The leading hadronic vacuum polarisation on the lattice

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Abstract. After discussing the relevance of a first principles theory-prediction of the hadronic vacuum polarisation for Standard Model tests, the theoretical challenges for its computation in lattice QCD are reviewed. New ideas that will potentially improve the quality of lattice simulations will be introduced and the status of ongoing simulations will be presented briefly.

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INTRODUCTION

The quest to measure the anomalous magnetic moment of the muon, a_μ, experimentally with ever increasing precision has started at CERN [1] decades ago and was later continued at BNL [2], yielding a precision of 0.5ppm [3]. In the Standard Model (SM) a_μ receives contributions most notably from QED but also from the Weak sector and from QCD. While the former two contributions can be computed in (high order) perturbation theory, the QCD contributions are non-perturbative and as such not computable analytically in a model independent way. As nicely summarised in [4], using unitarity and analyticity, the leading QCD-contribution (cf. figure 1) is currently determined from the experimental measurement of e^+e^--annihilation or hadronic τ-decays. With these experiment-based approaches one is able to predict $a_\mu^{\rm LHV}$ with a precision below 1%. While the former process seems to provide solid SM-predictions, a pure theory-prediction still seems worth-while. In particular since a_μ is sensitive to contributions from physics beyond the SM. In fact, the theory prediction currently differs from the experimental measurement by 3.2σ [4]. While not yet providing evidence for new physics, this tension is naturally causing excitement.

Here we report on progress in predicting $a_\mu^{\rm LHV}$ in lattice QCD. It will become clear that this type of calculations is still in its infancy [5, 6, 7, 8]. To this end the precise experimental prediction of $a_\mu^{\rm LHV}$ will in the near future remain a reference for assessing lattice computations.

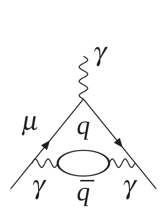

FIGURE 1. The leading hadronic contribution $a_\mu^{\rm LHV}$.

$a_\mu^{\rm LHV}$ ON THE LATTICE

The leading hadronic contribution is defined as

$$a_\mu^{\rm LHV} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dQ^2 K(Q^2)\big(\Pi(Q^2)-\Pi(0)\big), \qquad (1)$$

where the function $K(Q^2)$ parametrises the QED contribution to the diagram in figure 1 [9]. Note that the integration is over space-like momenta Q^2. The function $\Pi(Q^2)$ is the vacuum polarisation amplitude,

$$\begin{aligned} \Pi_{\mu\nu}(q) &= \int d^4x e^{iq(x-y)} \langle\, j_\mu^{\rm EM}(y)\, j_\nu^{\rm EM}(x)\rangle \\ &= (q_\mu q_\nu - q^2 g_{\mu\nu})\Pi(Q^2)\,, \end{aligned} \qquad (2)$$

where $Q^2 \equiv -q^2$. Simulations of lattice QCD always have to keep track of a number of systematic effects, most notably those stemming from the finite lattice-spacing, the finite volume and unphysically heavy quark masses. These effects have been studied deeply and a large body of tools by now allows one to estimate or to systematically control them at the level of precision required here. In this talk we present new ideas tailored to control systematic uncertainties arising from the presence of quark-disconnected diagrams, from the limited momentum resolution in finite volume field theory and from the contribution of vector resonances.

Quark disconnected diagrams and vector resonances The electro-magnetic current $j_\mu^{\rm EM}(x)$ in eq. (2) consists of a linear combination of flavour-diagonal quark-bilinear currents $\bar{q}(x)\gamma_\mu q(x)$, where $q = u, d, s$. The Wick contractions of eq. (2) therefore yield quark connected contributions

$$\langle \mathrm{Tr}\{S_q(x,y)\gamma_\mu S_q(y,x)\gamma_\nu\}\rangle\,, \tag{3}$$

and quark disconnected contributions

$$\langle \mathrm{Tr}\{S_q(y,y)\gamma_\mu\}\mathrm{Tr}\{S_q(x,x)\gamma_\nu\}\rangle\,, \tag{4}$$

where $S_q(x,y)$ is the quark propagator for the quark flavour q and the trace is over spin- and colour-indices. In lattice simulations the latter contribution is often neglected because of the huge overhead its computation causes [10].

A new method for predicting correlation functions consisting of quark-disconnected contributions was developed in [11]. By introducing valence quarks which are degenerate with the dynamical flavours, each Wick contraction can be rewritten in terms of a single fermionic correlation function defined in an unphysical theory. The physical result is recovered by summing over the correlation functions in the unphysical, partially quenched, theory [12, 13, 14]. In particular, the above expectation values eq. (3) and (4) remain unchanged if one of the quarks is replaced by a mass-degenerate partially quenched valence quark q'. This allows to express them in terms of individual two-point functions, the Wick-contractions of which yield either a connected or a disconnected two-point function:

$$\begin{aligned} C^{\mathrm{conn}}(y,x) &\equiv \langle \bar{q}(y)\gamma_\mu q'(y)\bar{q}'(x)\gamma_\nu q(x)\rangle\,, \\ C^{\mathrm{disc}}(y,x) &\equiv \langle \bar{q}'(y)\gamma_\mu q'(y)\bar{q}(x)\gamma_\nu q(x)\rangle\,. \end{aligned} \tag{5}$$

Within partially quenched chiral perturbation theory [12, 13, 14, 15, 16], expressions for the connected and the disconnected contributions to hadronic correlation functions can be computed. The case of the $N_f = 2$-theory without and with a partially quenched strange quark as well as the $N_f = 2+1$-theory were studied in [11]. In the $N_f = 2$-theory for example, the calculation at next-to-leading order in the effective theory predicts that the disconnected contribution reduces the connected contribution by only 10%.

The analytical prediction of quark-disconnected diagrams is work in progress. In particular, vector resonances which are not dynamical degrees of freedom in chiral perturbation theory turn out to be dominating $\Pi(Q^2)$ [6, 7]. It will be interesting to study the impact of vector resonances on the above predictions by including these degrees of freedom into the chiral Lagrangian [17]. However, while the effective theory for pions and kaons stands on solid grounds, this similar ansatz for vector mesons is a model.

Momentum resolution

Today, typical lattices extend over $L \approx 3$fm in the spatial directions (typically twice that large in the temporal direction, $T = 2L$). Besides vanishing momentum, the lowest hadron momentum therefore corresponds to the lowest non-vanishing Fourier-mode, $2\pi/T \approx 200$MeV. The plot in figure 2 shows the integral kernel of eq. (1), which is peaked at around the muon mass, $m_\mu \approx 106$MeV. The lattice data therefore needs to be extrapolated into this region. In order to compute $\Pi(Q^2)$ closer to the peak, the Mainz group [8] is applying twisted boundary conditions to the valence quark fields, $q(x+L\hat{i}) = e^{i\theta_i}q(x)$, which allows one to tune the offset of the Fourier Modes accessible in the lattice computation [18, 19, 20, 21, 22, 23, 24]. Naively, the twists applied to the quark-fields in the flavour-diagonal currents in eq. (2) will cancel (see e.g. [20]). However, the connected correlation function in eq. (5), $C^{\mathrm{conn.}}(y,x)$, is composed of flavour-off-diagonal currents. Hence, different twist-angles can be applied for the quark fields q and q', respectively [11]. This argument allows to apply partial twisting at least to the connected contribution to a_μ^{LHV}. The disconnected contribution can either be predicted using chiral perturbation theory (cf. above) or it can be computed for the usual Fourier momenta and then be interpolated using the ansatz provided by chiral perturbation theory. Figure 3 shows results for $\Pi(Q^2)$ by the Mainz group. Without twisting only the blue data points are accessible, while with twisting the red data points can be added. Besides providing data points closer to the region where the integral eq. (1) receives major contributions, the additional data points for larger values of the momentum will help in stabilising fits: In order to compute the result for the vacuum polarisation tensor eq. (2) one first fits an ansatz for its momentum-dependence to the data and then integrates it. The Mainz group uses models for the vector resonance, polynomials and Padé approximations as ansätze (cf. also ETM's study of parametrisations of vector resonances in [7]). An estimate of the systematic uncertainties is obtained from the spread of the respective fit-results. In our

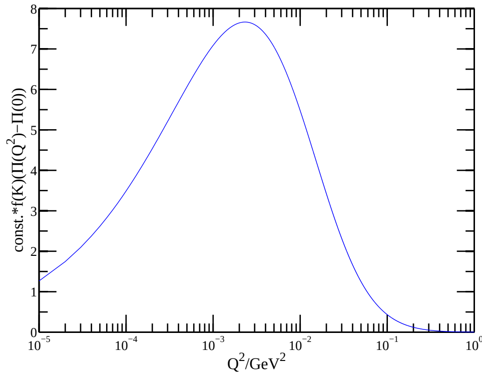

FIGURE 2. Shape of the integral kernel in eq. (1) assuming vector dominance for $\Pi(Q^2)$.

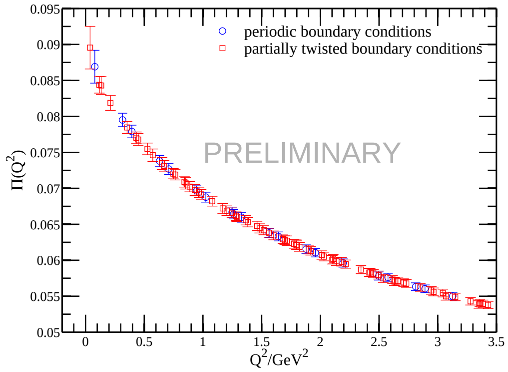

FIGURE 3. Results for $\Pi(Q^2)$ with (red) and without (blue) twisted boundary conditions.

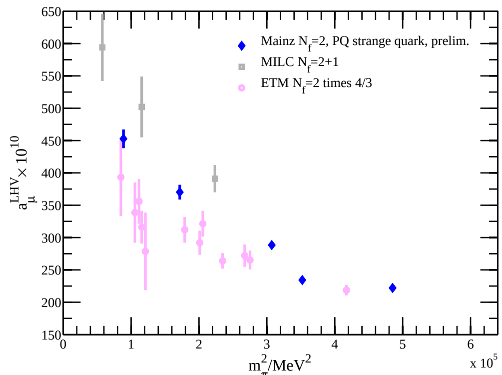

FIGURE 4. Comparison of results for a_μ^{LHV} as a function of the pion mass squared: Sources: our results (blue diamonds) are preliminary, MILC [6] (where we have chosen as the error bar the maximum spread between results from different q^2-parametrisations) and ETMC [7]. Note that the plotted results are for different values of the lattice cut-off and also of the lattice volume.

analysis the additional data points which we obtained by using partially twisted boundary conditions helped to reduce this spread significantly.

STATUS

Figure 4 shows results for the vacuum polarisation by ETM [7], MILC [6] and by the Mainz group. Note that Mainz is the only group using twisted boundary conditions while ETM is the only collaboration trying to compute the disconnected contribution directly. For comparison we quote the value obtained from the analysis of e^+e^--annihilation, $a_\mu^{\mathrm{LHV}} = 690(5) \times 10^{-10}$ [4]. Clearly the lattice data tend towards this value as the quark mass is reduced, but the extrapolation to the physical point is model-dependent and will only be dispensable once the simulation is carried out for physical quark masses.

In this talk we present the status of lattice computations of a_μ^{LHV}. While significant progress has been made recently systematic uncertainties are still not under sufficient control. Clearly, simulations at the physical point would be very desirable in order to be independent of model-based extrapolations of lattice data.

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