathbf{s}/2,t)\chi_\nu(\mathbf{r}-\mathbf{s}/2,t) \tag{42}$$

where the momentum $\mathbf{p}$ represents the kinetic momentum [43, 44], and $\mathcal{F}_I$ is the well-known gauge factor introduced first by Schwinger [37],

$$\mathcal{F}_I \;=\; \exp\{-ie_\nu\int_{\mathbf{r}-\mathbf{s}/2}^{\mathbf{r}+\mathbf{s}/2}\mathbf{A}(\mathbf{s}')\cdot d\mathbf{s}'\}. \tag{43}$$

The inverse transform of Eq. (42) is then

$$\chi_\nu^\dagger(\mathbf{r}+\mathbf{s}/2,t)\chi_\nu(\mathbf{r}-\mathbf{s}/2,t) = \int\frac{d\mathbf{p}}{2\pi\hbar^3}e^{-i\mathbf{p}\cdot\mathbf{s}}\exp\{-ie_\nu\int_{\mathbf{r}-\mathbf{s}/2}^{\mathbf{r}+\mathbf{s}/2}\mathbf{A}(\mathbf{s}')\cdot d\mathbf{s}'\}f_\nu(\mathbf{r},\mathbf{p};t). \tag{44}$$

From the Schrödinger equation (39), we construct $\chi_\nu^\dagger(\mathbf{r}_1,t)i\hbar\partial_t\chi_\nu(\mathbf{r}_2,t)$ and $\chi_\nu(\mathbf{r}_2,t)[-i\hbar\partial_t]\chi_\nu^\dagger(\mathbf{r}_1,t)$, subtract the two, and we get

$$\begin{aligned} i\hbar\partial_t[\chi_\nu^\dagger(\mathbf{r}_1,t)\chi_\nu(\mathbf{r}_2,t)] \;=\; &\Bigg[\left\{\frac{1}{2[E-e_\nu A^0(\mathbf{r}_2)]}\left(\frac{\hbar}{i}\nabla_{\mathbf{r}_2} - e_\nu\mathbf{A}(\mathbf{r}_2)\right)^2 + V(\mathbf{r}_2)\right\} \\ &-\left\{\frac{1}{2[E-e_\nu A^0(\mathbf{r}_1)]}\left(\frac{\hbar}{-i}\nabla_{\mathbf{r}_1} - e_\nu\mathbf{A}(\mathbf{r}_1)\right)^2 + V(\mathbf{r}_1)\right\}\Bigg]\chi_\nu^\dagger(\mathbf{r}_1,t)\chi_\nu(\mathbf{r}_2,t). \end{aligned} \tag{45}$$

The equation of motion for the Wigner function $f_\nu(\mathbf{r},\mathbf{p})$ can be obtained by substituting Eq. (44) into the above equation. The terms involving $V(\mathbf{r})$ give the result

$$\Big[V(\mathbf{r}_2)-V(\mathbf{r}_1)\Big]\chi_\nu^\dagger(\mathbf{r}+\mathbf{s}/2,t)\chi_\nu(\mathbf{r}-\mathbf{s}/2,t) = \int\frac{d\mathbf{p}}{2\pi\hbar^3}e^{-i\mathbf{p}\cdot\mathbf{s}}\mathcal{F}_I\frac{2}{\hbar}\sin\left\{\frac{\hbar}{2}\nabla_{\mathbf{p}}^f\cdot\nabla_{\mathbf{r}}^V\right\}V(\mathbf{r})f_\nu(\mathbf{r},\mathbf{p};t). \tag{46}$$

where $\nabla_{\mathbf{p}}^f$ applies only on f_ν and $\nabla_{\mathbf{r}}^V$ applies only on $V(\mathbf{r})$. This indicates that the quantum equation of motion for the Wigner function contains transcendental functions of the operators $\hbar\nabla_{\mathbf{p}}\cdot\nabla_{\mathbf{r}}$ applying on the potential and the

Wigner function. The expansion of the sine function will lead to a power series in $\hbar$. We see here that the equation of motion for the distribution function in kinetic theory corresponds only the lowest order approximation in such an expansion. We note also that the Wigner function is in general not identical to the distribution function because it can take on negative values [49, 50]. In the classical limit, the Wigner function can be confined to be positive and be identified with the phase space distribution function. To make connections with the classical kinetic theory approach, we shall take this limit of $\hbar \to 0$. The above equation is then approximated as

$$\left[V(\mathbf{r}_2) - V(\mathbf{r}_1)\right]\chi_\nu^\dagger(\mathbf{r}+\mathbf{s}/2,t)\chi_\nu(\mathbf{r}-\mathbf{s}/2,t) = \int \frac{d\mathbf{p}}{2\pi\hbar^3} e^{-i\mathbf{p}\cdot\mathbf{s}} \mathcal{F}_I[\nabla_\mathbf{r} V(\mathbf{r})]\cdot\nabla_\mathbf{p} f_\nu(\mathbf{r},\mathbf{p};t). \tag{47}$$

We shall take the weak-field limit so that we keep only terms first order in the external fields. Then the quantity $\nabla_\mathbf{r} V(\mathbf{r})$ is

$$\nabla_\mathbf{r} V(r) \;=\; \left\{\frac{(m+\mathcal{S})\nabla\mathcal{S}}{(E-e_\nu A^0)} + \frac{(m+\mathcal{S})^2 e_\nu \nabla A^0}{2(E-e_\nu A^0)^2} + \frac{\nabla e_\nu A^0}{2} + ..\right\}, \tag{48}$$

which goes to $\nabla\mathcal{S} + \nabla e_\nu A^0$ in the non-relativistic limit of $E \to m$.

To evaluate the other terms, we note that the variation of the gauge factor $\mathcal{F}_I$ arising from the variation of the end point $\mathbf{r}_2$ is given by

$$\delta_{\mathbf{r}_2} e^{-ie_\nu \int_{\mathbf{r}_2}^{\mathbf{r}_1} \mathbf{A}(\mathbf{s}')\cdot d\mathbf{s}'} = -ie^{-ie_\nu \int_{\mathbf{r}_2}^{\mathbf{r}_1} \mathbf{A}(\mathbf{s}')\cdot d\mathbf{s}'} \, e_\nu \delta_{\mathbf{r}_2}[\int_{\mathbf{r}_2}^{\mathbf{r}_1} \mathbf{A}(\mathbf{s}')\cdot d\mathbf{s}']. \tag{49}$$

The difference of the path integrals when one of the end points is varied from $\mathbf{r}_2$ to $\mathbf{r}_2 + \delta\mathbf{r}_2$ can be turned into a loop integral by noting that

$$\begin{aligned}
\delta_{\mathbf{r}_2}[\int_{\mathbf{r}_2}^{\mathbf{r}_1} \mathbf{A}(\mathbf{s}')\cdot d\mathbf{s}'] &= \left[\int_{\mathbf{r}_2+\delta\mathbf{r}_2}^{\mathbf{r}_1} - \int_{\mathbf{r}_2}^{\mathbf{r}_1}\right] \mathbf{A}(\mathbf{s}')\cdot d\mathbf{s}' \\
&= \left[\oint_{\mathbf{r}_2\to\mathbf{r}_2+\delta\mathbf{r}_2\to\mathbf{r}_1\to\mathbf{r}_2} - \int_{\mathbf{r}_2}^{\mathbf{r}_2+\delta\mathbf{r}_2}\right] \mathbf{A}(\mathbf{s}')\cdot d\mathbf{s}' \\
&= \left[\oint_{\mathbf{r}_2\to\mathbf{r}_2+\delta\mathbf{r}_2\to\mathbf{r}_1\to\mathbf{r}_2} \mathbf{A}(\mathbf{s}')\cdot d\mathbf{s}'\right] - \mathbf{A}(\mathbf{r}_2)\cdot\delta\mathbf{r}_2.
\end{aligned} \tag{50}$$

By Stokes' theorem, we can carry out the loop integral over a triangular area encircled by the loop and we have

$$\begin{aligned}
\oint_{\mathbf{r}_2\to\mathbf{r}_2+\delta\mathbf{r}_2\to\mathbf{r}_1\to\mathbf{r}_2} \mathbf{A}(\mathbf{s}')\cdot d\mathbf{s}' &= \int_{\text{area encircled by loop}} \nabla\times\mathbf{A}\cdot[\delta\mathbf{r}_2\times d\mathbf{s}'] \\
&\sim \mathbf{B}(\mathbf{r})\cdot\frac{[\delta\mathbf{r}_2\times\mathbf{s}]}{2} \sim \frac{1}{2}\delta\mathbf{r}_2\cdot[\mathbf{s}\times\mathbf{B}(\mathbf{r})].
\end{aligned} \tag{51}$$

Here a Taylor expansion of the $\mathbf{B}(\mathbf{r}+\mathbf{s}')$ field $\mathbf{B}(\mathbf{r}+\mathbf{s}')$ in powers of $\mathbf{s}'$ in the loop integral will lead to terms in power of $\hbar$ [43, 44]. Taking the $\mathbf{B}(\mathbf{r})$ field to be located at $\mathbf{r}$ in Eq. (51) represents the lowest-order $\hbar \to 0$ approximation in the expansion of $\mathbf{B}(\mathbf{r}+\mathbf{s}')$. As $\delta_{\mathbf{r}_2}\phi = \nabla_{\mathbf{r}_2}\phi\cdot\delta\mathbf{r}_2$, we obtain

$$\nabla_{\mathbf{r}_2} e^{-ie_\nu \int_{\mathbf{r}_2}^{\mathbf{r}_1} \mathbf{A}(\mathbf{s}')\cdot d\mathbf{s}'} \;=\; e^{-ie_\nu \int_{\mathbf{r}_2}^{\mathbf{r}_1} \mathbf{A}(\mathbf{s}')\cdot d\mathbf{s}'} i \left\{ e_\nu\mathbf{A}(\mathbf{r}_2) - \frac{1}{2}[\mathbf{s}\times e_\nu\mathbf{B}(\mathbf{r})]\right\}, \tag{52}$$

and

$$\left(\frac{\hbar}{i}\nabla_{\mathbf{r}_2} - e_\nu\mathbf{A}(\mathbf{r}_2)\right) e^{-ie_\nu \int_{\mathbf{r}_2}^{\mathbf{r}_1} \mathbf{A}(\mathbf{s}')\cdot d\mathbf{s}'} \;=\; e^{-ie_\nu \int_{\mathbf{r}_2}^{\mathbf{r}_1} \mathbf{A}(\mathbf{s}')\cdot d\mathbf{s}'} \left\{-\frac{1}{2}[\mathbf{s}\times e_\nu\mathbf{B}(\mathbf{r})]\right\}. \tag{53}$$

Therefore, we obtain

$$\begin{aligned}
&\left\{(\frac{\hbar}{i}\nabla_{\mathbf{r}_2} - e_\nu\mathbf{A}(\mathbf{r}_2))^2 - \{(\frac{\hbar}{-i}\nabla_{\mathbf{r}_1} - e_\nu\mathbf{A}(\mathbf{r}_2))^2\right\}\left[e^{-ie_\nu \int_{\mathbf{r}_2}^{\mathbf{r}_1} \mathbf{A}(\mathbf{s}')\cdot d\mathbf{s}'} e^{-i\mathbf{p}\cdot\mathbf{s}} f_\nu(\mathbf{r},\mathbf{p})\right] \\
= &\left\{e^{-ie_\nu \int_{\mathbf{r}_2}^{\mathbf{r}_1} \mathbf{A}(\mathbf{s}')\cdot d\mathbf{s}'}\right\}\left\{\left[\frac{\hbar}{i}\nabla_{\mathbf{r}_2} - \frac{1}{2}\mathbf{s}\times e_\nu\mathbf{B}(\mathbf{r})\right]^2 - \left[\frac{\hbar}{-i}\nabla_{\mathbf{r}_1} + \frac{1}{2}\mathbf{s}\times e_\nu\mathbf{B}(\mathbf{r})\right]^2\right\} e^{-i\mathbf{p}\cdot\mathbf{s}} f_\nu(\mathbf{r},\mathbf{p})
\end{aligned} \tag{54}$$

After some manipulation for the remaining terms, we get

$$\int \frac{d\mathbf{p}}{2\pi\hbar^3}\mathcal{F}_I e^{-i\mathbf{p}\cdot\mathbf{s}}\Big\{i\hbar\partial_t f_\nu(\mathbf{r},\mathbf{p}) + i\frac{\mathbf{p}}{E}\cdot\nabla_{\mathbf{r}} f_\nu(\mathbf{r},\mathbf{p}) + i(-\nabla_{\mathbf{r}}V + \frac{\mathbf{p}}{E}\times e_\nu\mathbf{B}(\mathbf{r}))\cdot\nabla_{\mathbf{p}} f_\nu(\mathbf{r},\mathbf{p})\Big\} = 0, \tag{55}$$

which is satisfied if

$$\partial_t f_\nu(\mathbf{r},\mathbf{p}) + \frac{\mathbf{p}}{E}\cdot\nabla_{\mathbf{r}} f_\nu(\mathbf{r},\mathbf{p}) + (-\nabla_{\mathbf{r}}V + \frac{\mathbf{p}}{E}\times e_\nu\mathbf{B}(\mathbf{r}))\cdot\nabla_{\mathbf{p}} f_\nu(\mathbf{r},\mathbf{p}) = 0. \tag{56}$$

This is just the equation of motion for the distribution function in kinetic theory for a collisionless fluid in weak fields [51].

Equation (56) is the equation of motion for the Wigner function of a single-particle state ν in external scalar and gauge fields. For a many-body system with different states a and particle types ν, the Wigner function is

$$f(\mathbf{r},\mathbf{p}) = \sum_{a\nu} n_{a\nu} f_{a\nu}(\mathbf{r},\mathbf{p}). \tag{57}$$

The equation of motion for the total distribution function $f(\mathbf{r},\mathbf{p})$ in the collisionless limit will be in the same form as Eq. (56) with $f_\nu(\mathbf{r},\mathbf{p})$ in Eq. (56) replaced by $f(\mathbf{r},\mathbf{p})$. The presence of other particles in different single-particle states allows one use the mean-fields as the external fields and to introduce the collision term by considering residual interactions between particles in different $a\nu$ states.

The exercise in this Section indicates that although the quantum probability density approach is particularly appropriate for cases of strongly interacting quantum systems, it also has a well-defined classical weak-coupling limit that coincides with the kinetic theory, from which hydrodynamics equations can also be formulated.

VI. SUMMARY AND DISCUSSIONS

We have generalized the formulation of Feshbach and Villars to write the Klein-Gordon equation as a set of two coupled time-dependent Schrödinger equations, for the particle and antiparticles components of the wave function. We have improved upon the formulation of Feshbach and Villars in this re-examination. In our new set of coupled time-dependent Schrödinger equations (7) the particle and the antiparticle components are better separated, and the kinetic energy operator in the equation for the antiparticle component is in the proper form of $(\mathbf{p}-e\mathbf{A})^2/2(E-e_\nu A^0)$, with a positive sign. The relativistic properties of the momentum variables in the coupled time-dependent Schrödinger equations are more apparent and their connection to relativistic hydrodynamics can be easily established.

We introduce amplitude and phase functions to cast the time-dependent Schrödinger equations into hydrodynamical form. We find that the equation of motion of the probability fluid of Klein-Gordon particles or antiparticles can be written in the form of relativistic hydrodynamics, with an additional quantum stress tensor. The other components of the hydrodynamical equation have their classical counterparts.

For simplicity, we have suppressed the pair-production degree of freedom in the present investigation. The pair production is however a quantum phenomenon that can be studied by using the Klein-Gordon equation, as was carried out in Ref. [29] to examine the Schwinger mechanism. The presence of the pair-production mechanism is a new feature in the hydrodynamical evolution. Since energy and momentum is diverted into pair production, the pair-production corresponds to a dissipative process, and will contribute to the viscosity of the fluid. Future investigations to include this pair-production in relativistic hydrodynamics will be of great interest.

As both the Schrödinger equation and the Klein-Gordon equation can be cast into a hydrodynamical form, one may inquire whether the Dirac equation can also be written in hydrodynamical form. It is well known that the Dirac equation can be reduced to a Klein-Gordon equation, with additional terms involving the spin and particle-antiparticle degrees of freedom. For a Dirac particle in an external field we have

$$\{\gamma^\nu(i\partial_\nu - eA_\nu) - (m+\mathcal{S})\}\psi = 0. \tag{58}$$

Upon multiplying this on the left by $\gamma^\nu(i\partial_\nu - eA_\nu) + (m+\mathcal{S})$, we obtain

$$\{(i\partial_\nu - eA_\nu)^2 - (m+\mathcal{S})^2 - i\alpha\cdot e\mathbf{E} + \sigma\cdot e\mathbf{B}(\mathbf{r}) - i[\gamma^\nu\partial_\nu\mathcal{S}]\}\psi = 0, \tag{59}$$

which is the Klein-Gordon equation with additional interactions. Thus, the Dirac equation can be likewise cast into a hydrodynamical form, following the procedures outlined in the present manuscript.

We have thus far discussed particles and antiparticles with opposite charges interacting in a gauge field. For the case of neutral particles, there is no interaction with the gauge field characterized by the charges $e_\pm$. However, relativistic doubling of states occurs and there are particles and antiparticles. How these neutral particles should be treated will depend on their interaction with the external field. If the interactions of the neutral particle and antiparticle with the external fields are identical, it will not be necessary to distinguish between a particle and an antiparticle. It then becomes possible to construct a simplified theory in which only one degree of freedom enters (say, the particle's), with a neutral antiparticle taken to be identical to its corresponding neutral particle. On the other hand, if their interactions with the external fields are different, the two degrees of freedom are distinct. One can then introduce additional quantum numbers to distinguish particles and antiparticles, and the present investigation containing different charges (or quantum numbers) for the new types of interaction will apply.

In the dynamics of a single-particle state in an external field, the external field can come internally from the single-particle state probability density as in the Gross-Pitaevskii equation [31–34] or from the self-consistent mean-field in a many-body system [14, 15]. A many-body system consists of a collection of various particles in their individual single-particle states. For example, we can examine a many-body system in the time-dependent Hartree approximation in which particle and antiparticles move in a time-dependent self-consistent mean-field generated by all other particles. The motion of each particle can be individually followed, as in the time-dependent mean-field approximation in a nuclear system [35]. The occupation numbers of these single-particle states will remain unchanged, if there are no additional residual interaction between the single particles. When particle residual interactions are allowed as in the extended time-dependent mean-field approximation [36], the occupation numbers will change and will approach an equilibrium distribution. As the fluid density is cumulative in nature the dynamics of the many-body system will rely on the cumulative effects from particles in individual states. Thus, there will be a total quantum stress tensor that is an important part of the equation of state of the many-body system. There will also be contributions from the mean fields to the equation of states of the relativistic fluid as in the nuclear fluid [15]. The present investigation of Klein-Gordon single-particle states should be useful in the study of the dynamics of a many-body system using relativistic hydrodynamics.

The foundation of hydrodynamics is usually presented within the framework of kinetic theory, we can compare the present quantum probability density approach to the usual kinetic theory approach by determining the equation of motion for the Wigner function using the time-dependent Schrodinger equation of the Klein-Gordon equation. Our comparison indicates that the equation of motion for the distribution function in an external field in the kinetic theory can be obtained from the quantum probability density approach in the limit of $\hbar \to 0$ and weak fields. Thus, the quantum probability approach extends the range of hydrodynamics applications to situations where quantum effects and/or strong interactions are important.

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