On A Cosmological Invariant as an Observational Probe in the Early Universe

Debashis Gangopadhyay*
S. N. Bose National Centre for Basic Sciences, JD Block, Sector-III, Salt Lake, Kolkata 700 098, India.

Somnath Mukherjee†
Dharsa Mihirlal Khan Institution[H.S], P.O:-New G.I.P Colony, Dist:-Howrah-711112, India.

k-essence scalar field models are usuually taken to have lagrangians of the form $\mathcal{L} = -V(\phi)F(X)$ with F some general function of $X = \nabla_\mu\phi\nabla^\mu\phi$. Under certain conditions this lagrangian in the context of the early universe can take the form of that of an oscillator with time dependent frequency. The Ermakov invariant for a time dependent oscillator in a cosmological scenario then leads to an invariant quadratic form involving the Hubble parameter and the logarithm of the scale factor. In principle, this invariant can lead to further observational probes for the early universe. Moreover, if such an invariant can be observationally verified then the presence of dark energy will also be indirectly confirmed.

PACS numbers: 98.80.Cq

1.Introduction

The motivation for this work lies in the existence of an invariant related to the time dependent oscillator, first obtained by Ermakov [1],[2],[3]. In the context of the $k-$essence lagrangian [4], the logarithm of the scale factor in a homogeneous universe at very early times after the big bang satisfies the equations of motion of an oscillator with time-dependent frequency [5]. The classical solutions of this theory are fully consistent with the inflationary scenario and a radiation dominated universe. A measure of temperature fluctuations can also be estimated using standard prescriptions [5]. In this work the focus will be on another interesting aspect of the time dependent oscillator ,*viz.*, the existence of invariants or first integrals of motion [1]. Here we show that as the $k-$ essence lagrangian takes the form of that of a time dependent oscillator,the invariant has cosmological analogues–in the classical as well as a quantum context. Classically one can construct an invariant quadratic form involving the Hubble parameter and the logarithm of the scale factor. Quantum expectation values of a function containing the scale factor and Hubble parameter can also be obtained. The quantum aspects will be discussed in subsequent publications.In this work we will limit ourselves to the classical aspects. Existence of this invariant implies possibilities of further observational probes in the early universe.

First a brief review is in order. Scherrer [6] showed that it is possible to unify the dark matter and dark energy components into a single scalar field model with the scalar field ϕ having a non-canonical kinetic term. These scalar fields are the $k-$essence fields which first appeared in models of inflation [7] and subsequently led to models of dark energy also [8]. The general form of such lagrangians is some function $F(X)$ with $X = \nabla_\mu\phi\nabla^\mu\phi$, and do not depend explicitly on ϕ to start with. In [6], a scaling relation was obtained ,*viz.* $X(\frac{dF}{dX})^2 = Ca(t)^{-6}$, C a constant (similar expression was also derived in [9]). [4] incorporates the scaling relation of [6] and in [5] it is shown how this lagrangian can be approximated to that of a time dependent oscillator for small scale factors in a certain epoch of the early universe. Literature on dark matter, dark energy and $k-$ essence can be found in [10].

The lagrangian L (or the pressure p) is taken as

$$\mathcal{L} = -V(\phi)F(X) \tag{1}$$

The energy density is

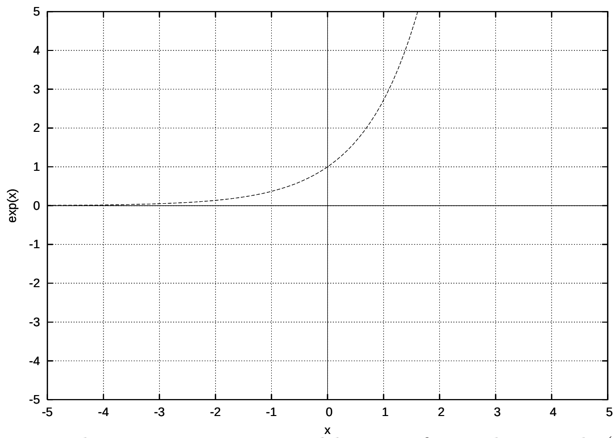
$$\rho = V(\phi)[F(X) - 2XF_X] \tag{2}$$

with $F_X \equiv \frac{dF}{dX}$. In this work, the scalar potential $V(\phi) = V$ is a (*positive*) constant and all the time variables $t \equiv t/t_0$, where t_0 is the present epoch and we are interested only in $t < 1$ scenarios. Also $a(t_1) < a(t_2)$ for $t_1 < t_2$ etc.

Using the scaling law and the zero-zero component of Einstein's field equations an expression for the lagrangian is obtained as follows.Take the Robertson-Walker (RW) metric : $ds^2 = c^2dt^2 - a^2(t)[\frac{dr^2}{(1-kr^2)} + r^2(d\theta^2 + sin^2\theta d\phi^2)]$. where $k(= 0, 1 \; or -1)$ is the curvature constant. The zero-zero component of Einstein's equation reads: $R_{00} - \frac{1}{2}g_{00}R = -\kappa T_{00}$. This gives with the RW metric $\frac{k}{a^2} + H^2 = \frac{8\pi G}{3}\rho$. For $k = 0$, and a homogeneous and isotropic spacetime (i.e. $\phi(t, \mathbf{x}) = \phi(\mathbf{t})$) (1) becomes

$$\mathcal{L} = -c_1\dot{q}^2 - c_2\dot{\phi}e^{-3q} \tag{3}$$

with $q(t) = ln\ a(t)$, $c_1 = 3(8\pi G)^{-1}$, $c_2 = 2V\sqrt{C}$, and two generalised coordinates $q(t)$ and $\phi(t)$. (3) has a kinetic term for q and an interaction term. There is no kinetic term for ϕ.



Note that $a = e^q$. It is readily seen from the graph (in the figure $x \equiv q$) of the exponential function that in the region $-1 < q < 0$ one has , $a = e^q < 1$. Hence in this region q small (i.e. $|q| < 1$) means a is also small (i.e $a < 1$). Moreover, in this region a grows from $e^{-1} = 0.367879$ to $e^0 = 1$. So within this region a grows as q grows. So smaller values of q mean that we are going back to smaller values of a i.e. to earlier epochs. *Throughout this work we will restrict ourselves to this domain i.e.* $-1 < q < 0$. *In this domain* a *is small when* q *is small and* $|a| < 1$ *for* $|q| < 1$. So expand the exponential in (3), keep terms upto $O(q^2)$ and replace q by $q + \frac{1}{3}$ to get [5]

$$\mathcal{L} = -\frac{M}{2}[\dot{q}^2 + 12\pi G g(t) q^2] - (\frac{1}{2})g(t) \tag{4}$$

where $M = \frac{3}{4\pi G}$, $g(t) = 2\sqrt{C}V\dot{\phi}$, and we use $c = 1$ (c is speed of light). Put $12\pi G g(t) = -\Omega^2(t)$. This means

$$\phi(t) = -\frac{1}{24\pi G\sqrt{C}V}\int dt\Omega^2(t) \tag{5}$$

(4) now becomes

$$\mathcal{L} = -\frac{M}{2}[\dot{q}^2 - \Omega^2(t)q^2] - (\frac{1}{2})g(t) \tag{6}$$

Now, the term $\frac{1}{2}g(t)$ is a total time derivative and thus has no contribution to the equations of motion and hence ignorable. Then(6) becomes

$$\mathcal{L} = -\frac{M}{2}[\dot{q}^2 - \Omega^2(t)q^2] \tag{7}$$

Ignoring the overall negative sign in all subsequent discussions, we have a time dependent oscillator for $q(t) = ln\ a(t)$.

2. Ermakov Invariant in a cosmological context

The Hamiltonian $\mathcal{H}$ corresponding to (7) is

$$\mathcal{H} = \frac{M}{2}[p^2 + \Omega^2(t)q^2] = \frac{M}{2}[H^2 + \Omega^2(t)q^2] \tag{8}$$

where $p = \dot{q} = \frac{\dot{a}}{a} = H$, H is the Hubble parameter. Following Ermakov [1], [2] , one can immediately write down the invariant I as

$$I = \frac{1}{2}[\rho^{-2}q^2 + \left(\rho H - \frac{1}{M}\dot{\rho}q\right)^2] \tag{9}$$

where $\rho(t)$ satisfies Ermakov's equation

$$\frac{1}{M^2}\ddot{\rho} + \Omega^2\rho - \rho^{-3} = 0 \tag{10}$$

Putting in the values of M and simplifying , one gets

$$I = \mathcal{A}(t)(ln\ a(t))^2 - \mathcal{B}(t)(ln\ a(t))H(t) + \mathcal{C}(t)H^2 \tag{11}$$

with

$$\mathcal{A}(t) = \frac{\rho^{-2}(t)}{2} + \frac{32\pi^2 G^2}{9}(\dot{\rho}(t))^2,$$
$$\mathcal{B}(t) = \frac{8\pi G\rho(t)\dot{\rho}(t)}{3},$$
$$\mathcal{C}(t) = \frac{\rho^2(t)}{2} \tag{12}$$

I is an invariant for the Hamiltonian $\mathcal{H}$ in the sense:

$$\frac{dI}{dt} = \frac{\partial I}{\partial t} + [I, \mathcal{H}]_{\mathrm{Poisson}\ bracket} = 0 \tag{13}$$

Therefore, in the early universe one can write down an invariant quadratic form in the Hubble parameter and the logarithm of the scale factor with time dependent coefficients. These coefficients are functions of the solutions of the Ermakov equation.

Let us now determine what type of solutions of ρ are possible. Note that this is determined solely through (10) and depend on the constant $M = \frac{3}{4\pi G}$ and the frequency $\Omega(t)$ which in turn is determined by the scalar field ϕ. So a choice for $\Omega(t)$ must ensure that a solution for ρ exists and one also gets a scalar field consistent with cosmological scenarios.

Case a

Consider a scalar field potential $V = \frac{1}{2}m^2\phi^2$ where m is the mass of the scalar field. If one assumes a scenario where $\dot{\phi}^2 >> V$ i.e. the kinetic energy is large compared to the potential energy then a solution for the scalar field is [14]

$$\phi(t) = const. - (12\pi)^{-1/2}\ lnt \tag{14}$$

Choosing $\Omega(t) = t^{-1/2}$ and using (5) gives

$$\phi(t) = \phi_0 - \frac{1}{24\pi G\sqrt{C}V}ln\ t \tag{15}$$

where the constant of integration has been identified as ϕ_0. Comparing (14) and (15) will fix the constant ϕ_0 and $\sqrt{C}$. For this choice of $\Omega(t) = t^{-1/2}$ the general solution for ρ is [2]

$$\rho(t) = \gamma_1\pi M t^{1/2}\Big[A^2 Y_1^2(2Mt^{1/2}) + B^2 J_1^2(2Mt^{1/2})$$
$$+2\gamma_2(A^2B^2 - \frac{1}{\pi^2 M^2})^{1/2} J_1(2Mt^{1/2})Y_1(2Mt^{1/2})\Big]^{1/2} \tag{16}$$

Here $\gamma_1 = \pm 1$, $\gamma_2 = \pm 1$, A, B are arbitrary complex constants, J_1, Y_1 are Bessel functions of the first and second kinds respectively. Let us take $\gamma_1 = \gamma_2 = +1$.

Thus this choice of Ω is consistent with an ultrahard equation of state and a scalar field with a logarithmic dependence on time [14]. Note that the dominance of kinetic energy is a natural choice for $k-$essence scalar fields.

Now we show that the Ermakov invariant (11) is a powerful tool to estimate the scale factor. For $t \to 0$, $J_\alpha(x) \to \frac{1}{\Gamma(\alpha+1)}(\frac{x}{2})^\alpha \quad ; Y_\alpha(x) \to -\frac{\Gamma(\alpha)}{\pi}(\frac{2}{x})^\alpha \quad ; 0 < x \leq \sqrt(\alpha+1); \quad \alpha > 0$. In our case $\alpha = 1$. Using these (16) takes the form

$$\rho(t) = (\pi M)^{1/2} t^{3/2} + (\pi M)^{-3/2} t^{-1/2} \tag{17}$$

One can now determine $A(t), B(t), C(t)$. For small times only the inverse powers of t will dominate. Therefore keeping only $O(t^{-2})$ and $O(t^{-3})$ terms, (11) becomes ($q = ln \ \ a(t)$):

$$\frac{q\dot{q}}{2\pi^3 M^4 t^2} + \frac{q^2}{8\pi^3 M^5 t^3} \approx I \tag{18}$$

and the solution for q is

$$q(t) = \left[A_0 t^{-1/2M} + A_1 t^3\right]^{1/2} \tag{19}$$

So the scale factor is

$$a(t) = e^{[A_0 t^{-1/2M} + A_1 t^3]^{1/2}} \sim e^{A_0^{1/2} t^{-1/4M}} \tag{20}$$

and the solution is consistent with the inflationary scenario. Here A_0 is an arbitrary constant of integration and $A_1 = \frac{8\pi^3 M^5 I}{6M+1}$. Note that as $a \to e^0 = 1$, $t \to [\frac{-2A_0}{A_1}]^{2M/(6M+1)}$. So A_0 should be chosen to be negative. (Here we have illustrated solutions for ($\gamma_1 = 1, \gamma_2 = 1$). Solutions for ($\gamma_1 = -1, \gamma_2 = \pm 1$) in the cosmological context will be discussed in subsequent publications).

Case b

Now suppose we choose Ω to be a constant. Then (5) gives

$$\phi(t) = \phi_i - \frac{1}{24\pi G\sqrt{C}V} t \tag{21}$$

For a constant Ω the general solution for ρ is

$$\rho(t) = \gamma_1 \Omega^{-1}\Big[A^2 cos^2(M\Omega t) + B^2 sin^2(M\Omega t) + 2\gamma_2 (A^2B^2 - \Omega^2)^{1/2} sin(\Omega M t) cos(\Omega M t)\Big]^{1/2} \tag{22}$$

Compare this with the attractor solution in [14] :

$$\phi_{atr}(t) \approx \phi_i - \frac{m}{\sqrt{12}\pi}(t - t_i) \tag{23}$$

Here the trajectory joins the attractor where it is flat at $|\phi| >> 1$ and afterwards the solution describes a stage of accelerated expansion [14] However, $\Omega = constant$ is *not* a natural choice for $k-$essence fields because it implies that the potential energy now dominates. The discussion of this case is merely for illustrative purposes.

The Ermakov invariant also exists in the quantum context [2].The invariant I , now an operator, is a constant of motion for the quantum system for any ρ that satisfies (10). So we now have the Heisenberg equation of motion for I as $\frac{dI}{dt} = \frac{\partial I}{\partial t} + \frac{1}{i\hbar}[I, H] = 0$. Creation and annihilation operators can be constructed and normalised eigenstates of I exist. Note that the hamiltonian corresponding to (7) is $\mathcal{H}(t) = \frac{p^2}{2M} + \frac{1}{2}M\Omega^2(t)q^2$. If $\psi_n(q,t)$ be the eigenfunctions of the invariant operator I, then $\langle\psi_n|\mathcal{H}(t)|\psi_n\rangle = \frac{M}{2}(\rho^{-2} + \Omega^2\rho^2 + \frac{1}{M^2}\dot{\rho}^2)(n + \frac{1}{2})\hbar$ where $n = 0, 1, 2......$ So in a quantum context, this invariant can also be used to estimate the quantum expectation value of a function involving the scale factor and the Hubble parameter. An analogue of the Berry's phase in early universe can also be defined as follows [15]. When the time dependent parameters of a quantum system evolving adiabatically in time executes a complete loop in parameter space, the wavefunction (in addition to its dynamic phase) picks up a geometric phase. In the Ermakov context, this phase factor is given by $\gamma_n(\mathcal{C}) = -(1/2)(n + 1/2)\int_0^T(\rho\ddot{\rho} - \dot{\rho}^2)$. where ρ satisfies (10). These aspects will be discussed in subsequent publications.

3. Conclusion: An observational probe in the early universe

The basic conclusion of this work is that the Ermakov invariant in a cosmological context (11) can be used as an observational probe in the early universe in the domain *viz.* $-1 < q(= ln \ \ a(t)) < 1$ in the following way. Observationally or otherwise, the Hubble parameter and the scale factor is known over a substantially large period of time. If these can be known for periods within the domain under consideration, the validity of (11) can be tested. Alternatively, knowing either of the two *viz.* the Hubble parameter or the scale factor, will enable the other to be determined using (11) and (12). As everything is based on a particular form of the dark energy lagrangian, any vindication of (11) implies an indirect proof of the presence of dark energy as a principal constituent of the universe.

The author would like to thank the Centre For Astroparticle Physics and Space Science, Bose Institute, Kolkata, for a sabbatical tenure during which this work has been done.

* debashis@bose.res.in
† sompresi@gmail.com

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