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Perturbative renormalization factors and $\mathcal{O}(a^2)$ corrections for lattice 4-fermion operators with improved fermion/gluon actions

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Abstract

In this work we calculate the corrections to the amputated Green's functions of 4-fermion operators, in 1-loop Lattice Perturbation theory. One of the novel aspects of our calculations is that they are carried out to second order in the lattice spacing, $\mathcal{O}(a^2)$.

We employ the Wilson/clover action for massless fermions (also applicable for the twisted mass action in the chiral limit) and a family of Symanzik improved actions for gluons. Our calculations have been carried out in a general covariant gauge. Results have been obtained for several popular choices of values for the Symanzik coefficients (Plaquette, Tree-level Symanzik, Iwasaki, TILW and DBW2 action).

While our Green's function calculations regard any pointlike 4-fermion operators which do not mix with lower dimension ones, we pay particular attention to $\Delta F = 2$ operators, both Parity Conserving and Parity Violating (F stands for flavour: S, C, B). By appropriately projecting those bare Green's functions we compute the perturbative renormalization constants for a complete basis of 4-fermion operators and we study their mixing pattern. For some of the actions considered here, even $\mathcal{O}(a^0)$ results did not exist in the literature to date. The correction terms which we calculate (along with our previous $\mathcal{O}(a^2)$ calculation of Z_Ψ [1–3]) are essential ingredients for minimizing the lattice artifacts which are present in non-perturbative evaluations of renormalization constants with the RI′-MOM method.

Our perturbative results, for the matrix elements of $\Delta F = 2$ operators and for the corresponding renormalization matrices, depend on a large number of parameters: coupling constant, number of colors, lattice spacing, external momentum, clover parameter, Symanzik coefficients, gauge parameter. To make these results most easily accessible to the reader, we have included them in the distribution package of this paper, as an ASCII file named: 4-fermi.m; the file is best perused as Mathematica input.

The main results of this work have been applied to improve non-perturbative estimates of the B_K-parameter in $N_F = 2$ twisted mass lattice QCD [4].

I. INTRODUCTION

A number of flavour-changing processes are currently under study in Lattice simulations. Among the most common examples are the decay $K \to \pi\pi$ and K^0–$\bar{K}^0$ oscillations. From experimental evidence, we know that these weak processes violate the CP symmetry. In theory, the calculation of the amount of CP violation in K^0–$\bar{K}^0$ oscillations requires the knowledge of the kaon B_K parameter.

The parameter B_K is obtained from the $\Delta S = 2$ weak matrix element:

$$B_K = \frac{\langle \bar{K}^0 | \hat{O}^{\Delta S=2} | K^0 \rangle}{\frac{8}{3} \langle \bar{K}^0 | \bar{s}\gamma_\mu d | 0 \rangle \, \langle 0 | \bar{s}\gamma_\mu d | K^0 \rangle}\,, \tag{1}$$

where s and d stand for strange and down quarks, and $\hat{O}^{\Delta S=2}$ is the effective 4-quark interaction renormalized operator, corresponding to the bare operator:

$$O^{\Delta S=2} = (\bar{s}\gamma^{\rm L}_\mu d)(\bar{s}\gamma^{\rm L}_\mu d), \qquad \gamma^{\rm L}_\mu = \gamma_\mu(1\!\!1 - \gamma_5). \tag{2}$$

The above operator splits into parity-even and parity-odd parts; in standard notation: $O^{\Delta S=2} = O^{\Delta S=2}_{VV+AA} - O^{\Delta S=2}_{VA+AV}$. Since the above weak process is simulated in the framework of Lattice QCD, where Parity is a symmetry, the parity-odd part gives no contribution to the K^0–$\bar{K}^0$ matrix element. Thus, we conclude that B_K can be extracted from the correlator ($x_0 > 0$, $y_0 < 0$):

$$C_{KOK}(x,y) = \langle (\bar{d}\gamma_5 s)(x) \hat{O}^{\Delta S=2}_{VV+AA}(0) (\bar{d}\gamma_5 s)(y) \rangle, \qquad O^{\Delta S=2}_{VV+AA} = (\bar{s}\gamma_\mu d)(\bar{s}\gamma_\mu d) + (\bar{s}\gamma_\mu\gamma_5 d)(\bar{s}\gamma_\mu\gamma_5 d)\,, \tag{3}$$

where $O^{\Delta S=2}_{VV+AA}$ is the bare operator and $\hat{O}^{\Delta S=2}_{VV+AA}$ is the corresponding renormalized operator.

Our results are immediately applicable to other $\Delta F = 2$ processes of great phenomenological interest, such as $D - \bar{D}$ or $B - \bar{B}$ mixing. They are also useful in new physics models (i.e. beyond the standard model), because there the complete basis of 4-fermion operators contributes to neutral meson mixing amplitudes; this is the case for instance of SUSY models (see e.g. [5]). For this, one needs to study more general operators of the form

$$\mathcal{O}_{XY} \equiv (\bar{s}\, X\, d)(\bar{s}\, Y\, d) \tag{4}$$

where X and Y are general Dirac matrices (see Eq. (8)).

With Wilson fermions on the lattice, the explicit breaking of chiral symmetry also induces a mixing between the Standard Model operator in Eq. (2) and the other $\Delta S = 2$ operators of the basis Eq. (4). A strategy which allows to avoid this mixing and, at the same time, guarantees automatic $\mathcal{O}(a)$-improvement of the four fermion operators has been proposed in [6] and it makes use of twisted and Osterwalder-Seiler fermions. In this approach, for the

B_K computation, in place of the operator in Eq. (3) a four-quark operator with a different flavour content (s, d, s', d'), and with $\Delta S = \Delta s + \Delta s' = 2$ is considered, namely [6]

$$\mathcal{O}^{\Delta S=2}_{VV+AA} = (\bar{s}\gamma_\mu d)(\bar{s}'\gamma_\mu d') + (\bar{s}\gamma_\mu\gamma_5 d)(\bar{s}'\gamma_\mu\gamma_5 d') + (\bar{s}\gamma_\mu d')(\bar{s}'\gamma_\mu d) + (\bar{s}\gamma_\mu\gamma_5 d')(\bar{s}'\gamma_\mu\gamma_5 d)\,, \quad (5)$$

where now the correlator is given by: $C_{K\mathcal{O}K'}(x,y) = \langle(\bar{d}\gamma_5 s)(x)\, 2\mathcal{O}^{\Delta S=2}_{VV+AA}(0)(\bar{d}'\gamma_5 s')(y)\rangle$. Making use of Wick's theorem one checks the equality: $C_{K\mathcal{O}K'}(x,y) = C_{KOK}(x,y)$, which means that both correlators contain the same physical information.

The aforementioned matrix elements are very sensitive to various systematic errors. A major issue facing Lattice Gauge Theory, since its early days, has been the reduction of effects induced by the finiteness of lattice spacing a, in order to better approach the elusive continuum limit.

In order to obtain reliable non-perturbative estimates of physical quantities it is essential to keep under control the $\mathcal{O}(a)$ systematic errors in simulations or, additionally, reduce the lattice artifacts in numerical results. Such a reduction, regarding renormalization functions, can be achieved by subtracting appropriately the $\mathcal{O}(a^2)$ perturbative correction terms presented in this paper, from corresponding non-perturbative results.

In this paper we address the perturbative aspects of this problem from a very general point of view. In particular, we study the bare 4-point amputated Green's function of the most general pointlike 4-fermion operators with four distinct flavours[1]. Although the computational procedure laid out in the paper is applicable to all orders in the lattice spacing, we focus on two different results:

1. The perturbative 1-loop evaluation of renormalization factors for a variety of 4-fermion operators. These factors can be used to renormalize 4-fermion operators computed non-perturbatively with any fermion/gluon Wilson-like improved action. This part can be considered as an extension of other computations of 4-fermion operator renormalization [7].

2. The evaluation of $\mathcal{O}(a^2)$ contributions to the aforementioned 1-loop computations. These are very useful to improve non-perturbative estimates for the same renormalization factors [4], since they can reduce lattice artifacts, leading to more reliable determinations.

[1] For $\Delta S = 1$ operators with flavour structure $(\bar{s}Xd)(\bar{q}Yq)$ penguin contractions induce a power divergent mixing of the four fermion operators with lower dimension operators. This case is not considered in the present paper.

In Section II we define the general 4-fermion operators and describe the setup of the computation. The calculations are carried out up to 1-loop in Lattice Perturbation theory and up to $\mathcal{O}(a^2)$ in lattice spacing. In the same Section we also present simplified expressions for the three Feynman diagrams, which constitute the building blocks of the whole calculation. In addition, we address certain difficulties which are associated to the $\mathcal{O}(a^2)$ computation. In Section III we switch to the evaluation of the renormalization matrices for the 4-fermion operators. In particular, we focus on the complete basis of 20 four-fermion operators of dimension six which do not need power subtractions (i.e. mixing occurs only with other operators of equal dimensions). In the last Section, we summarize the main results of this work, and discuss how non-perturbative estimates are being improved by subtracting our $\mathcal{O}(a^2)$ correction terms.

II. AMPUTATED GREEN'S FUNCTIONS OF 4-FERMION $\Delta S = \Delta s + \Delta s' = 2$ OPERATORS.

Here we evaluate, up to $\mathcal{O}(a^2)$, the 1-loop matrix element of the 4-fermion operators (the superscript letter F stands for Fierz.):

$$\mathcal{O}_{XY} \equiv (\bar{s}\, X\, d)(\bar{s}'\, Y\, d') \equiv \sum_x \sum_{c,d} \sum_{k_1,k_2,k_3,k_4} \Big(\bar{s}^c_{k_1}(x)\, X_{k_1k_2}\, d^c_{k_2}(x)\Big)\Big(\bar{s}'^d_{k_3}(x)\, Y_{k_3k_4}\, d'^d_{k_4}(x)\Big) \tag{6}$$

$$\mathcal{O}^F_{XY} \equiv (\bar{s}\, X\, d')(\bar{s}'\, Y\, d) \equiv \sum_x \sum_{c,d} \sum_{k_1,k_2,k_3,k_4} \Big(\bar{s}^c_{k_1}(x)\, X_{k_1k_2}\, d'^c_{k_2}(x)\Big)\Big(\bar{s}'^d_{k_3}(x)\, Y_{k_3k_4}\, d^d_{k_4}(x)\Big) \tag{7}$$

with a generic initial state: $\bar{d}'^{a_4}_{i_4}(p_4)\, s'^{a_3}_{i_3}(p_3)|0\rangle$, and a generic final state: $\langle 0|\bar{d}^{a_2}_{i_2}(p_2)\, s^{a_1}_{i_1}(p_1)$. Spin indices are denoted by i, k, and color indices by a, c, d, while X and Y correspond to the following set of products of the Dirac matrices:

$$X, Y = \{1\!\!1, \gamma^5, \gamma_\mu, \gamma_\mu\gamma^5, \sigma_{\mu\nu}, \gamma^5\sigma_{\mu\nu}\} \equiv \{S, P, V, A, T, \tilde{T}\}; \qquad \sigma_{\mu\nu} = \frac{1}{2}[\gamma_\mu, \gamma_\nu]. \tag{8}$$

Our calculations are performed using massless fermions described by the Wilson/clover action. By taking $m_f = 0$, our results are identical also for the twisted mass and the Osterwalder-Seiler actions in the chiral limit (in the so called twisted mass basis). For gluons we employ a 3-parameter family of Symanzik improved actions, which comprises all common gluon actions (Plaquette, tree-level Symanzik, Iwasaki, DBW2, Lüscher-Weisz). Conventions and notations for the actions, as well as algebraic manipulations involving the evaluation of 1-loop Feynman diagrams (up to $\mathcal{O}(a^2)$), are described in detail in Ref. [1].

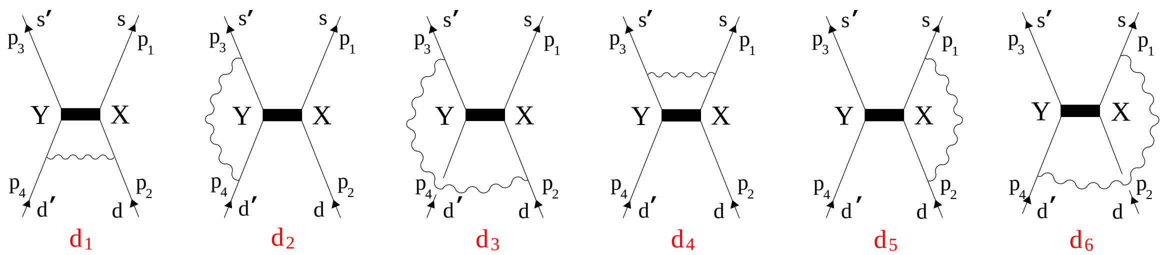

FIG. 1: 1-loop diagrams contributing to the amputated Green's function of the 4-fermion operator $\mathcal{O}^{XY}$. Wavy (solid) lines represent gluons (fermions).

To establish notation and normalization, let us first write the tree-level expression for the amputated Green's functions of the operators $\mathcal{O}_{XY}$ and $\mathcal{O}^F_{XY}$:

$$\Lambda^{XY}_{tree}(p_1,p_2,p_3,p_4,r_s,r_d,r_{s'},r_{d'})^{a_1a_2a_3a_4}_{i_1i_2i_3i_4} = X_{i_1i_2}\,Y_{i_3i_4}\,\delta_{a_1a_2}\,\delta_{a_3a_4}\,, \tag{9}$$

$$(\Lambda^F)^{XY}_{tree}(p_1,p_2,p_3,p_4,r_s,r_d,r_{s'},r_{d'})^{a_1a_2a_3a_4}_{i_1i_2i_3i_4} = -X_{i_1i_4}\,Y_{i_3i_2}\,\delta_{a_1a_4}\,\delta_{a_3a_2}\,, \tag{10}$$

where r is the Wilson parameter, one for each flavour.

We continue with the first quantum corrections. There are twelve 1-loop diagrams that enter our 4-fermion calculation, six for each operator $\mathcal{O}_{XY}$, $\mathcal{O}^F_{XY}$. The diagrams $d_1 - d_6$ corresponding to the operator $\mathcal{O}_{XY}$ are illustrated in Fig. 1. The other six diagrams, $d^F_1 - d^F_6$, involved in the Green's function of $\mathcal{O}^F_{XY}$ are similar to $d_1 - d_6$, and may be obtained from $d_1 - d_6$ by interchanging the fermionic fields d and d' along with their momenta, color and spin indices, and respective Wilson parameters.

The only diagrams that need to be calculated from first principles are d_1, d_2 and d_3, while the rest can be expressed in terms of the first three. This is a result of the symmetries between the diagrams (d_1, d_4), (d_2, d_5) and (d_3, d_6). Diagrams d_4, d_5 and d_6 can be expressed as diagrams d_1, d_2 and d_3 by exchanging the external quark legs and X, Y, if necessary. In particular, the expressions for the amputated Green's functions $\Lambda^{XY}_{d_4} - \Lambda^{XY}_{d_6}$ can be obtained via the following relations:

$$\Lambda^{XY}_{d_4}(p_1,p_2,p_3,p_4,r_s,r_d,r_{s'},r_{d'})^{a_1a_2a_3a_4}_{i_1i_2i_3i_4} = \left(\Lambda^{XY}_{d_1}(-p_2,-p_1,-p_4,-p_3,r_d,r_s,r_{d'},r_{s'})^{a_2a_1a_4a_3}_{i_2i_1i_4i_3}\right)^\star, \tag{11}$$

$$\Lambda^{XY}_{d_5}(p_1,p_2,p_3,p_4,r_s,r_d,r_{s'},r_{d'})^{a_1a_2a_3a_4}_{i_1i_2i_3i_4} = \Lambda^{YX}_{d_2}(p_3,p_4,p_1,p_2,r_{s'},r_{d'},r_s,r_d)^{a_3a_4a_1a_2}_{i_3i_4i_1i_2}, \tag{12}$$

$$\Lambda^{XY}_{d_6}(p_1,p_2,p_3,p_4,r_s,r_d,r_{s'},r_{d'})^{a_1a_2a_3a_4}_{i_1i_2i_3i_4} = \Lambda^{YX}_{d_3}(p_3,p_4,p_1,p_2,r_{s'},r_{d'},r_s,r_d)^{a_3a_4a_1a_2}_{i_3i_4i_1i_2}. \tag{13}$$

Once we have constructed $\Lambda^{XY}_{d_4} - \Lambda^{XY}_{d_6}$ we can use relation:

$$(\Lambda^F)^{XY}_{d_j}(p_1,p_2,p_3,p_4,r_s,r_d,r_{s'},r_{d'})^{a_1a_2a_3a_4}_{i_1i_2i_3i_4} = -\Lambda^{XY}_{d_j}(p_1,p_4,p_3,p_2,r_s,r_{d'},r_{s'},r_d)^{a_1a_4a_3a_2}_{i_1i_4i_3i_2}, \tag{14}$$

to derive the expressions for $(\Lambda^F)^{XY}_{d_i}$ $(i = 1, \cdots, 6)$. From the amputated Green's functions for all twelve diagrams we can write down the total 1-loop expressions for the operators $\mathcal{O}_{XY}$ and $\mathcal{O}^F_{XY}$:

$$\Lambda^{XY}_{1-loop} = \sum_{j=1}^{6} \Lambda^{XY}_{d_j}, \qquad (\Lambda^F)^{XY}_{1-loop} = \sum_{j=1}^{6} (\Lambda^F)^{XY}_{d_j}. \tag{15}$$

In our algebraic expressions for the 1-loop amputated Green's functions $\Lambda^{XY}_{d_1}$, $\Lambda^{XY}_{d_2}$ and $\Lambda^{XY}_{d_3}$ we kept the Wilson parameters for each quark field distinct, that is: r_s, r_d, $r_{s'}$, $r_{d'}$ for the quark fields s, d, s' and d' respectively. For the required numerical integration of the algebraic expressions corresponding to each Feynman diagram, we are forced to choose the value for each r parameter.In the numerical results presented in this paper we set:

$$r_s = r_d = r_{s'} = r_{d'} = 1. \tag{16}$$

Concerning the external momenta p_i (shown explicitly in Fig. 1) we have chosen to evaluate the amputated Green's functions at the renormalization point:

$$p_1 = p_2 = p_3 = p_4 \equiv p. \tag{17}$$

It is easy and not time consuming to repeat the calculations for other choices of Wilson parameters and for other renormalization prescriptions. The final 1-loop expressions for $\Lambda^{XY}_{d_1}$, $\Lambda^{XY}_{d_2}$ and $\Lambda^{XY}_{d_3}$, up to $\mathcal{O}(a^2)$, are obtained as a function of: the coupling constant g, clover parameter $c_{\rm SW}$, number of colors N_c, lattice spacing a, external momentum p and gauge parameter λ.

As an example we present the results for $\Lambda^{XY}_{d_1}$ and for the special choices: $c_{\rm SW} = 0$, $\lambda = 0$ (Landau Gauge), $r_s = r_d = r_{s'} = r_{d'} = 1$, and tree-level Symanzik improved action:

$$\Lambda^{XY}_{d_1}(p)^{a_1\,a_2\,a_3\,a_4}_{i_1\,i_2\,i_3\,i_4} = \frac{g^2}{16\pi^2}\left(\delta_{a_1\,a_4}\delta_{a_3\,a_2} - \frac{\delta_{a_1\,a_2}\delta_{a_3\,a_4}}{N_c}\right) \times \left\{ (\Lambda_{\mathcal{O}(a^0)})^{XY}_{d_1} + a(\Lambda_{\mathcal{O}(a^1)})^{XY}_{d_1} + a^2(\Lambda_{\mathcal{O}(a^2)})^{XY}_{d_1} \right\}, \tag{18}$$

where:

$$\begin{aligned}(\Lambda_{\mathcal{O}(a^0)})^{XY}_{d_1} &= X_{i_1\,i_2} Y_{i_3\,i_4}\left[-\frac{1}{2}\ln(a^2p^2) - 0.05294144(3)\right] + \sum_\mu (X\gamma^\mu)_{i_1\,i_2}(Y\gamma^\mu)_{i_3\,i_4}\,[-0.507914049(6)] \\ &+ \sum_{\mu,\nu}(X\gamma^\mu\gamma^\nu)_{i_1\,i_2}(Y\gamma^\mu\gamma^\nu)_{i_3\,i_4}\left[\frac{1}{8}\ln(a^2p^2) + 0.018598520(2)\right] \\ &+ \sum_{\mu,\nu,\rho}(X\gamma^\mu\gamma^\rho)_{i_1\,i_2}(Y\gamma^\nu\gamma^\rho)_{i_3\,i_4}\left[0.397715726853\frac{p_\mu p_\nu}{p^2}\right].\end{aligned} \tag{19}$$

The $\mathcal{O}(a^1)$ and $\mathcal{O}(a^2)$ contributions of Eq. (18) along with the complete results for all diagrams for tree-level Symanzik improved gluons, $c_{\rm SW} \neq 0$ and $\lambda \neq 0$, are presented in

Appendix A; the reader can find similar expressions for other gluon actions in electronic form (4-fermi.m). We note in passing that in diagram 3 the dependence on external momentum has the same terms as in diagram 2, with identical numerical coefficients; the difference between the two diagrams lies in the structure of color and gamma matrices multiplying each term.

The setup presented up to this point applies to both $\mathcal{O}(a^0)$ and $\mathcal{O}(a^2)$ calculation. For the $\mathcal{O}(a^2)$ case additional difficulties arise in extracting correctly the full $\mathcal{O}(a^2)$ dependence. The crucial point of our calculation is the correct extraction of the full $\mathcal{O}(a^2)$ dependence from loop integrands with strong IR divergences (convergent only beyond 6 dimensions). The singularities are isolated using the procedure explained in Ref. [1]. In order to reduce the number of strong IR divergent integrals, appearing in diagram d_1, we have inserted the identity below into selected 3-point functions:

$$1 = \frac{1}{\widehat{a\,p}^2}\Big(\widehat{k+a\,p}^2 + \widehat{k-a\,p}^2 - 2\hat{k}^2 + 16\sum_\sigma \sin(k_\sigma)^2\sin(ap_\sigma)^2\Big), \tag{20}$$

where $\hat{q}^2 = 4\sum_\mu \sin^2(\frac{q_\mu}{2})$ and $k\,(p)$ is the loop (external) momentum. Repeated use of Eq. (20) reduces the 3-point functions to either 2-point functions or more convergent expressions. The factor $1/\widehat{a\,p}^2$ in Eq. (20) can be treated by Taylor expansion. For our calculations it was necessary only to $\mathcal{O}(a^0)$:

$$\frac{1}{\widehat{a\,p}^2} = \frac{1}{a^2\,p^2} + \frac{\sum_\sigma p_\sigma^4}{(p^2)^2} + \mathcal{O}(a^2\,p^2). \tag{21}$$

Here we present one of the four integrals with strong IR divergences that enter this calculation:

$$\begin{aligned}\int_{-\pi}^{\pi}\frac{d^4k}{(2\pi)^4}\frac{\sin(k_\mu)\,\sin(k_\nu)}{\hat{k}^2\,\widehat{k+a\,p}^2\,\widehat{k-a\,p}^2} \;=\;& \delta_{\mu\nu}\Big(0.002457072288 - \frac{\ln(a^2p^2)}{64\pi^2}\Big) + 0.001870841540\frac{p_\mu p_\nu}{p^2}\\ &+\; a^2\Bigg[\delta_{\mu\nu}\Bigg(p^2\Big(0.00055270353(6) - \frac{\ln(a^2p^2)}{512\pi^2}\Big)\\ &\qquad - p_\mu^2\Big(0.0001282022(1) + \frac{\ln(a^2\,p^2)}{768\pi^2}\Big) + 0.000157122310\,\frac{\sum_\sigma p_\sigma^4}{p^2}\Bigg)\\ &\qquad + p_\mu\,p_\nu\Bigg(-0.00029731225(4) + \frac{\ln(a^2p^2)}{768\pi^2} - 0.000047949674\frac{(p_\mu^2+p_\nu^2)}{p^2}\\ &\qquad + 0.000268598599\frac{\sum_\sigma p_\sigma^4}{(p^2)^2}\Bigg)\Bigg] + \mathcal{O}(a^4\,p^4).\end{aligned} \tag{22}$$

The results for the other three integrals can be found in Ref. [1]. Integrands with simple IR divergences (convergent beyond 4 dimensions) can be handled by well-known techniques.

III. MIXING AND RENORMALIZATION OF $\mathcal{O}_{XY}$ AND $\mathcal{O}^F_{XY}$ ON THE LATTICE.

The matrix element $\langle\bar{K}^0|O^{\Delta S=2}_{VV+AA}|K^0\rangle$ is very sensitive to various systematic errors. The main roots of this problem are: **a)** $\mathcal{O}(a)$ systematic errors due to numerical integration, **b)** with Wilson-like fermions, the operator $O^{\Delta S=2}_{VV+AA}$ mixes with other 4-fermion $\Delta S = 2$ operators of dimension six. Mixing with operators of lower dimensionality is impossible because there is no candidate $\Delta S = 2$ operator.

In order to address these problems we have calculated the mixing pattern (renormalization matrices) of the Parity Conserving and Parity Violating 4-fermion $\Delta S = 2$ operators (defined below), by using the amputated Green's functions obtained in the previous section. A more extensive theoretical background and non-perturbative results, concerning renormalization matrices of 4-fermion operators, can be found in Ref. [8] (see also [6, 9, 10]). Next we summarize all important relations from Ref. [8] needed for the present calculation.

One can construct a complete basis of 20 independent operators which have the symmetries of the generic QCD Wilson lattice action (Parity P, Charge conjugation C, Flavour exchange symmetry $S{\equiv}(d \leftrightarrow d')$, Flavour Switching symmetries $S'{\equiv}(s \leftrightarrow d, s' \leftrightarrow d')$ and $S''{\equiv}(s \leftrightarrow d', d \leftrightarrow s')$), with 4 degenerate quarks. This basis can be decomposed into smaller independent bases according to the discrete symmetries P, S, CPS', CPS''. Following the notation of Ref. [8] we have 10 Parity Conserving operators, Q, ($P{=}+1$, $S{=}\pm 1$) and 10 Parity Violating operators, $\mathcal{Q}$, ($P{=}-1$, $S{=}\pm 1$):

$$
\left\{\begin{array}{l}
Q_1^{S=\pm1} \equiv \frac{1}{2}\left[\mathcal{O}_{VV} \pm \mathcal{O}^F_{VV}\right] + \frac{1}{2}\left[\mathcal{O}_{AA} \pm \mathcal{O}^F_{AA}\right], \\
Q_2^{S=\pm1} \equiv \frac{1}{2}\left[\mathcal{O}_{VV} \pm \mathcal{O}^F_{VV}\right] - \frac{1}{2}\left[\mathcal{O}_{AA} \pm \mathcal{O}^F_{AA}\right], \\
Q_3^{S=\pm1} \equiv \frac{1}{2}\left[\mathcal{O}_{SS} \pm \mathcal{O}^F_{SS}\right] - \frac{1}{2}\left[\mathcal{O}_{PP} \pm \mathcal{O}^F_{PP}\right], \\
Q_4^{S=\pm1} \equiv \frac{1}{2}\left[\mathcal{O}_{SS} \pm \mathcal{O}^F_{SS}\right] + \frac{1}{2}\left[\mathcal{O}_{PP} \pm \mathcal{O}^F_{PP}\right], \\
Q_5^{S=\pm1} \equiv \frac{1}{2}\left[\mathcal{O}_{TT} \pm \mathcal{O}^F_{TT}\right],
\end{array}\right.
\quad
\begin{array}{l}
\left\{ \mathcal{Q}_1^{S=\pm1} \equiv \frac{1}{2}\left[\mathcal{O}_{VA} \pm \mathcal{O}^F_{VA}\right] + \frac{1}{2}\left[\mathcal{O}_{AV} \pm \mathcal{O}^F_{AV}\right], \right. \\
\left\{\begin{array}{l}
\mathcal{Q}_2^{S=\pm1} \equiv \frac{1}{2}\left[\mathcal{O}_{VA} \pm \mathcal{O}^F_{VA}\right] - \frac{1}{2}\left[\mathcal{O}_{AV} \pm \mathcal{O}^F_{AV}\right], \\
\mathcal{Q}_3^{S=\pm1} \equiv \frac{1}{2}\left[\mathcal{O}_{PS} \pm \mathcal{O}^F_{PS}\right] - \frac{1}{2}\left[\mathcal{O}_{SP} \pm \mathcal{O}^F_{SP}\right],
\end{array}\right. \\
\left\{\begin{array}{l}
\mathcal{Q}_4^{S=\pm1} \equiv \frac{1}{2}\left[\mathcal{O}_{PS} \pm \mathcal{O}^F_{PS}\right] + \frac{1}{2}\left[\mathcal{O}_{SP} \pm \mathcal{O}^F_{SP}\right], \\
\mathcal{Q}_5^{S=\pm1} \equiv \frac{1}{2}\left[\mathcal{O}_{T\tilde{T}} \pm \mathcal{O}^F_{T\tilde{T}}\right].
\end{array}\right.
\end{array}
\tag{23}
$$

Summation over all independent Lorentz indices (if any), of the Dirac matrices, is implied. The operators shown above are grouped together according to their mixing pattern. This implies that the renormalization matrices $Z^{S=\pm1}$ ($\mathcal{Z}^{S=\pm1}$), for the Parity Conserving (Violating) operators, have the form:

$$Z^{S=\pm1} = \begin{pmatrix} Z_{11} & Z_{12} & Z_{13} & Z_{14} & Z_{15} \\ Z_{21} & Z_{22} & Z_{23} & Z_{24} & Z_{25} \\ Z_{31} & Z_{32} & Z_{33} & Z_{34} & Z_{35} \\ Z_{41} & Z_{42} & Z_{43} & Z_{44} & Z_{45} \\ Z_{51} & Z_{52} & Z_{53} & Z_{54} & Z_{55} \end{pmatrix}^{S=\pm1} \quad , \quad \mathcal{Z}^{S=\pm1} = \begin{pmatrix} \mathcal{Z}_{11} & 0 & 0 & 0 & 0 \\ 0 & \mathcal{Z}_{22} & \mathcal{Z}_{23} & 0 & 0 \\ 0 & \mathcal{Z}_{32} & \mathcal{Z}_{33} & 0 & 0 \\ 0 & 0 & 0 & \mathcal{Z}_{44} & \mathcal{Z}_{45} \\ 0 & 0 & 0 & \mathcal{Z}_{54} & \mathcal{Z}_{55} \end{pmatrix}^{S=\pm1} . \tag{24}$$

Now the renormalized Parity Conserving (Violating) operators, $\hat{Q}^{S=\pm1}$ ($\hat{\mathcal{Q}}^{S=\pm1}$), are defined via the equations:

$$\hat{Q}_l^{S=\pm1} = Z_{lm}^{S=\pm1} \cdot Q_m^{S=\pm1}, \quad \hat{\mathcal{Q}}_l^{S=\pm1} = \mathcal{Z}_{lm}^{S=\pm1} \cdot \mathcal{Q}_m^{S=\pm1}, \tag{25}$$

where $l, m = 1, \ldots, 5$ (a sum over m is implied). The renormalized amputated Green's functions $\hat{L}^{S=\pm1}$ ($\hat{\mathcal{L}}^{S=\pm1}$) corresponding to $Q^{S=\pm1}$ ($\mathcal{Q}^{S=\pm1}$), are given in terms of their bare counterparts $L^{S=\pm1}$ ($\mathcal{L}^{S=\pm1}$) through:

$$\hat{L}_l^{S=\pm1} = Z_\Psi^{-2} Z_{lm}^{S=\pm1} \cdot L_m^{S=\pm1}, \quad \hat{\mathcal{L}}_l^{S=\pm1} = Z_\Psi^{-2} \mathcal{Z}_{lm}^{S=\pm1} \cdot \mathcal{L}_m^{S=\pm1}, \tag{26}$$

where Z_Ψ is the quark field renormalization constant. In order to obtain Z_Ψ for a given renormalization prescription, one must make use of the inverse fermion propagator, S^{-1}, calculated (up to 1-loop and up to $\mathcal{O}(a^2)$ for massless Wilson/clover fermions and Symanzik improved gluons) in Ref. [1].

The renormalization matrices $Z^{S=\pm1}$ ($\mathcal{Z}^{S=\pm1}$), are computed using the appropriate Parity Conserving (Violating) Projectors $P^{S=\pm1}$ ($\mathcal{P}^{S=\pm1}$):

$$\begin{aligned}
P_1^{S=\pm1} &\equiv +\frac{\Pi_{VV} + \Pi_{AA}}{64 N_c (N_c \pm 1)}, & \mathcal{P}_1^{S=\pm1} &\equiv -\frac{\Pi_{VA} + \Pi_{AV}}{64 N_c (N_c \pm 1)}, \\
P_2^{S=\pm1} &\equiv +\frac{\Pi_{VV} - \Pi_{AA}}{64 (N_c^2 - 1)} \pm \frac{\Pi_{SS} - \Pi_{PP}}{32 N_c (N_c^2 - 1)}, & \mathcal{P}_2^{S=\pm1} &\equiv -\frac{\Pi_{VA} - \Pi_{AV}}{64 (N_c^2 - 1)} \mp \frac{\Pi_{SP} - \Pi_{PS}}{32 N_c (N_c^2 - 1)}, \\
P_3^{S=\pm1} &\equiv \pm\frac{\Pi_{VV} - \Pi_{AA}}{32 N_c (N_c^2 - 1)} + \frac{\Pi_{SS} - \Pi_{PP}}{16 (N_c^2 - 1)}, & \mathcal{P}_3^{S=\pm1} &\equiv \mp\frac{\Pi_{VA} - \Pi_{AV}}{32 N_c (N_c^2 - 1)} - \frac{\Pi_{SP} - \Pi_{PS}}{16 (N_c^2 - 1)}, \\
P_4^{S=\pm1} &\equiv +\frac{\Pi_{SS} + \Pi_{PP}}{\frac{32 N_c (N_c^2 - 1)}{2N_c \pm 1}} \mp \frac{\Pi_{TT}}{32 N_c (N_c^2 - 1)}, & \mathcal{P}_4^{S=\pm1} &\equiv +\frac{\Pi_{SP} + \Pi_{PS}}{\frac{32 N_c (N_c^2 - 1)}{2N_c \pm 1}} \mp \frac{\Pi_{T\tilde{T}}}{32 N_c (N_c^2 - 1)}, \\
P_5^{S=\pm1} &\equiv \mp\frac{\Pi_{SS} + \Pi_{PP}}{32 N_c (N_c^2 - 1)} + \frac{\Pi_{TT}}{\frac{96 N_c (N_c^2 - 1)}{2N_c \mp 1}}, & \mathcal{P}_5^{S=\pm1} &\equiv \mp\frac{\Pi_{SP} + \Pi_{PS}}{32 N_c (N_c^2 - 1)} + \frac{\Pi_{T\tilde{T}}}{\frac{96 N_c (N_c^2 - 1)}{2N_c \mp 1}},
\end{aligned}$$

where $\Pi_{XY} \equiv (X_{i_2 i_1} \otimes Y_{i_4 i_3})\delta_{a_2 a_1}\delta_{a_4 a_3}$. Again, summation is implied over all independent Lorentz indices (if any) of the Dirac matrices. The above Projectors are chosen to obey the following orthogonality conditions:

$$\mathrm{Tr}(P_l^{S=\pm1} \cdot L_{m\,(tree)}^{S=\pm1}) = \delta_{lm}, \quad \mathrm{Tr}(\mathcal{P}_l^{S=\pm1} \cdot \mathcal{L}_{m\,(tree)}^{S=\pm1}) = \delta_{lm}, \tag{27}$$

where the trace is taken over spin and color indices, and $L_{(tree)}^{S=\pm1}$, $\mathcal{L}_{(tree)}^{S=\pm1}$ are the tree-level amputated Green's functions of the operators $Q^{S=\pm1}$, $\mathcal{Q}^{S=\pm1}$ respectively.

Consistently with the RI′ schemes, one may impose the renormalization conditions:

$$\mathrm{Tr}(P_l^{S=\pm1} \cdot \hat{L}_m^{S=\pm1}) = \delta_{lm}, \quad \mathrm{Tr}(\mathcal{P}_l^{S=\pm1} \cdot \hat{\mathcal{L}}_m^{S=\pm1}) = \delta_{lm}. \tag{28}$$

These conditions should be imposed at a given renormalization scale, μ. Note, however, that due to the presence of Lorentz non-invariant quantities, such as $\sum_\rho p_\rho^4$, which enter the Greens functions at $\mathcal{O}(a^2)$ and beyond, the renormalization factors computed in the RI′-MOM scheme are also affected through finite cutoff effects by the choice of the direction for the external momentum.

By inserting Eqs. (26) in the above relations, we obtain the renormalization matrices $Z^{S=\pm1}$, $\mathcal{Z}^{S=\pm1}$ in terms of known quantities:

$$Z^{S=\pm1} = Z_\Psi^2 \left[\left(D^{S=\pm1} \right)^T \right]^{-1}, \quad \mathcal{Z}^{S=\pm1} = Z_\Psi^2 \left[\left(\mathcal{D}^{S=\pm1} \right)^T \right]^{-1}, \tag{29}$$

where:

$$D_{lm}^{S=\pm1} \equiv \mathrm{Tr}(P_l^{S=\pm1} \cdot L_m^{S=\pm1}), \quad \mathcal{D}_{lm}^{S=\pm1} \equiv \mathrm{Tr}(\mathcal{P}_l^{S=\pm1} \cdot \mathcal{L}_m^{S=\pm1}). \tag{30}$$

Note that $D^{S=\pm1}$ and $\mathcal{D}^{S=\pm1}$ have the same matrix structure as $Z^{S=\pm1}$ and $\mathcal{Z}^{S=\pm1}$ respectively. For convenience we express them as:

$$D^{S=\pm1} = 1\!\!1 + \frac{g^2}{16\,\pi^2} \begin{pmatrix} d_{11}^\pm & d_{12}^\pm & d_{13}^\pm & d_{14}^\pm & d_{15}^\pm \\ d_{21}^\pm & d_{22}^\pm & d_{23}^\pm & d_{24}^\pm & d_{25}^\pm \\ d_{31}^\pm & d_{32}^\pm & d_{33}^\pm & d_{34}^\pm & d_{35}^\pm \\ d_{41}^\pm & d_{42}^\pm & d_{43}^\pm & d_{44}^\pm & d_{45}^\pm \\ d_{51}^\pm & d_{52}^\pm & d_{53}^\pm & d_{54}^\pm & d_{55}^\pm \end{pmatrix} + \mathcal{O}(g^4) \tag{31}$$

$$\mathcal{D}^{S=\pm 1} = 1\!\!1 + \frac{g^2}{16\,\pi^2} \begin{pmatrix} \delta^{\pm}_{11} & 0 & 0 & 0 & 0 \\ 0 & \delta^{\pm}_{22} & \delta^{\pm}_{23} & 0 & 0 \\ 0 & \delta^{\pm}_{32} & \delta^{\pm}_{33} & 0 & 0 \\ 0 & 0 & 0 & \delta^{\pm}_{44} & \delta^{\pm}_{45} \\ 0 & 0 & 0 & \delta^{\pm}_{54} & \delta^{\pm}_{55} \end{pmatrix} + \mathcal{O}(g^4) \tag{32}$$

In the parity violating case, as explained in Ref. [8] (Section 5.3), an equality holds between two pairs of matrix elements:

$$\delta^{+}_{22} = +\delta^{-}_{22}, \tag{33}$$

$$\delta^{+}_{23} = -\delta^{-}_{23}, \tag{34}$$

$$\delta^{+}_{32} = -\delta^{-}_{32}, \tag{35}$$

$$\delta^{+}_{33} = +\delta^{-}_{33}. \tag{36}$$

In addition, for the parity conserving projection the matrix elements d^{+}_{53}, d^{-}_{53} give zero at the 1-loop of perturbative theory:

$$d^{+}_{53} = 0, \tag{37}$$

$$d^{-}_{53} = 0. \tag{38}$$

The matrix elements of Eqs. (31)-(32) have the following simple and generic form:

$$\begin{aligned} d^{\pm}_{l,m} &= d^{\pm(0,1)}_{l,m} + c_{\rm SW}\, d^{\pm(0,2)}_{l,m} + c^2_{\rm SW}\, d^{\pm(0,3)}_{l,m} + \lambda\, d^{\pm(0,4)}_{l,m} + \big(d^{\pm(0,5)}_{l,m} + \lambda\, d^{\pm(0,6)}_{l,m}\big) \ln(a^2p^2) \\ &+ a^2 \Big[p^2 \big(d^{\pm(2,3)}_{l,m} + c_{\rm SW}\, d^{\pm(2,4)}_{l,m} + c^2_{\rm SW}\, d^{\pm(2,5)}_{l,m} + \lambda\, d^{\pm(2,6)}_{l,m}\big) \\ &\quad + p^2 \ln(a^2p^2)\big(d^{\pm(2,7)}_{l,m} + c_{\rm SW}\, d^{\pm(2,8)}_{l,m} + c^2_{\rm SW}\, d^{\pm(2,9)}_{l,m} + \lambda\, d^{\pm(2,10)}_{l,m}\big) \\ &\quad + \frac{\sum_\mu p_\mu^4}{p^2}\big(d^{\pm(2,1)}_{l,m} + \lambda\, d^{\pm(2,2)}_{l,m}\big)\Big] + \mathcal{O}(a^3), \end{aligned} \tag{39}$$

$$\begin{aligned} \delta^{\pm}_{l,m} &= \delta^{\pm(0,1)}_{l,m} + c_{\rm SW}\, \delta^{\pm(0,2)}_{l,m} + c^2_{\rm SW}\, \delta^{\pm(0,3)}_{l,m} + \lambda\, \delta^{\pm(0,4)}_{l,m} + \big(\delta^{\pm(0,5)}_{l,m} + \lambda\, \delta^{\pm(0,6)}_{l,m}\big) \ln(a^2p^2) \\ &+ a^2 \Big[p^2 \big(\delta^{\pm(2,3)}_{l,m} + c_{\rm SW}\, \delta^{\pm(2,4)}_{l,m} + c^2_{\rm SW}\, \delta^{\pm(2,5)}_{l,m} + \lambda\, \delta^{\pm(2,6)}_{l,m}\big) \\ &\quad + p^2 \ln(a^2p^2)\big(\delta^{\pm(2,7)}_{l,m} + c_{\rm SW}\, \delta^{\pm(2,8)}_{l,m} + c^2_{\rm SW}\, \delta^{\pm(2,9)}_{l,m} + \lambda\, \delta^{\pm(2,10)}_{l,m}\big) \\ &\quad + \frac{\sum_\mu p_\mu^4}{p^2}\big(\delta^{\pm(2,1)}_{l,m} + \lambda\, \delta^{\pm(2,2)}_{l,m}\big)\Big] + \mathcal{O}(a^3). \end{aligned} \tag{40}$$

The quantities $d_{l,m}^{\pm(i,j)}$ and $\delta_{l,m}^{\pm(i,j)}$ appearing above are numerical coefficients depending on the number of colors N_c and the Symanzik parameters for each gluon action we have considered; the index i denotes the power of the lattice spacing a that they multiply. Due to extremely lengthy results we provide the quantities $d_{l,m}^{\pm(i,j)}$, $\delta_{l,m}^{\pm(i,j)}$ (Tables I - VIII) only for the special choices: $N_c = 3$, $r_s = r_d = r_{s'} = r_{d'} = 1$, and tree-level Symanzik improved action. In all Tables the systematic errors in parentheses come from the extrapolation ($L \to \infty$) over finite lattice sizes. The full set of results is provided in the distribution package of this paper as an ASCII file named: 4-fermi.m; the file is best perused as Mathematica input; for notation see Appendix B.

The perturbative renormalization constants (Eqs. (29)) can be computed directly using the $\mathcal{O}(a^0)$ coefficients: $d_{l,m}^{\pm(0,j)}$ and $\delta_{l,m}^{\pm(0,j)}$ (first line of Eqs. (40) - (41)). The renormalization factor of the fermion field, Z_q, is also required and in the RI′-MOM scheme it reads [3]

$$Z_q = 1 + \frac{g^2\, C_F}{16\,\pi^2}\Big[\epsilon^{(1)} + c_{\rm SW}\,\epsilon^{(2)} + c_{\rm SW}^2\,\epsilon^{(3)} + \lambda\,\epsilon^{(4)} - \lambda\,\ln(a^2 p^2)\Big]\,, \tag{41}$$

where for the tree-level Symanzik improved action

$$\epsilon^{(1)} = -13.0232725(2)\,, \tag{42}$$

$$\epsilon^{(2)} = 1.242202721(2)\,, \tag{43}$$

$$\epsilon^{(3)} = 2.01542508(3)\,, \tag{44}$$

$$\epsilon^{(4)} = 4.79200964(9)\,. \tag{45}$$

As an example, we provide the exact expression for $\mathcal{Z}_{VA+AV}^{S=\pm1}$, up to 1-loop approximation:

$$\begin{aligned}\mathcal{Z}_{VA+AV}^{S=+1} = 1 \; - \; & \frac{g^2}{16\,\pi^2}\left[\delta_{1,1}^{+(0,1)} + c_{\rm SW}\,\delta_{1,1}^{+(0,2)} + c_{\rm SW}^2\,\delta_{1,1}^{+(0,3)} + \lambda\,\delta_{1,1}^{+(0,4)} + 2\,\ln(a^2p^2)\right] \\ & + \frac{g^2\,C_F}{16\,\pi^2}\,2\left[\epsilon^{(1)} + c_{\rm SW}\,\epsilon^{(2)} + c_{\rm SW}^2\,\epsilon^{(3)} + \lambda\,\epsilon^{(4)} - \lambda\,\ln(a^2p^2)\right].\end{aligned} \tag{46}$$

This costant is the one relevant for the renormalization of B_K in the twisted mass/Osterwalder-Seiler approach [6], implemented in Ref. [4][2]. For the tree-level Symanzik improved action adopted in the calculation of Ref. [6], with $c_{\rm SW} = 0$ and in the Landau gauge, Eq. (46) reads:

$$Z_{VA+AV} = 1 - \frac{g^2}{16\pi^2}\left[2\ln(a^2p^2) + 42.3359\right]. \tag{47}$$

While the RI′-MOM scheme allows for a non-perturbative renormalization procedure of the lattice operators, the Wilson coefficients entering the effective weak Hamiltonian for neutral meson mixing, both in the Standard Model and beyond, are often computed in the $\overline{\rm MS}$ scheme. For convenience, we then also provide here the formulae relating the operators renormalized in the RI′-MOM to those renormalized in the $\overline{\rm MS}$ scheme, at the next-to-leading order (i.e. 1-loop). This relation does not depend on the chosen regularization and it may be conveniently computed using, for instance, continuum dimensional regularization.

We restrict our attention to the $\Delta F = 2$ Parity Conserving operators, which are relevant for neutral meson mixing. These are the 5 operators $Q^{S=+1}_{1,\ldots,5}$ of Eq.(23) with $S = +1$.[3] For these operators, the conversion from the RI′-MOM to the $\overline{\rm MS}$ scheme can be written in the form

$$\left(Q^{S=+1}\right)^{\overline{\rm MS}}_{l} = \left(1 + \frac{g^2}{16\pi^2}\Delta r\right)_{lm}\left(Q^{S=+1}\right)^{\rm RI'-MOM}_{m} \tag{48}$$

where Δr is a 5×5 matrix. This matrix is independent of the choice of the regularization, i.e. it is the same for instance for continuum dimensional regularization and for the lattice regularization. The chiral symmetry of QCD also implies that the same matrix Δr is also valid for the Parity Violating sector (though the operators $\mathcal{Q}_2$ and $\mathcal{Q}_2$ vanish in the $\Delta F = 2$

[2] In Ref. [4], indeed, the lattice regularization chosen for the valence quarks s, s', d, d' is maximally twisted Wilson fermions with $r_s = r_{s'} = r_d = -r_{d'}$. In this case, the parity conserving operator $Q_1 = O_{VV+AA}$, besides being free from wrong chirality mixings, admits the same renormalization constant (here called $\mathcal{Z}_{VA+AV}$) as the operator $\mathcal{Q}_1 = O_{VA+AV}$ regularized with untwisted Wilson quarks (i.e. with $r_s = r_{s'} = r_d = r_{d'} + 1$).

[3] Note that in the $\Delta F = 2$ case the operators with $S = -1$ vanish identically, since $O^F_{XY} = O_{XY}$. Moreover, there are only three Parity Violating operators, since $\mathcal{Q}^{S=+1}_2$ and $\mathcal{Q}^{S=+1}_3$ also vanish.

case).

When dealing with four-fermion operators, the (modified) minimal subtraction prescription in dimensional regularization is not sufficient however to univocally specify the renormalization scheme. Different $\overline{\rm MS}$ schemes can be defined, which differ for the definition of the so called evanescent operators. The scheme usually adopted in the analysis of $K-\bar{K}$ mixing is the $\overline{\rm MS}$ scheme defined for instance in Ref. [11], for which the 1-loop conversion matrix Δr of Eq. (48) reads:

$$\Delta r = \begin{pmatrix} -\frac{14}{3}+8\ln 2 & 0 & 0 & 0 & 0 \\ 0 & -\frac{2}{3}-\frac{2}{3}\ln 2 & -4-4\ln 2 & 0 & 0 \\ 0 & 1-\ln 2 & \frac{34}{3}-\frac{2}{3}\ln 2 & 0 & 0 \\ 0 & 0 & 0 & \frac{10}{3}+\frac{10}{3}\ln 2 & -\frac{1}{18}+\frac{7}{18}\ln 2 \\ 0 & 0 & 0 & \frac{56}{3}\ln 2 & -\frac{16}{9}+\frac{58}{9}\ln 2 \end{pmatrix} . \tag{49}$$

For $B-\bar{B}$ mixing, instead, the $\overline{\rm MS}$ scheme of Ref. [12] is more commonly adopted. The corresponding matrix Δr differ from the one given in Eq. (49) only in the $Q_{4,5}^{S=+1}$ sector, which in this case reads

$$(\Delta r)_{Q_4-Q5} = \begin{pmatrix} \frac{43}{6}+\frac{10}{3}\ln 2 & -\frac{7}{72}+\frac{7}{18}\ln 2 \\ \frac{58}{3}+\frac{56}{3}\ln 2 & -\frac{65}{18}+\frac{58}{9}\ln 2 \end{pmatrix} . \tag{50}$$

In order to correct, to $\mathcal{O}(a^2)$, non-perturbative estimates for the renormalization constants of 4-fermion operators one should take into account the $\mathcal{O}(a^2)$ corrections of Eqs. (40) - (41), as well as the $\mathcal{O}(a^2)$ terms of the fermion propagator [1]. The exact terms that need to be subtracted from the non-perturbative Z_q, computed in the RI′-MOM scheme are provided in Ref. [3] for general action parameters and in Ref. [2] for tree-level Symanzik improved gluons, $c_{\rm SW}=0$, Landau gauge.

IV. CONCLUSION

The calculations presented regard all 4-fermion operators of the form: $\bar{s}\, X\, d\, \bar{s}'\, Y\, d'$ where X, Y are generic Dirac matrices. Our results have explicit dependence on: p, a, g, $c_{\rm SW}$, λ, N_c, and implicit depencence on the Symanzik parameters, c_i. Thus, the numerical results are presented for a selection of currently used values of c_i.

In a recent paper [4] the present results on the perturbative renormalization of $\Delta F = 2$ operators have been combined with numerical simulation data in order to determine non-perturbative renormalization coefficients with better precision. This allowed us to extract physical values for B_K with reduced lattice artifacts.

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Appendix A: Analytic expressions

In general, the final 1-loop expressions for $\Lambda^{XY}_{d_1}$, $\Lambda^{XY}_{d_2}$ and $\Lambda^{XY}_{d_3}$, up to $\mathcal{O}(a^2)$, are obtained as a function of: the coupling constant g, clover parameter c_{SW}, number of colors N_c, lattice spacing a, external momentum p, and gauge parameter λ. The specific values $\lambda = 1\,(0)$ correspond to the Feynman (Landau) gauge. Here we present the results for $\Lambda^{XY}_{d_1}$, $\Lambda^{XY}_{d_2}$ and $\Lambda^{XY}_{d_3}$ for the special choices: $r_s = r_d = r_{s'} = r_{d'} = 1$, and tree-level Symanzik improved gluon action.

Diagram d1

$$\Lambda^{XY}_{d_1}(p)^{a_1\,a_2\,a_3\,a_4}_{i_1\,i_2\,i_3\,i_4} = \frac{g^2}{16\pi^2}\left(\delta_{a_1\,a_4}\delta_{a_3\,a_2} - \frac{\delta_{a_1\,a_2}\delta_{a_3\,a_4}}{N_c}\right)\times\left\{(\Lambda_{\mathcal{O}(a^0)})^{XY}_{d_1} + a^1(\Lambda_{\mathcal{O}(a^1)})^{XY}_{d_1} + a^2(\Lambda_{\mathcal{O}(a^2)})^{XY}_{d_1}\right\}, \quad \text{(A1)}$$

$$\begin{aligned}
(\Lambda_{\mathcal{O}(a^0)})^{XY}_{d_1} =\ & X_{i_1\,i_2}Y_{i_3\,i_4}\Big[-0.05294144(3) + 0.737558970(1)\,c_{\rm SW} + 0.238486988(3)\,c_{\rm SW}^2 \\
& \qquad -2.100573331(5)\,\lambda + \frac{1}{2}(-1+\lambda)\ln(a^2p^2)\Big] \\
& + \sum_\mu (X\gamma^\mu)_{i_1\,i_2}(Y\gamma^\mu)_{i_3\,i_4}\Big[-0.507914049(6) + 0.55316919(1)\,c_{\rm SW} - 0.194516637(3)\,c_{\rm SW}^2\Big] \\
& + \sum_{\mu,\nu} (X\gamma^\mu\gamma^\nu)_{i_1\,i_2}(Y\gamma^\mu\gamma^\nu)_{i_3\,i_4}\Big[0.018598520(2) - 0.1843897425(8)\,c_{\rm SW} - 0.0596217473(8)\,c_{\rm SW}^2 \\
& \qquad + \frac{1}{8}\ln(a^2p^2)\Big] \\
& + \sum_{\mu,\nu,\rho} (X\gamma^\mu\gamma^\rho)_{i_1\,i_2}(Y\gamma^\nu\gamma^\rho)_{i_3\,i_4}\Big[\frac{p_\mu p_\nu}{p^2}\Big(0.397715726853 + 0.147715726853\,\lambda\Big)\Big],
\end{aligned} \quad \text{(A2)}$$

$$\begin{aligned}
(\Lambda_{\mathcal{O}(a^1)})^{XY}_{d_1} =\ & \sum_\mu \Big(X_{i_1\,i_2}(Y\gamma^\mu)_{i_3\,i_4} + (X\gamma^\mu)_{i_1\,i_2}\,Y_{i_3\,i_4}\Big)\times\Big[ip_\mu\Big(0.09460083(1) - 0.065711182(4)\,c_{\rm SW} \\
& \qquad -0.059929106(1)\,c_{\rm SW}^2 + 0.438508366(3)\,\lambda \\
& \qquad + \frac{1}{4}(-1 + c_{\rm SW} - \lambda)\ln(a^2p^2)\Big)\Big] \\
& + \sum_{\mu,\nu} \Big((X\gamma^\mu\gamma^\nu)_{i_1\,i_2}(Y\gamma^\nu)_{i_3\,i_4} + (X\gamma^\nu)_{i_1\,i_2}(Y\gamma^\mu\gamma^\nu)_{i_3\,i_4}\Big)\times\Big[ip_\mu\Big(0.1692905881(6) + 0.010283104(5)\,c_{\rm SW} \\
& \qquad -0.0680031615(8)\,c_{\rm SW}^2 + 0.073857863427\,\lambda \\
& \qquad + \frac{1}{16}(1 + 3\,c_{\rm SW})\ln(a^2p^2)\Big)\Big] \\
& + \sum_{\mu,\nu,\rho} \Big((X\gamma^\mu\gamma^\nu\gamma^\rho)_{i_1\,i_2}(Y\gamma^\nu\gamma^\rho)_{i_3\,i_4} + (X\gamma^\nu\gamma^\rho)_{i_1\,i_2}(Y\gamma^\mu\gamma^\nu\gamma^\rho)_{i_3\,i_4}\Big)\times\Big[ip_\mu\Big(-0.0279443091(3)\,c_{\rm SW} \\
& \qquad +0.0319830668(5)\,c_{\rm SW}^2 \\
& \qquad -\frac{1}{16}\,c_{\rm SW}\ln(a^2p^2)\Big)\Big],
\end{aligned} \quad \text{(A3)}$$

$$
\begin{aligned}
(\Lambda_{\mathcal{O}(a^2)})^{XY}_{d_1} = X_{i_1\,i_2} Y_{i_3\,i_4} \Big[& p^2 \Big(1.32362251(5) - 0.43684285(3)\, c_{\rm SW} - 0.0208665277(5)\, c_{\rm SW}^2 \\
& + 0.64073441(3)\,\lambda + \frac{1}{72}(-17 + 9\, c_{\rm SW} - 9\,\lambda)\ln(a^2 p^2) \Big) \\
& + \frac{\sum_\sigma p_\sigma^4}{p^2} \Big(0.06213648(8) - 0.07400055(8)\,\lambda \Big) \Big] \\
+ \sum_\mu (X\gamma^\mu)_{i_1\,i_2} (Y\gamma^\mu)_{i_3\,i_4} \Big[& p^2 \Big(0.059895142(8) - 0.241755150(3)\, c_{\rm SW} + 0.114731816(7)\, c_{\rm SW}^2 \\
& - 0.036928931713\,\lambda + \frac{1}{48}(-7 + 11\, c_{\rm SW} - 4\, c_{\rm SW}^2)\ln(a^2 p^2) \Big) \\
& + p_\mu^2 \Big(1.01694823(2) - 0.44474062(1)\, c_{\rm SW} - 0.033265121(3)\, c_{\rm SW}^2 \Big) \Big] \\
+ \sum_{\mu,\nu} \Big((X\gamma^\mu\gamma^\nu)_{i_1\,i_2} Y_{i_3\,i_4} + X_{i_1\,i_2} (Y\gamma^\mu\gamma^\nu)_{i_3\,i_4} \Big) \times \Big[& \frac{p_\nu p_\mu^3}{p^2} \Big(0.00592406(2) - 0.00295805(2)\,\lambda \Big) \Big] \\
+ \sum_{\mu,\nu} (X\gamma^\mu)_{i_1\,i_2} (Y\gamma^\nu)_{i_3\,i_4} \Big[& p_\mu p_\nu \Big(- 0.19915360(1) + 0.212823513(3)\, c_{\rm SW} + 0.033028338(2)\, c_{\rm SW}^2 \\
& + 0.1600141922(8)\,\lambda + \frac{1}{24}(-4 + 5\, c_{\rm SW} + 2\, c_{\rm SW}^2 - 3\,\lambda)\ln(a^2 p^2) \Big) \Big] \\
+ \sum_{\mu,\nu} (X\gamma^\mu\gamma^\nu)_{i_1\,i_2} (Y\gamma^\mu\gamma^\nu)_{i_3\,i_4} \Big[& p^2 \Big(- 0.08962805(1) + 0.0769373498(3)\, c_{\rm SW} + 0.0067184623(3)\, c_{\rm SW}^2 \\
& + \frac{1}{240}(7 - 5\, c_{\rm SW})\ln(a^2 p^2) \Big) \\
& + p_\mu^2 \Big(+ 0.16608907(6) + 0.07446360(2)\, c_{\rm SW} - 0.0087763322(3)\, c_{\rm SW}^2 \\
& - \frac{29}{180}\ln(a^2 p^2) \Big) \\
& + \frac{\sum_\sigma p_\sigma^4}{p^2} \Big(- 0.048180849735 \Big) \Big] \\
+ \sum_{\mu,\nu,\rho} (X\gamma^\mu\gamma^\rho)_{i_1\,i_2} (Y\gamma^\nu\gamma^\rho)_{i_3\,i_4} \Big[& p_\mu p_\nu \Big(- 0.21865904(4) + 0.054629909(8)\, c_{\rm SW} + 0.00276900638(7)\, c_{\rm SW}^2 \\
& - 0.082411837(6)\,\lambda + \frac{1}{1440}(164 - 60\, c_{\rm SW} + 45\,\lambda)\ln(a^2 p^2) \Big) \\
& + \frac{(p_\mu^3 p_\nu + p_\mu p_\nu^3)}{p^2} \Big(- 0.110138789528 - 0.024619287809\,\lambda \Big) \\
& + p_\mu p_\nu \frac{\sum_\sigma p_\sigma^4}{(p^2)^2} \Big(0.140961390102 + 0.045240352404\,\lambda \Big) \\
& + \frac{p_\mu p_\nu p_\rho^2}{p^2} \Big(- 0.477634781(8) - 0.083831642(8)\,\lambda \Big) \Big] \\
+ \sum_{\mu,\nu,\rho} (X\gamma^\mu\gamma^\nu\gamma^\rho)_{i_1\,i_2} (Y\gamma^\mu\gamma^\nu\gamma^\rho)_{i_3\,i_4} \Big[& p_\mu^2 \Big(0.00385492408(3)\, c_{\rm SW}^2 \Big) \Big] \\
+ \sum (X\gamma^\mu\gamma^\rho\gamma^\sigma)_{i_1\,i_2} (Y\gamma^\nu\gamma^\rho\gamma^\sigma)_{i_3\,i_4} \Big[& p_\mu p_\nu \Big(- 0.0209503296(6)\, c_{\rm SW}^2 - \frac{1}{32}\, c_{\rm SW}^2 \ln(a^2 p^2) \Big) \Big] .
\end{aligned}
\tag{A4}
$$

Diagram d2

$$\Lambda^{XY}_{d_2}(p)^{a_1\,a_2\,a_3\,a_4}_{i_1\,i_2\,i_3\,i_4} = \frac{g^2}{16\pi^2}\delta_{a_1\,a_2}\delta_{a_3\,a_4}\left(N_c - \frac{1}{N_c}\right) \times \left\{(\Lambda_{\mathcal{O}(a^0)})^{XY}_{d_2} + a^1(\Lambda_{\mathcal{O}(a^1)})^{XY}_{d_2} + a^2(\Lambda_{\mathcal{O}(a^2)})^{XY}_{d_2}\right\}, \quad \text{(A5)}$$

$$\begin{aligned}
(\Lambda_{\mathcal{O}(a^0)})^{XY}_{d_2} &= X_{i_1\,i_2}Y_{i_3\,i_4}\Big[1.2904478(4) + 0.737558970(1)\,c_{\rm SW} + 0.238486988(3)\,c^2_{\rm SW} \\
&\qquad +2.3960046(4)\,\lambda + \frac{1}{2}(-1-\lambda)\ln(a^2p^2)\Big] \\
&+ \sum_{\mu} X_{i_1\,i_2}(\gamma^\mu Y\gamma^\mu)_{i_3\,i_4}\Big[-0.507914047(8) + 0.55316917(2)\,c_{\rm SW} - 0.194516638(9)\,c^2_{\rm SW}\Big] \\
&+ \sum_{\mu,\nu} X_{i_1\,i_2}(\gamma^\mu\gamma^\nu Y\gamma^\mu\gamma^\nu)_{i_3\,i_4}\Big[-0.129117207(2) - 0.1843897425(8)\,c_{\rm SW} - 0.0596217473(8)\,c^2_{\rm SW} \\
&\qquad + \frac{1}{8}\ln(a^2p^2)\Big] \\
&+ \sum_{\mu,\nu,\rho} X_{i_1\,i_2}(\gamma^\mu\gamma^\rho Y\gamma^\nu\gamma^\rho)_{i_3\,i_4}\Big[\frac{p_\mu p_\nu}{p^2}\Big(-\frac{1}{4}\lambda\Big)\Big],
\end{aligned} \quad \text{(A6)}$$

$$\begin{aligned}
(\Lambda_{\mathcal{O}(a^1)})^{XY}_{d_2} &= \sum_{\mu}\Big(X_{i_1\,i_2}(Y\gamma^\mu)_{i_3\,i_4} + X_{i_1\,i_2}(\gamma^\mu Y)_{i_3\,i_4}\Big) \times \Big[ip_\mu\Big(0.37785613(9) - 0.56675680(2)\,c_{\rm SW} \\
&\qquad -0.160026205(2)\,c^2_{\rm SW} - 0.48393977(9)\,\lambda \\
&\qquad +\frac{1}{8}(-3 + 5\,c_{\rm SW} + 2\,\lambda)\ln(a^2p^2)\Big)\Big] \\
&+ \sum_{\mu,\nu}\Big(X_{i_1\,i_2}(\gamma^\mu\gamma^\nu Y\gamma^\nu)_{i_3\,i_4} + X_{i_1\,i_2}(\gamma^\nu Y\gamma^\nu\gamma^\mu)_{i_3\,i_4}\Big) \times \Big[ip_\mu\Big(-0.073251555(1) - 0.001704761(4)\,c_{\rm SW} \\
&\qquad +0.032093938(2)\,c^2_{\rm SW} - \frac{1}{8}\lambda \\
&\qquad +\frac{1}{16}(3 - c_{\rm SW})\ln(a^2p^2)\Big)\Big] \\
&+ \sum_{\mu,\nu,\rho}\Big(X_{i_1\,i_2}(\gamma^\mu\gamma^\nu\gamma^\rho Y\gamma^\nu\gamma^\rho)_{i_3\,i_4} + X_{i_1\,i_2}(\gamma^\nu\gamma^\rho Y\gamma^\nu\gamma^\rho\gamma^\mu)_{i_3\,i_4}\Big) \times \Big[ip_\mu\Big(0.0459135542(5)\,c_{\rm SW} \\
&\qquad +0.0319830668(5)\,c^2_{\rm SW} \\
&\qquad -\frac{1}{16}\,c_{\rm SW}\ln(a^2p^2)\Big)\Big],
\end{aligned} \quad \text{(A7)}$$

$$
\begin{aligned}
(\Lambda_{\mathcal{O}(a^2)})^{XY}_{d_2} = {} & X_{i_1\,i_2} Y_{i_3\,i_4} \Big[p^2 \Big(0.7374671(6) - 0.24301094(4)\, c_{\rm SW} - 0.0096054476(8)\, c_{\rm SW}^2 \\
& \qquad -0.4696085(6)\, \lambda + \frac{1}{120}(-23 + 5\, c_{\rm SW} + 15\, \lambda) \ln(a^2 p^2) \Big) \\
& \qquad + \frac{\sum_\sigma p_\sigma^4}{p^2} \Big(\frac{1}{90}(-77 + 15\, \lambda) \Big) \Big] \\
& + \sum_\mu X_{i_1\,i_2} (\gamma^\mu Y \gamma^\mu)_{i_3\,i_4} \Big[p^2 \Big(0.04610701(4) - 0.19136171(3)\, c_{\rm SW} + 0.02347831(4)\, c_{\rm SW}^2 \\
& \qquad -0.06249999(1)\, \lambda + \frac{1}{16}(1 + c_{\rm SW}) \ln(a^2 p^2) \Big) \\
& \qquad + p_\mu^2 \Big(0.17251518(3) - 0.19211806(3)\, c_{\rm SW} + 0.01744902(2)\, c_{\rm SW}^2 \Big) \Big] \\
& + \sum_{\mu,\nu} \Big(X_{i_1\,i_2} (Y \gamma^\mu \gamma^\nu)_{i_3\,i_4} + X_{i_1\,i_2} (\gamma^\nu \gamma^\mu Y)_{i_3\,i_4} \Big) \times \Big[\frac{p_\nu p_\mu^3}{p^2} \Big(\frac{101}{288} \Big) \Big] \\
& + \sum_{\mu,\nu} X_{i_1\,i_2} (\gamma^\mu Y \gamma^\nu)_{i_3\,i_4} \Big[p_\mu p_\nu \Big(0.05513763(3) - 0.005630284(8)\, c_{\rm SW} - 0.072690409(7)\, c_{\rm SW}^2 \\
& \qquad -0.10887203(3)\, \lambda + \frac{1}{8}(-2 + c_{\rm SW} + c_{\rm SW}^2 + \lambda) \ln(a^2 p^2) \Big) \Big] \\
& + \sum_{\mu,\nu} X_{i_1\,i_2} (\gamma^\mu \gamma^\nu Y \gamma^\mu \gamma^\nu)_{i_3\,i_4} \Big[p^2 \Big(-0.05064893(1) + 0.0394316274(6)\, c_{\rm SW} + 0.00332360968(9)\, c_{\rm SW}^2 \\
& \qquad + \frac{1}{720}(13 - 15\, c_{\rm SW}) \ln(a^2 p^2) \Big) \\
& \qquad + p_\mu^2 \Big(+0.05383442(9) + 0.124311493(7)\, c_{\rm SW} - 0.0051958638(1)\, c_{\rm SW}^2 \\
& \qquad - \frac{1}{15} \ln(a^2 p^2) \Big) \\
& \qquad + \frac{\sum_\sigma p_\sigma^4}{p^2} \Big(-\frac{1}{240} \Big) \Big] \\
& + \sum_{\mu,\nu,\rho} X_{i_1\,i_2} (\gamma^\mu \gamma^\rho Y \gamma^\rho \gamma^\nu)_{i_3\,i_4} \Big[p_\mu p_\nu \Big(-0.03270359(5) - 0.039026988(4)\, c_{\rm SW} + 0.0015068706(2)\, c_{\rm SW}^2 \\
& \qquad -0.06926696(4)\, \lambda + \frac{1}{1440}(28 + 60\, c_{\rm SW} + 45\, \lambda) \ln(a^2 p^2) \Big) \\
& \qquad + \frac{(p_\mu^3 p_\nu + p_\mu p_\nu^3)}{p^2} \Big(\frac{1}{960}(41 - 40\, \lambda) \Big) \\
& \qquad + p_\mu p_\nu \frac{\sum_\sigma p_\sigma^4}{(p^2)^2} \Big(\frac{1}{960}(7 + 25\, \lambda) \Big) \\
& \qquad + \frac{p_\mu p_\nu p_\rho^2}{p^2} \Big(\frac{1}{288}(-40 - 9\, \lambda) \Big) \Big] \\
& + \sum_{\mu,\nu,\rho} X_{i_1\,i_2} (\gamma^\mu \gamma^\nu \gamma^\rho Y \gamma^\mu \gamma^\nu \gamma^\rho)_{i_3\,i_4} \Big[p_\mu^2 \Big(0.00385492795(2)\, c_{\rm SW}^2 \Big) \Big] \\
& + \sum X_{i_1\,i_2} (\gamma^\mu \gamma^\rho \gamma^\sigma Y \gamma^\nu \gamma^\rho \gamma^\sigma)_{i_3\,i_4} \Big[p_\mu p_\nu \Big(0.015978597(1)\, c_{\rm SW}^2 - \frac{1}{32}\, c_{\rm SW}^2 \ln(a^2 p^2) \Big) \Big].
\end{aligned}
\tag{A8}
$$

Diagram d3

$$\Lambda^{XY}_{d_3}(p)^{a_1\,a_2\,a_3\,a_4}_{i_1\,i_2\,i_3\,i_4} = \frac{g^2}{16\pi^2}\left(\delta_{a_1\,a_4}\delta_{a_3\,a_2} - \frac{\delta_{a_1\,a_2}\delta_{a_3\,a_4}}{N_c}\right)\times\left\{(\Lambda_{\mathcal{O}(a^0)})^{XY}_{d_3} + a^1(\Lambda_{\mathcal{O}(a^1)})^{XY}_{d_3} + a^2(\Lambda_{\mathcal{O}(a^2)})^{XY}_{d_3}\right\}, \quad \text{(A9)}$$

$$\begin{aligned}
(\Lambda_{\mathcal{O}(a^0)})^{XY}_{d_3} &= X_{i_1\,i_2}Y_{i_3\,i_4}\Big[1.2904478(4) + 0.737558970(1)\,c_{\rm SW} + 0.238486988(3)\,c_{\rm SW}^2 \\
&\qquad +2.3960046(4)\,\lambda + \frac{1}{2}(-1-\lambda)\ln(a^2p^2)\Big] \\
&+\sum_{\mu}(X\gamma^\mu)_{i_1\,i_2}(\gamma^\mu Y)_{i_3\,i_4}\Big[-0.507914047(8) + 0.55316917(2)\,c_{\rm SW} - 0.194516638(9)\,c_{\rm SW}^2\Big] \\
&+\sum_{\mu,\nu}(X\gamma^\mu\gamma^\nu)_{i_1\,i_2}(\gamma^\mu\gamma^\nu Y)_{i_3\,i_4}\Big[-0.129117207(2) - 0.1843897425(8)\,c_{\rm SW} - 0.0596217473(8)\,c_{\rm SW}^2 \\
&\qquad +\frac{1}{8}\ln(a^2p^2)\Big] \\
&+\sum_{\mu,\nu,\rho}(X\gamma^\mu\gamma^\rho)_{i_1\,i_2}(\gamma^\nu\gamma^\rho Y)_{i_3\,i_4}\Big[\frac{p_\mu p_\nu}{p^2}\Big(-\frac{1}{4}\lambda\Big)\Big],
\end{aligned} \quad \text{(A10)}$$

$$\begin{aligned}
(\Lambda_{\mathcal{O}(a^1)})^{XY}_{d_3} &= \sum_{\mu}\Big((X\gamma^\mu)_{i_1\,i_2}Y_{i_3\,i_4} + X_{i_1\,i_2}(\gamma^\mu Y)_{i_3\,i_4}\Big)\times\Big[ip_\mu\Big(0.37785613(9) - 0.56675680(2)\,c_{\rm SW} \\
&\qquad -0.160026205(2)\,c_{\rm SW}^2 - 0.48393977(9)\,\lambda \\
&\qquad +\frac{1}{8}(-3+5\,c_{\rm SW}+2\,\lambda)\ln(a^2p^2)\Big)\Big] \\
&+\sum_{\mu,\nu}\Big((X\gamma^\nu)_{i_1\,i_2}(\gamma^\mu\gamma^\nu Y)_{i_3\,i_4} + (X\gamma^\nu\gamma^\mu)_{i_1\,i_2}(\gamma^\nu Y)_{i_3\,i_4}\Big)\times\Big[ip_\mu\Big(-0.073251555(1) - 0.001704761(4)\,c_{\rm SW} \\
&\qquad +0.032093938(2)\,c_{\rm SW}^2 - \frac{1}{8}\lambda \\
&\qquad +\frac{1}{16}(3-c_{\rm SW})\ln(a^2p^2)\Big)\Big] \\
&+\sum_{\mu,\nu,\rho}\Big((X\gamma^\mu\gamma^\nu\gamma^\rho)_{i_1\,i_2}(\gamma^\nu\gamma^\rho Y)_{i_3\,i_4} + (X\gamma^\nu\gamma^\rho)_{i_1\,i_2}(\gamma^\nu\gamma^\rho\gamma^\mu Y)_{i_3\,i_4}\Big)\times\Big[ip_\mu\Big(0.0459135542(5)\,c_{\rm SW} \\
&\qquad +0.0319830668(5)\,c_{\rm SW}^2 \\
&\qquad -\frac{1}{16}c_{\rm SW}\ln(a^2p^2)\Big)\Big],
\end{aligned} \quad \text{(A11)}$$

$$
\begin{aligned}
(\Lambda_{\mathcal{O}(a^2)})^{XY}_{d_3} = X_{i_1\,i_2} Y_{i_3\,i_4} \Big[p^2 \Big(& 0.7374671(6) - 0.24301094(4)\, c_{\rm SW} - 0.0096054476(8)\, c_{\rm SW}^2 \\
& -0.4696085(6)\,\lambda + \frac{1}{120}(-23 + 5\, c_{\rm SW} + 15\,\lambda) \ln(a^2 p^2) \Big) \\
& + \frac{\sum_\sigma p_\sigma^4}{p^2} \Big(\frac{1}{90}(-77 + 15\,\lambda) \Big) \Big] \\
+ \sum_\mu (X\gamma^\mu)_{i_1\,i_2} (\gamma^\mu Y)_{i_3\,i_4} \Big[p^2 \Big(& 0.04610701(4) - 0.19136171(3)\, c_{\rm SW} + 0.02347831(4)\, c_{\rm SW}^2 \\
& -0.06249999(1)\,\lambda + \frac{1}{16}(1 + \, c_{\rm SW}) \ln(a^2 p^2) \Big) \\
& + p_\mu^2 \Big(0.17251518(3) - 0.19211806(3)\, c_{\rm SW} + 0.01744902(2)\, c_{\rm SW}^2 \Big) \Big] \\
+ \sum_{\mu,\nu} \Big((X\gamma^\mu\gamma^\nu)_{i_1\,i_2} Y_{i_3\,i_4} + X_{i_1\,i_2} (\gamma^\nu\gamma^\mu Y)_{i_3\,i_4} \Big) & \times \Big[\frac{p_\nu p_\mu^3}{p^2} \Big(\frac{101}{288} \Big) \Big] \\
+ \sum_{\mu,\nu} (X\gamma^\mu)_{i_1\,i_2} (\gamma^\nu Y)_{i_3\,i_4} \Big[p_\mu p_\nu \Big(& 0.05513763(3) - 0.005630284(8)\, c_{\rm SW} - 0.072690409(7)\, c_{\rm SW}^2 \\
& -0.10887203(3)\,\lambda + \frac{1}{8}(-2 + \, c_{\rm SW} + \, c_{\rm SW}^2 + \lambda) \ln(a^2 p^2) \Big) \Big] \\
+ \sum_{\mu,\nu} (X\gamma^\mu\gamma^\nu)_{i_1\,i_2} (\gamma^\mu\gamma^\nu Y)_{i_3\,i_4} \Big[p^2 \Big(& -0.05064893(1) + 0.0394316274(6)\, c_{\rm SW} + 0.00332360968(9)\, c_{\rm SW}^2 \\
& + \frac{1}{720}(13 - 15\, c_{\rm SW}) \ln(a^2 p^2) \Big) \\
& + p_\mu^2 \Big(+ 0.05383442(9) + 0.124311493(7)\, c_{\rm SW} - 0.0051958638(1)\, c_{\rm SW}^2 \\
& - \frac{1}{15} \ln(a^2 p^2) \Big) \\
& + \frac{\sum_\sigma p_\sigma^4}{p^2} \Big(-\frac{1}{240} \Big) \Big] \\
+ \sum_{\mu,\nu,\rho} (X\gamma^\rho\gamma^\mu)_{i_1\,i_2} (\gamma^\nu\gamma^\rho Y)_{i_3\,i_4} \Big[p_\mu p_\nu \Big(& -0.03270359(5) - 0.039026988(4)\, c_{\rm SW} + 0.0015068706(2)\, c_{\rm SW}^2 \\
& -0.06926696(4)\,\lambda + \frac{1}{1440}(28 + 60\, c_{\rm SW} + 45\,\lambda) \ln(a^2 p^2) \Big) \\
& + \frac{(p_\mu^3 p_\nu + p_\mu p_\nu^3)}{p^2} \Big(\frac{1}{960}(41 - 40\,\lambda) \Big) \\
& + p_\mu p_\nu \frac{\sum_\sigma p_\sigma^4}{(p^2)^2} \Big(\frac{1}{960}(7 + 25\,\lambda) \Big) \\
& + \frac{p_\mu p_\nu p_\rho^2}{p^2} \Big(\frac{1}{288}(-40 - 9\,\lambda) \Big) \Big] \\
+ \sum_{\mu,\nu,\rho} (X\gamma^\mu\gamma^\nu\gamma^\rho)_{i_1\,i_2} (\gamma^\mu\gamma^\nu\gamma^\rho Y)_{i_3\,i_4} & \Big[p_\mu^2 \Big(0.00385492795(2)\, c_{\rm SW}^2 \Big) \Big] \\
+ \sum (X\gamma^\mu\gamma^\rho\gamma^\sigma)_{i_1\,i_2} (\gamma^\nu\gamma^\rho\gamma^\sigma Y)_{i_3\,i_4} & \Big[p_\mu p_\nu \Big(0.015978597(1)\, c_{\rm SW}^2 - \frac{1}{32}\, c_{\rm SW}^2 \ln(a^2 p^2) \Big) \Big]. \qquad \text{(A12)}
\end{aligned}
$$

Appendix B: Notation in ASCII file: 4-fermi.m

The full body of our results can be accessed online through the file 4-fermi.m, which is a Mathematica input file. It includes the expressions for the three Feynman diagrams:

- $\Lambda^{XY}_{d_1}$: d1[action,csw,lambda,Nc,g,aL]
- $\Lambda^{XY}_{d_2}$: d2[action,csw,lambda,Nc,g,aL]
- $\Lambda^{XY}_{d_3}$: d3[action,csw,lambda,Nc,g,aL]

from which one can construct the matrix elements of any 4-fermion operator of the above form. Each expression depends on the variables:

- action: Selection of improved gauge action as follows, 1 $\rightarrow$ Plaquette, 2 $\rightarrow$ Tree Level Symanzik, 3 $\rightarrow$ TILW ($\beta\, c_0 = 8.60$), 4 $\rightarrow$ TILW ($\beta\, c_0 = 8.45$), 5 $\rightarrow$ TILW ($\beta\, c_0 = 8.30$), 6 $\rightarrow$ TILW ($\beta\, c_0 = 8.20$), 7 $\rightarrow$ TILW ($\beta\, c_0 = 8.10$), 8 $\rightarrow$ TILW ($\beta\, c_0 = 8.00$), 9 $\rightarrow$ Iwasaki, 10 $\rightarrow$ DBW2
- csw: clover parameter
- lambda: gauge parameter (Landau/Feynman/Generic correspond to 0/1/lambda)
- Nc: number of colors
- g: coupling constant
- aL: lattice spacing

In particular, the quantities of interest in that file are the renornalization matrices for the 10 Parity Conserving operators, and 10 Parity Violating operators, which read:

- $Z^{S=+1}$: PCplus[action,csw,lambda,Nc,g,aL,p2,p4][Projector,LGreen]
- $Z^{S=-1}$: PCminus[action,csw,lambda,Nc,g,aL,p2,p4][Projector,LGreen]
- $\mathcal{Z}^{S=+1}$: PVplus[action,csw,lambda,Nc,g,aL,p2,p4][Projector,LGreen]
- $\mathcal{Z}^{S=-1}$: PVminus[action,csw,lambda,Nc,g,aL,p2,p4][Projector,LGreen] .

The additinal variables are

- p2: $\sum_{i=1}^{4} p_i^2$

- p4: $\sum_{i=1}^{4} p_i^4$
- Projector: the index l of Section III (1 to 5)
- LGreen: the index m of Section III (1 to 5)

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Parity Conserving, Flavour Exchange Symmetry Plus

$(l,\, m)$	$d^{+(0,1)}_{l,m}$	$d^{+(0,2)}_{l,m}$	$d^{+(0,3)}_{l,m}$	$d^{+(0,4)}_{l,m}$	$d^{+(0,5)}_{l,m}$	$d^{+(0,6)}_{l,m}$	$d^{+(2,1)}_{l,m}$	$d^{+(2,2)}_{l,m}$
(1, 1)	7.607190(2)	-2.95023588(3)	-0.95394796(3)	12.293750(2)	2	-8/3	-2.79899092(5)	0.39295395(5)
(1, 2)	5.41774985(8)	-5.9004712(3)	2.0748441(1)	0	0	0	0	0
(1, 3)	-0.67721873(1)	0.73755889(5)	-0.25935552(2)	0	0	0	0	0
(1, 4)	-0.67721873(1)	0.73755892(2)	-0.259355516(6)	0	0	0	0	0
(1, 5)	-2.03165620(4)	2.21267677(7)	-0.77806655(2)	0	0	0	0	0
(2, 1)	7.4494060(1)	-8.1131478(4)	2.8529107(1)	0	0	0	0	0
(2, 2)	4.331175(2)	-3.68779471(5)	0.30109257(1)	10.816593(2)	1	-8/3	-0.87361423(2)	0.57016434(2)
(2, 3)	-0.8545378(4)	-0.368779461(8)	0.129677758(2)	-0.1931470(4)	0	0	0.89270365(2)	0.08962151(2)
(2, 4)	0.338609366(5)	-0.36877945(2)	0.129677758(6)	0	0	0	0	0
(2, 5)	-1.01582810(1)	1.10633834(5)	-0.38903328(2)	0	0	0	0	0
(3, 1)	1.35443746(2)	-1.47511779(6)	0.51871103(2)	0	0	0	0	0
(3, 2)	-9.615778(1)	-10.32582548(4)	-2.34313283(3)	-3.772588(1)	6	0	1.70275904(9)	0.92098603(9)
(3, 3)	13.627614(2)	9.58826675(6)	4.59385837(5)	15.316593(2)	-8	-8/3	1.92846910(2)	-0.27358566(2)
(3, 4)	-8.8038435(2)	9.5882656(6)	-3.3716217(2)	0	0	0	0	0
(3, 5)	-6.09496858(8)	6.6380301(3)	-2.3341997(1)	0	0	0	0	0
(4, 1)	-2.70887493(5)	2.9502357(1)	-1.03742206(3)	0	0	0	0	0
(4, 2)	9.4810622(1)	-10.3258245(4)	3.6309772(1)	0	0	0	0	0
(4, 3)	-10.8354997(2)	11.8009423(6)	-4.1496883(2)	0	0	0	0	0
(4, 4)	10.269733(2)	7.37558970(4)	2.38486989(5)	14.816594(2)	-5	-8/3	2.01567488(3)	0.02849782(3)
(4, 5)	9.732709(2)	7.37558970(3)	2.38486988(5)	3.477157(2)	-5	0	-1.07855468(7)	-0.76054387(7)
(5, 1)	-2.70887493(5)	2.9502357(1)	-1.03742206(2)	0	0	0	0	0
(5, 2)	-1.35443746(2)	1.47511779(6)	-0.51871103(2)	0	0	0	0	0
(5, 3)	0	0	0	0	0	0	0	0
(5, 4)	1.1783609(6)	-0.49170598(2)	-0.15899133(2)	0.1590523(6)	1/3	0	-0.98220341(2)	-0.06601462(2)
(5, 5)	2.297078(2)	-8.35900166(4)	-2.70285255(5)	10.134698(2)	17/3	-8/3	-3.06215787(3)	0.83396857(3)

TABLE I: *The coefficients* $d^{+(0,1)}_{l,m} - d^{+(0,6)}_{l,m}$ *and* $d^{+(2,1)}_{l,m} - d^{+(2,2)}_{l,m}$.

Parity Conserving, Flavour Exchange Symmetry Plus

$(l,\ m)$	$d_{l,m}^{+(2,3)}$	$d_{l,m}^{+(2,4)}$	$d_{l,m}^{+(2,5)}$	$d_{l,m}^{+(2,6)}$	$d_{l,m}^{+(2,7)}$	$d_{l,m}^{+(2,8)}$	$d_{l,m}^{+(2,9)}$	$d_{l,m}^{+(2,10)}$
(1, 1)	2.642227(4)	1.7473718(2)	0.08346609(1)	-3.289500(4)	-19/18	-1/2	0	25/24
(1, 2)	-1.0988822(5)	2.5685206(3)	-0.9493554(4)	0.9569921(2)	0	-1	1	-1/3
(1, 3)	0.13736030(9)	-0.32106507(4)	0.11866942(8)	-0.11962402(4)	0	1/8	-1/8	1/24
(1, 4)	0.35245840(2)	-0.399645902(8)	0.06172133(2)	0.0040994899(4)	-1/4	3/8	-1/4	-1/24
(1, 5)	1.05737519(6)	-1.19893771(2)	0.18516399(5)	0.012298470(1)	-3/4	9/8	-3/4	-1/8
(2, 1)	-1.5109631(5)	3.5317158(3)	-1.3053636(4)	1.3158642(3)	0	-11/8	11/8	-11/24
(2, 2)	0.789419(4)	1.84106755(5)	-0.37712545(3)	-2.739903(4)	1/18	-11/8	3/8	41/48
(2, 3)	-1.6447718(6)	0.199822951(3)	-0.065051477(5)	-0.0214962(6)	11/24	-3/16	1/16	-1/24
(2, 4)	-0.06868015(3)	0.16053253(1)	-0.01966766(2)	0.05981201(1)	0	-1/16	0	-1/48
(2, 5)	0.20604045(9)	-0.48159760(4)	0.05900298(7)	-0.17943602(4)	0	3/16	0	1/16
(3, 1)	-0.2747206(1)	0.64213013(6)	-0.2373388(1)	0.23924803(5)	0	-1/4	1/4	-1/12
(3, 2)	-7.416224(3)	4.6520712(2)	-0.18110545(2)	0.870210(3)	5/3	-9/4	1/4	-13/24
(3, 3)	2.045124(4)	-3.9381018(3)	-0.49577614(3)	-4.174196(4)	11/36	7/8	3/8	17/12
(3, 4)	1.785683(1)	-4.1738459(5)	0.5113591(7)	-1.5551122(4)	0	13/8	0	13/24
(3, 5)	1.2362427(5)	-2.8895856(3)	0.3540179(4)	-1.0766161(2)	0	9/8	0	3/8
(4, 1)	1.40983359(8)	-1.59858361(4)	0.24688532(7)	0.016397959(2)	-1	3/2	-1	-1/6
(4, 2)	-1.9230442(7)	4.4949109(3)	-0.5506945(6)	1.6747362(3)	0	-7/4	0	-7/12
(4, 3)	2.197764(1)	-5.1370411(5)	0.6293651(7)	-1.9139843(4)	0	2	0	2/3
(4, 4)	-0.286209(4)	-2.8633160(3)	-0.023092586(8)	-4.273362(4)	1/18	5/4	0	59/48
(4, 5)	4.602710(3)	-3.0948716(3)	-0.05164224(1)	-0.928529(3)	-7/9	5/4	0	9/16
(5, 1)	1.40983359(8)	-1.59858361(3)	0.24688532(6)	0.016397959(2)	-1	3/2	-1	-1/6
(5, 2)	0.2747206(1)	-0.64213013(6)	0.0786706(1)	-0.23924803(5)	0	1/4	0	1/12
(5, 3)	0	0	0	0	0	0	0	0
(5, 4)	1.255191(1)	0.2526361(1)	0.009152736(3)	0.009222(1)	-17/54	-1/12	0	1/16
(5, 5)	0.828945(4)	4.0632560(3)	0.12704697(1)	-2.661259(4)	-23/27	-17/12	0	35/48

TABLE II: *The coefficients* $d_{l,m}^{+(2,3)} - d_{l,m}^{+(2,10)}$.

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$(l,\, m)$	$d^{-(0,1)}_{l,m}$	$d^{-(0,2)}_{l,m}$	$d^{-(0,3)}_{l,m}$	$d^{-(0,4)}_{l,m}$	$d^{-(0,5)}_{l,m}$	$d^{-(0,6)}_{l,m}$	$d^{-(2,1)}_{l,m}$	$d^{-(2,2)}_{l,m}$
(1, 1)	-2.830716(3)	5.90047176(3)	1.90789592(4)	9.748573(3)	-4	-8/3	2.12575961(5)	0.46409210(5)
(1, 2)	5.41774985(8)	-5.9004712(3)	2.0748441(1)	0	0	0	0	0
(1, 3)	1.35443746(2)	-1.47511779(6)	0.51871103(2)	0	0	0	0	0
(1, 4)	1.35443746(2)	-1.47511784(3)	0.518711032(7)	0	0	0	0	0
(1, 5)	4.06331239(5)	-4.42535353(9)	1.55613310(2)	0	0	0	0	0
(2, 1)	3.38609366(9)	-3.6877945(3)	1.2967776(1)	0	0	0	0	0
(2, 2)	0.267862(2)	0.73755883(5)	-1.25504053(1)	10.816593(2)	1	-8/3	-0.87361423(2)	0.57016434(2)
(2, 3)	1.5317566(4)	-0.368779461(8)	0.129677758(2)	0.1931470(4)	0	0	-0.89270365(2)	-0.08962151(2)
(2, 4)	0.338609366(5)	-0.36877945(2)	0.129677758(6)	0	0	0	0	0
(2, 5)	-1.01582810(1)	1.10633834(5)	-0.38903328(2)	0	0	0	0	0
(3, 1)	1.35443746(2)	-1.47511779(6)	0.51871103(2)	0	0	0	0	0
(3, 2)	12.324653(1)	7.37558979(4)	3.38055490(3)	3.772588(1)	-6	0	-1.70275904(9)	-0.92098603(9)
(3, 3)	9.564301(2)	14.01362029(6)	3.03772527(5)	15.316593(2)	-8	-8/3	1.92846910(2)	-0.27358566(2)
(3, 4)	-12.8671559(2)	14.0136190(6)	-4.9277548(2)	0	0	0	0	0
(3, 5)	6.09496858(8)	-6.6380301(3)	2.3341997(1)	0	0	0	0	0
(4, 1)	5.41774986(5)	-5.9004714(1)	2.07484413(3)	0	0	0	0	0
(4, 2)	-6.7721873(1)	7.3755889(4)	-2.5935552(1)	0	0	0	0	0
(4, 3)	-10.8354997(2)	11.8009423(6)	-4.1496883(2)	0	0	0	0	0
(4, 4)	12.922182(2)	16.22629734(4)	5.24671376(5)	15.816593(2)	-11	-8/3	1.84126332(3)	-0.57566914(3)
(4, 5)	-9.555272(2)	-1.47511794(3)	-0.47697397(5)	-3.068020(2)	1	0	2.15255184(7)	0.47726124(7)
(5, 1)	5.41774986(6)	-5.9004714(1)	2.07484413(3)	0	0	0	0	0
(5, 2)	-1.35443746(2)	1.47511779(6)	-0.51871103(2)	0	0	0	0	0
(5, 3)	0	0	0	0	0	0	0	0
(5, 4)	-1.1192154(6)	2.45852990(2)	0.79495663(2)	-0.0226732(6)	-5/3	0	1.34020247(2)	-0.02841292(2)
(5, 5)	-3.777376(2)	0.49170598(4)	0.15899132(5)	8.771247(2)	-1/3	-8/3	0.16287196(3)	0.68000502(3)

TABLE III: *The coefficients* $d^{-(0,1)}_{l,m} - d^{-(0,6)}_{l,m}$ *and* $d^{-(2,1)}_{l,m} - d^{-(2,2)}_{l,m}$.

Parity Conserving, Flavour Exchange Symmetry Minus

$(l,\ m)$	$d_{l,m}^{-(2,3)}$	$d_{l,m}^{-(2,4)}$	$d_{l,m}^{-(2,5)}$	$d_{l,m}^{-(2,6)}$	$d_{l,m}^{-(2,7)}$	$d_{l,m}^{-(2,8)}$	$d_{l,m}^{-(2,9)}$	$d_{l,m}^{-(2,10)}$
(1, 1)	3.611581(4)	-3.4947437(3)	-0.16693217(1)	-2.043017(4)	-5/9	1	0	5/12
(1, 2)	-1.0988822(5)	2.5685206(3)	-0.9493554(4)	0.9569921(2)	0	-1	1	-1/3
(1, 3)	-0.2747206(1)	0.64213013(6)	-0.2373388(1)	0.23924803(5)	0	-1/4	1/4	-1/12
(1, 4)	-0.70491680(2)	0.79929180(1)	-0.12344266(2)	-0.0081989797(6)	1/2	-3/4	1/2	1/12
(1, 5)	-2.11475039(7)	2.39787541(3)	-0.37032798(6)	-0.024596939(2)	3/2	-9/4	3/2	1/4
(2, 1)	-0.6868013(5)	1.6053254(3)	-0.5933471(4)	0.5981201(2)	0	-5/8	5/8	-5/24
(2, 2)	2.904169(4)	-0.55680786(5)	0.40349227(3)	-2.715306(4)	-13/9	7/8	-3/8	29/48
(2, 3)	1.2923134(6)	0.199822951(3)	-0.065051477(5)	0.0173967(6)	-5/24	-3/16	1/16	1/12
(2, 4)	-0.06868015(3)	0.16053253(1)	-0.01966766(2)	0.05981201(1)	0	-1/16	0	-1/48
(2, 5)	0.20604045(9)	-0.48159760(4)	0.05900298(7)	-0.17943602(4)	0	3/16	0	1/16
(3, 1)	-0.2747206(1)	0.64213013(6)	-0.2373388(1)	0.23924803(5)	0	-1/4	1/4	-1/12
(3, 2)	6.006391(3)	-3.0534875(2)	-0.33930637(2)	-0.886608(3)	-2/3	3/4	1/4	17/24
(3, 3)	4.159875(4)	-6.3359772(3)	0.28484158(3)	-4.149599(4)	-43/36	25/8	-3/8	7/6
(3, 4)	2.609845(1)	-6.1002363(5)	0.7473711(7)	-2.2728563(4)	0	19/8	0	19/24
(3, 5)	-1.2362427(5)	2.8895856(3)	-0.3540179(4)	1.0766161(2)	0	-9/8	0	-3/8
(4, 1)	-2.81966719(8)	3.19716722(4)	-0.49377065(7)	-0.032795919(2)	2	-3	2	1/3
(4, 2)	1.3736030(7)	-3.2106507(3)	0.3933532(6)	-1.1962402(3)	0	5/4	0	5/12
(4, 3)	2.197764(1)	-5.1370411(5)	0.6293651(7)	-1.9139843(4)	0	2	0	2/3
(4, 4)	6.491207(4)	-7.4107631(3)	-0.187841964(8)	-4.050432(4)	-17/18	11/4	0	65/48
(4, 5)	-2.042489(3)	0.0632400(3)	-0.05819067(1)	1.051220(3)	5/9	-1/4	0	-9/16
(5, 1)	-2.8196672(1)	3.19716722(4)	-0.49377065(8)	-0.032795919(2)	2	-3	2	1/3
(5, 2)	0.2747206(1)	-0.64213013(6)	0.0786706(1)	-0.23924803(5)	0	1/4	0	1/12
(5, 3)	0	0	0	0	0	0	0	0
(5, 4)	-0.401784(1)	-1.2631800(1)	-0.045763706(3)	0.031675(1)	13/54	5/12	0	-1/16
(5, 5)	3.734320(4)	-0.9473033(3)	-0.09480166(1)	-1.755961(4)	-23/27	1/12	0	17/48

TABLE IV: *The coefficients* $d_{l,m}^{-(2,3)} - d_{l,m}^{-(2,10)}$.

Parity Violating, Flavour Exchange Symmetry Plus

$(l,\, m)$	$\delta^{+(0,1)}_{l,m}$	$\delta^{+(0,2)}_{l,m}$	$\delta^{+(0,3)}_{l,m}$	$\delta^{+(0,4)}_{l,m}$	$\delta^{+(0,5)}_{l,m}$	$\delta^{+(0,6)}_{l,m}$	$\delta^{+(2,1)}_{l,m}$	$\delta^{+(2,2)}_{l,m}$
(1, 1)	7.607190(2)	-2.95023588(3)	-0.95394796(3)	12.293750(2)	2	-8/3	-2.79899092(5)	0.39295395(5)
(2, 2)	2.299519(2)	-1.475117940(5)	-0.476973978(5)	10.816593(2)	1	-8/3	-0.87361423(2)	0.57016434(2)
(2, 3)	-1.1931472(4)	0	0	-0.1931470(4)	0	0	0.89270365(2)	0.08962151(2)
(3, 2)	-10.970215(1)	-8.85070764(3)	-2.86184387(3)	-3.772588(1)	6	0	1.70275904(9)	0.92098603(9)
(3, 3)	11.595958(2)	11.80094352(4)	3.81579182(4)	15.316593(2)	-8	-8/3	1.92846910(2)	-0.27358566(2)
(4, 4)	10.269733(2)	7.37558970(4)	2.38486989(5)	14.816594(2)	-5	-8/3	2.01567488(3)	0.02849782(3)
(4, 5)	9.732709(2)	7.37558970(3)	2.38486988(5)	3.477157(2)	-5	0	-1.07855468(7)	-0.76054387(7)
(5, 4)	1.1783609(6)	-0.49170598(2)	-0.15899133(2)	0.1590523(6)	1/3	0	-0.98220341(2)	-0.06601462(2)
(5, 5)	2.297078(2)	-8.35900166(4)	-2.70285255(5)	10.134698(2)	17/3	-8/3	-3.06215787(3)	0.83396857(3)

TABLE V: *The coefficients* $\delta^{+(0,1)}_{l,m} - \delta^{+(0,6)}_{l,m}$ *and* $\delta^{+(2,1)}_{l,m} - \delta^{+(2,2)}_{l,m}$.

Parity Violating, Flavour Exchange Symmetry Plus

$(l,\, m)$	$\delta^{+(2,3)}_{l,m}$	$\delta^{+(2,4)}_{l,m}$	$\delta^{+(2,5)}_{l,m}$	$\delta^{+(2,6)}_{l,m}$	$\delta^{+(2,7)}_{l,m}$	$\delta^{+(2,8)}_{l,m}$	$\delta^{+(2,9)}_{l,m}$	$\delta^{+(2,10)}_{l,m}$
(1, 1)	2.642227(4)	1.7473718(2)	0.08346609(1)	-3.289500(4)	-19/18	-1/2	0	25/24
(2, 2)	1.846794(4)	0.64212984(5)	0.0131834100(7)	-2.727604(4)	-25/36	-1/4	0	35/48
(2, 3)	-1.4685426(6)	0	0	-0.0194464(6)	1/3	0	0	-1/16
(3, 2)	-6.711307(3)	3.8527794(2)	0.079100460(4)	0.878409(3)	7/6	-3/2	0	-5/8
(3, 3)	3.102499(4)	-5.1370395(3)	-0.105467280(7)	-4.161897(4)	-4/9	2	0	31/24
(4, 4)	-0.286209(4)	-2.8633160(3)	-0.023092586(8)	-4.273362(4)	1/18	5/4	0	59/48
(4, 5)	4.602710(3)	-3.0948716(3)	-0.05164224(1)	-0.928529(3)	-7/9	5/4	0	9/16
(5, 4)	1.255191(1)	0.2526361(1)	0.009152736(3)	0.009222(1)	-17/54	-1/12	0	1/16
(5, 5)	0.828945(4)	4.0632560(3)	0.12704697(1)	-2.661259(4)	-23/27	-17/12	0	35/48

TABLE VI: *The coefficients* $\delta^{+(2,3)}_{l,m} - \delta^{+(2,10)}_{l,m}$.

Parity Violating, Flavour Exchange Symmetry Minus

$(l,\, m)$	$\delta_{l,m}^{-(0,1)}$	$\delta_{l,m}^{-(0,2)}$	$\delta_{l,m}^{-(0,3)}$	$\delta_{l,m}^{-(0,4)}$	$\delta_{l,m}^{-(0,5)}$	$\delta_{l,m}^{-(0,6)}$	$\delta_{l,m}^{-(2,1)}$	$\delta_{l,m}^{-(2,2)}$
(1, 1)	-2.830716(3)	5.90047176(3)	1.90789592(4)	9.748573(3)	-4	-8/3	2.12575961(5)	0.46409210(5)
(2, 2)	2.299519(2)	-1.475117940(5)	-0.476973978(5)	10.816593(2)	1	-8/3	-0.87361423(2)	0.57016434(2)
(2, 3)	1.1931472(4)	0	0	0.1931470(4)	0	0	-0.89270365(2)	-0.08962151(2)
(3, 2)	10.970215(1)	8.85070764(3)	2.86184387(3)	3.772588(1)	-6	0	-1.70275904(9)	-0.92098603(9)
(3, 3)	11.595958(2)	11.80094352(4)	3.81579182(4)	15.316593(2)	-8	-8/3	1.92846910(2)	-0.27358566(2)
(4, 4)	12.922182(2)	16.22629734(4)	5.24671376(5)	15.816593(2)	-11	-8/3	1.84126332(3)	-0.57566914(3)
(4, 5)	-9.555272(2)	-1.47511794(3)	-0.47697397(5)	-3.068020(2)	1	0	2.15255184(7)	0.47726124(7)
(5, 4)	-1.1192154(6)	2.45852990(2)	0.79495663(2)	-0.0226732(6)	-5/3	0	1.34020247(2)	-0.02841292(2)
(5, 5)	-3.777376(2)	0.49170598(4)	0.15899132(5)	8.771247(2)	-1/3	-8/3	0.16287196(3)	0.68000502(3)

TABLE VII: *The coefficients* $\delta_{l,m}^{-(0,1)} - \delta_{l,m}^{-(0,6)}$ *and* $\delta_{l,m}^{-(2,1)} - \delta_{l,m}^{-(2,2)}$.

Parity Violating, Flavour Exchange Symmetry Minus

$(l,\, m)$	$\delta_{l,m}^{-(2,3)}$	$\delta_{l,m}^{-(2,4)}$	$\delta_{l,m}^{-(2,5)}$	$\delta_{l,m}^{-(2,6)}$	$\delta_{l,m}^{-(2,7)}$	$\delta_{l,m}^{-(2,8)}$	$\delta_{l,m}^{-(2,9)}$	$\delta_{l,m}^{-(2,10)}$
(1, 1)	3.611581(4)	-3.4947437(3)	-0.16693217(1)	-2.043017(4)	-5/9	1	0	5/12
(2, 2)	1.846794(4)	0.64212984(5)	0.0131834100(7)	-2.727604(4)	-25/36	-1/4	0	35/48
(2, 3)	1.4685426(6)	0	0	0.0194464(6)	-1/3	0	0	1/16
(3, 2)	6.711307(3)	-3.8527794(2)	-0.079100460(4)	-0.878409(3)	-7/6	3/2	0	5/8
(3, 3)	3.102499(4)	-5.1370395(3)	-0.105467280(7)	-4.161897(4)	-4/9	2	0	31/24
(4, 4)	6.491207(4)	-7.4107631(3)	-0.187841964(8)	-4.050432(4)	-17/18	11/4	0	65/48
(4, 5)	-2.042489(3)	0.0632400(3)	-0.05819067(1)	1.051220(3)	5/9	-1/4	0	-9/16
(5, 4)	-0.401784(1)	-1.2631800(1)	-0.045763706(3)	0.031675(1)	13/54	5/12	0	-1/16
(5, 5)	3.734320(4)	-0.9473033(3)	-0.09480166(1)	-1.755961(4)	-23/27	1/12	0	17/48

TABLE VIII: *The coefficients* $\delta_{l,m}^{-(2,3)} - \delta_{l,m}^{-(2,10)}$.