The Odd-Parity CMB Bispectrum

Marc Kamionkowski[1] and Tarun Souradeep[2]
[1] *California Institute of Technology, Mail Code 350-17, Pasadena, CA 91125 and*
[2] *Inter-University Centre for Astronomy and Astrophysics, Pune 411007, India*
(Dated: October 22, 2010)

Measurement of the cosmic microwave background (CMB) bispectrum, or three-point correlation function, has now become one of the principle efforts in early-Universe cosmology. Here we show that there is a odd-parity component of the CMB bispectrum that has been hitherto unexplored. We argue that odd-parity temperature-polarization bispectra can arise, in principle, through weak lensing of the CMB by chiral gravitational waves or through cosmological birefringence, although the signals will be small even in the best-case scenarios. Measurement of these bispectra requires only modest modifications to the usual data-analysis algorithms. They may be useful as a consistency test in searches for the usual bispectrum and to search for surprises in the data.

PACS numbers:

I. INTRODUCTION

The simplest single-field slow-roll (SFSR) inflationary models assumed in the now-standard cosmological model predict departures from Gaussianity to be undetectably small [1]. Yet no theorist believes these models to be the entire story, and many beyond-SFSR models predict departures from Gaussianity to be larger [2] and possibly detectable with current or forthcoming CMB experiments. Still, the variety of beyond-SFSR models, and the heterogeneity of their non-Gaussian predictions, is huge, and no consensus exists on the likely form of beyond-SFSR physics. Given the centrality of this question for physics, however, it is important to leave no stone unturned; no prospective signal easily obtainable with existing data, no matter how likely or unlikely, should be overlooked.

The principle effort in the search for non-Gaussianity is measurement of the cosmic microwave background (CMB) bispectrum [3, 4], the three-point correlation function in harmonic space. Given the small bispectrum signals anticipated, the full bispectrum is not measured. Rather, a specific model is compared against the data to constrain the non-Gaussian amplitude in that particular model. The working-horse model for such analyses has been the local model [1, 4], but the bispectra associated with a variety other models [5] have also been considered.

The purpose of this paper is to point out that there is an entirely different class of bispectra that have been hitherto unexplored. All bispectrum analyses that have been done so far assume the bispectrum to be even-parity. There is, however, an entirely different class of bispectra that are odd-parity. Although a odd-parity temperature bispectrum cannot arise from a projection of a three-dimensional density bispectrum, odd-parity temperature and temperature-polarization bispectra can arise, at least in principle, from lensing by gravitational waves or from cosmological birefringence.

Although these signals are small—perhaps unobservably so—they may be worth pursuing for at least two reasons: (1) The analyses required to determine the standard (even-parity) bispectrum amplitudes are complicated. For example, Ref. [6] claimed evidence for a non-Gaussian signal in WMAP data, in disagreement with other null searches [7]. Modifications of the standard analyses to include measurement of odd-parity bispectra should be simple and straightforward, and they thus provide, with the expectation of a vanishing signal, a valuable null test, and thus consistency check, for the standard searches. And (2) there may be new parity-violating physics, beyond what we have envisioned here, that might give rise to such signals. It is with these motivations that we now explore odd-parity bispectra.

To begin, recall that a CMB experiment provides a measurement of the temperature $T(\hat{n})$ as a function of position $\hat{n}$ on the sky. The temperature can be re-written in terms of spherical-harmonic coefficients $a_{lm} = \int d^2\hat{n}\, T(\hat{n}) Y_{lm}^*(\hat{n})$. The rotationally-invariant CMB power spectrum is $C_l = \left\langle |a_{lm}|^2 \right\rangle$, where the angle brackets denote an average over all realizations. The bispectrum is given by

$$B_{l_1 l_2 l_3}^{m_1 m_2 m_3} \equiv \left\langle a_{l_1 m_1} a_{l_2 m_2} a_{l_3 m_3} \right\rangle . \tag{1}$$

Here we always choose $l_1 \leq l_2 \leq l_3$, in contrast to most of the literature, which assumes the bispectrum to be symmetric in l_1, l_2, l_3, as symmetrization wipes out the inherently antisymmetric signals we consider here. The rotationally-invariant, or angle-averaged, bispectrum is

$$B_{l_1 l_2 l_3} \equiv \sum_{m_1 m_2 m_3} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} B_{l_1 l_2 l_3}^{m_1 m_2 m_3}, \tag{2}$$

where the quantities in parentheses are Wigner-3j symbols. The bispectrum must satisfy the triangle conditions and selection rules, $m_1 + m_2 + m_3 = 0$ and $|l_i - l_j| \leq l_k \leq |l_i + l_j|$.

The third condition usually assumed in the CMB bispectrum literature is $l_1 + l_2 + l_3 =$even. This has nothing to do with the restrictions of angular-momentum addition encoded in the Clebsch-Gordan coefficients. Indeed, one can add, for example, two angular-momentum states with quantum numbers $l_2 = 4$ and $l_3 = 5$ to obtain a total-angular-momentum state with $l_1 = 2$. The restriction $l_1+l_2+l_3$ =even is a consequence of the assumption of parity invariance. Since the a_{lm} have parity $(-1)^l$, the bispectrum will have odd parity unless $l_1+l_2+l_3$=even.

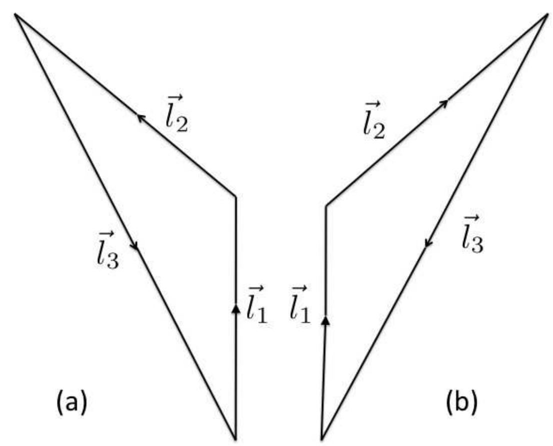

FIG. 1: Here we plot two Fourier triangles with $l_1 < l_2 < l_3$ on a small patch of sky. The two have opposite handedness: in (a) the cross product $\vec{l}_1 \times \vec{l}_2$ comes out of the page, while in (b) the cross product goes into the page. The even-parity bispectrum (that with $l_1 + l_2 + l_3$ =even) weights both of these triangles similarly. The odd-parity bispectrum (configurations with $l_1+l_2+l_3$=odd) takes on different signs for the two different triangles.

The distinction between odd- and even-parity configurations can be understood heuristically for multipole moments $l_i \gg 1$. On a patch of sky sufficiently small to be approximated as flat, the three (l_i, m_i) modes then become three plane waves with wavevectors $\vec{l}_1, \vec{l}_2, \vec{l}_3$. The conditions imposed on (l_i, m_i) by the Clebsch-Gordan coefficients then become a restriction $\vec{l}_1 + \vec{l}_2 + \vec{l}_3 = 0$. The bispectrum then depends on the product of three Fourier coefficients $T_{\vec{l}_i}$ for configurations in which the three wavevectors sum to zero. Two examples of such triangles are shown in Fig. 1, where we have labeled the triangle sides such that $l_1 < l_2 < l_3$. The two triangles are mirror images of each other. An even-parity bispectrum (that with $l_1 + l_2 + l_3$ =even) is the same for both of these triangles. An odd-parity bispectrum (configurations with $l_1 + l_2 + l_3$=odd) takes on different signs for the two different triangles.

Interestingly enough, an odd-parity CMB bispectrum cannot arise as a projection of a parity-violating density, or potential, bispectrum, as the distinction between right- and left-handed triangles does not exist in three spatial dimensions. To see this, note that triangle (a) in Fig. 1 is the same as triangle (b) if we look at it from the other side of the page. In other words, in two spatial dimensions, we can construct a scalar $(\vec{l}_1 \cdot \vec{l}_2)$ from two vectors and also a pseudoscalar $(\vec{l}_1 \times \vec{l}_2)$. However, in three spatial dimensions, we can only construct the scalar $\vec{l}_1 \cdot \vec{l}_2$ from two vectors. The three-dimensional spatial bispectrum therefore has no odd-parity configurations. Thus, the condition $l_1 + l_2 + l_3$ =even on the bispectrum follows simply if we assume that the CMB map is a projection of a three-dimensional scalar field.

Still, a parity-violating CMB temperature bispectrum might alternatively arise, for example, if there is a bispectrum for tensor perturbations (gravitational waves); in this case, the polarization of one of the the gravitational waves may provide an additional vector with which to construct parity-violating correlations. Lensing by gravitational waves provides a specific example. A gravitational wave produces a lensing pattern that couples two large-l moments a_{lm} due to density perturbations, but these two are then correlated with the low-l moment a_{lm} due to the gravitational wave itself [8]. This is thus effectively a three-point correlation, and if the gravitational-wave background is chiral—if there is an asymmetry in the amplitude of right- versus left-handed gravitational waves—then the bispectrum may be parity violating [9].

Other examples can be obtained for three-point correlations that involve the CMB polarization, as well as the temperature. The polarization map is described in terms of spherical-harmonic coefficients a^E_{lm} and a^B_{lm} for the gradient (E mode) and curl (B mode) components of the polarization [10], in addition to the temperature coefficients, which we now call a^T_{lm}. The parity of the T and E coefficients are $(-1)^l$, while the parity of the B coefficients are $(-1)^{l+1}$. There are now ten three-point correlations that can be considered (TTT, TTE, TTB, TEE, TBB, TEB, EEE, EEB, EBB, and BBB), and there are even-parity and odd-parity parts for each, the parity being determined by $(-1)^{k+\sum_i l_i}$, where k is the number of B-mode coefficients [11]. For example, $\langle a^T_{l_1 m_1} a^E_{l_2 m_2} a^B_{l_3 m_3} \rangle$ has even parity for $l_1 + l_2 + l_3$ =odd and odd parity for $l_1 + l_2 + l_3$ =even.

Suppose that there are no gravitational waves and thus no B modes at the surface of last scatter. Density perturbations will still induce temperature fluctuations and E modes of the polarization. If there is a nonzero three-dimensional bispectrum, for example, of the local-model form, then there will be even-parity temperature-polarization bispectra induced; i.e., there will be TTT, TTE, TEE, and EEE bispectra with $l_1 + l_2 + l_3$=even. Now suppose that there is a quintessence field ϕ that couples to the pseudoscalar of electromagnetism through a Lagrangian term $(\phi/M_*)F\tilde{F}$, where F and $\tilde{F}$ are the electromagnetic-field-strength tensor and its dual, respectively [12]. The time evolution of ϕ then leads to a rotation, by an angle $\alpha = (\Delta\phi)/M_*$, of the linear polarization of each CMB photon as it propagates from the surface of last scatter [13]. This rotation then converts some of the E mode into a B mode [14]. If $\alpha \ll 1$, then these induced B-mode spherical-harmonic coefficients are $a^B_{lm} \simeq 2\alpha a^E_{lm}$. This thus induces, to linear order in α, TTB, TEB, and EEB bispectra with $l_1 + l_2 + l_3$=even. But since the parity of the a^B_{lm} coefficients is opposite to those of the a^T_{lm} or a^E_{lm}, the induced TTB, TEB, and

EEB bispectra are parity odd. Of course, cosmological birefringence will also induce parity-violating TB and EB power spectra. Current constraints [7, 15] to the rotation angle α from these power spectra, combined with current constraints to the spatial bispectrum, guarantee that odd-parity bispectra induced by cosmological birefringence should be small.

Now that we have discussed some physical mechanisms that might induce odd-parity bispectra, we now discuss measurement of these signals. Implementation of steps in the data analysis to extract these odd-parity bispectra should be straightforward once the analysis pipeline for obtaining the even-parity bispectra are in place. We illustrate with the temperature bispectrum. It is convenient to work with the reduced bispectrum, $b_{l_1l_2l_3} \equiv B_{l_1l_2l_3}/G_{l_1l_2l_3}$, where

$$G_{l_1l_2l_3} \equiv \sqrt{\frac{(2l_1+1)(2l_2+1)(2l_3+1)}{4\pi}} \begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix}. \quad (3)$$

These reduced bispectra, for a given combination of $l_1+l_2+l_3$=even, can be estimated from the map from

$$\widehat{b_{l_1l_2l_3}} = G^{-1}_{l_1l_2l_3} \sum_{m_1m_2m_3} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} a_{l_1m_1}a_{l_2m_2}a_{l_3m_3}, \quad (4)$$

with variance $\left\langle \left(\widehat{b_{l_1l_2l_3}}\right)^2 \right\rangle = (G_{l_1l_2l_3})^{-2}$.

Since measurement of each $b_{l_1l_2l_3}$ will be extremely noisy, one generally assumes a particular model for the bispectrum and then estimates the parameter that quantifies the non-Gaussianity. For example, in the local model [4], $b_{l_1l_2l_3} = 6f_{\rm nl}(C_{l_1}C_{l_2} + \text{perm})$, where $f_{\rm nl}$ is the non-Gaussianity parameter. The minimum-variance estimator for $f_{\rm nl}$ is then

$$\begin{aligned} \widehat{f_{\rm nl}} &= \sigma^2_{f_{\rm nl}} \sum_{l_1<l_2<l_3} \frac{6G_{l_1l_2l_3}(C_{l_1}C_{l_2}+\text{perms})}{C^m_{l_1}C^m_{l_2}C^m_{l_3}} \\ &\quad \times \sum_{m_1m_2m_3} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} a_{l_1m_1}a_{l_2m_2}a_{l_3m_3}, \end{aligned} \quad (5)$$

where C^m_l is the power spectrum for the map (including noise), and

$$\sigma^{-2}_{f_{\rm nl}} = \sum_{l_1<l_2<l_3} \frac{[6G_{l_l l_2l_3}(C_{l_1}C_{l_2}+\text{perms})]^2}{C^m_{l_1}C^m_{l_2}C^m_{l_3}}, \quad (6)$$

is the inverse variance to $\widehat{f_{\rm nl}}$. Note that we have approximated and simplified by restricting $l_1 < l_2 < l_3$, and note further that the sums in Eqs. (5) and (6) extend only over multipole moments $l_1+l_2+l_3$=even.

Measurement of the odd-parity bispectrum is similar, except that we now sum over configurations with $l_1+l_2+l_3$ =odd. The only subtlety is that the factors $G_{l_1l_2l_3}$ vanish for $l_1+l_2+l_3$ = odd. To remedy this situation, we use identities of Wigner-3j symbols [16] to redefine

$$\begin{aligned} G_{l_1l_2l_3} &\equiv \frac{\sqrt{l_3(l_3+1)l_2(l_2+1)}}{[l_1(l_1+1)-l_2(l_2+1)-l_3(l_3+1)]} \\ &\times \sqrt{\frac{(2l_1+1)(2l_2+1)(2l_3+1)}{4\pi}} \begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & -1 & 1 \end{pmatrix}. \end{aligned} \quad (7)$$

This matches Eq. (3) for $l_1+l_2+l_3$=even, but remains non-zero otherwise. The definition in Eq. (7) is actually what appears in the bispectrum induced by weak lensing by chiral gravitational waves [9].

With this replacement, one can then define, for example, an estimator for an odd-parity bispectrum with a given l dependence (e.g., the local-model form) through

$$\begin{aligned} \widehat{f_{\rm nl}}^{\rm odd} &= \sigma^2_{f_{\rm nl}} \sum_{l_1<l_2<l_3} \frac{6G_{l_1l_2l_3}(C_{l_1}C_{l_2}+\text{perms})}{C^m_{l_1}C^m_{l_2}C^m_{l_3}} \\ &\times \sum_{m_1m_2m_3} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} a_{l_1m_1}a_{l_2m_2}a_{l_3m_3}, \end{aligned} \quad (8)$$

where now the sum is over $l_1+l_2+l_3$=odd. The variance to this estimator is again given by Eq. (6), but now summing over $l_1+l_2+l_3$=odd, and it should be numerically comparable to the even-parity variance. Implementation of steps to measure $\widehat{f_{\rm nl}}^{\rm odd}$ in an analysis routine that measures $\widehat{f_{\rm nl}}$ should be simple and straightforward.

Some further insight can be gained by considering the form of estimators for the amplitude of an odd-parity bispectrum in the flat-sky limit. We illustrate with a parity-breaking extension of the local model. As discussed above, the bispectrum, usually written as a function $B(l_1,l_2,l_3)$ of the three wavevector magnitudes, can alternatively be written, taking $l_1 < l_2 < l_3$, as a function $B(\vec{l}_1,\vec{l}_2)$ of the two shortest wavevectors. The usual local model can then be generalized to

$$B(\vec{l}_1,\vec{l}_2) = 2\left[f_{\rm nl} + f^{\rm odd}_{\rm nl}\frac{\vec{l}_1\times\vec{l}_2}{l_1l_2}\right](C_{l_1}C_{l_2}+\text{perms}), \quad (9)$$

where $f^{\rm odd}_{\rm nl}$ is an odd-parity non-Gaussian amplitude. The minimum-variance estimator for the usual $f_{\rm nl}$ can then be written in terms of a sum (see, e.g., Ref. [17]),

$$\widehat{f_{\rm nl}} \propto \sum \frac{T_{\vec{l}_1}T_{\vec{l}_2}T_{\vec{l}_3}6(C_{l_1}C_{l_2}+\text{perms})}{C^m_{l_1}C^m_{l_2}C^m_{l_3}}, \quad (10)$$

over all triangles $\vec{l}_1+\vec{l}_2+\vec{l}_3$ with $l_1 < l_2 < l_3$. The minimum-variance estimator for the odd-parity amplitude $f^{\rm odd}_{\rm nl}$ can then be written analogously as

$$\widehat{f^{\rm odd}_{\rm nl}} \propto \sum \frac{T_{\vec{l}_1}T_{\vec{l}_2}T_{\vec{l}_3}6(C_{l_1}C_{l_2}+\text{perms})}{C^m_{l_1}C^m_{l_2}C^m_{l_3}}\frac{\vec{l}_1\times\vec{l}_2}{l_1l_2}, \quad (11)$$

over the same triangles. In other words, it is the same as the usual estimator except that it differences, rather than sums, triangles of different handedness. Thus, the odd-parity-bispectrum estimator is a null test for the usual even-parity bispectrum.

To summarize, we have shown that there is a broad class of odd-parity CMB temperature-polarization bispectra that have been hitherto overlooked but that can be easily measured with the data. We provided two examples of cosmological physics that could, in principle at least, produce nonvanishing odd-parity bispectra. Realistically, though, the bispectra in these examples will probably be too small to be observed. Still, measurement of these odd-parity three-point correlations should be pursued. They may provide a valuable consistency test for the complicated analyses employed to measure the usual bispectrum amplitude, a null test for bispectrum measurements analogous to measurements of the curl [19] in weak-lensing analyses. And who knows? Maybe there is new parity-violating physics we have not yet foreseen that might give rise to such signals. Detection of such a cosmological signal would, needless to say, be remarkable.

MK thanks the support of the Miller Institute for Basic Research in Science at the University of California, Berkeley, where part of this work was completed. MK was supported at Caltech by DoE DE-FG03-92-ER40701, NASA NNX10AD04G, and the Gordon and Betty Moore Foundation. TS acknowledges support from Swarnajayanti grant, DST, India and the visit to Caltech during which the work was initiated.

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