

# Factorization of correlations in two-dimensional percolation on the plane and torus

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**Abstract.** Recently, Delfino and Viti have examined the factorization of the three-point density correlation function  $P_3$  at the percolation point in terms of the two-point density correlation functions  $P_2$ . According to conformal invariance, this factorization is exact on the infinite plane, such that the ratio  $R(z_1, z_2, z_3) = P_3(z_1, z_2, z_3)/[P_2(z_1, z_2)P_2(z_1, z_3)P_2(z_2, z_3)]^{1/2}$  is not only universal but also a constant, independent of the  $z_i$ , and in fact an operator product expansion (OPE) coefficient. Delfino and Viti analytically calculate its value (1.022013...) for percolation, in agreement with the numerical value 1.022 found previously in a study of  $R$  on the conformally equivalent cylinder. In this paper we confirm the factorization on the plane numerically using periodic lattices (tori) of very large size, which locally approximate a plane. We also investigate the general behavior of  $R$  on the torus, and find a minimum value of  $R \approx 1.0132$  when the three points are maximally separated. In addition, we present a simplified expression for  $R$  on the plane as a function of the SLE parameter  $\kappa$ .

## 1. Introduction

The study of correlations in percolation provides insight into the nature of the percolation process. The well-known two-point density correlation function  $P_2(z_1, z_2)$  as a function of the locations of the points  $z_1$  and  $z_2$  behaves as

$$P_2(z_1, z_2) \sim |z_1 - z_2|^{2(D-d)} \quad (1)$$

for large  $|z_1 - z_2|$ , on a  $d$ -dimensional percolating system at the critical point  $p_c$ , where  $D$  is the fractal dimension, which has the universal value  $91/48$  in two dimensions. However, the coefficient to (1) depends upon the model of percolation and also vanishes in the continuum limit where the lattice spacing goes to zero, and is thus non-universal.

In order to study higher-order correlations, the present authors considered the ratio [1]

$$R(z_1, z_2, z_3) := \frac{P_3(z_1, z_2, z_3)}{\sqrt{P_2(z_1, z_2)P_2(z_1, z_3)P_2(z_2, z_3)}}, \quad (2)$$

where  $P_3(z_1, z_2, z_3)$  is the three-point density correlation function. With the ratio defined this way (including the square root in the denominator), the lattice factors cancel out and the quantity  $R(z_1, z_2, z_3)$  converges to a universal function in the continuum limit. It was shown in [1, 2] via conformal field theory that if  $z_1$  and  $z_2$  reside on the boundary of a (compact) bounded or half-infinite system and  $z_3$  is on the boundary or inside it, then, in the continuum limit,  $R$  is a constant independent of  $z_1$ ,  $z_2$ , and  $z_3$  and equal to

$$C_0 := 2^{7/2}\pi^{5/2}3^{-3/4}\Gamma(1/3)^{-9/2} = 1.0299268\dots \quad (3)$$

This behavior was termed factorization, i.e., the three-point function factors into a product of square roots of two-point functions, multiplied by a constant. In [3], this concept was generalized to the case where correlations between intervals on the boundary of a rectangle and a single point  $z_1$  inside was studied. There, the factorization is not exact, but depends upon the distance from the bounding intervals and the boundary conditions (free or wired—a wired interval means that all sites are constrained to belong to one cluster). Far from the bounding intervals,  $R$  once again approaches  $C_0$ . Related recent work includes Refs. [4, 5, 6, 7, 8, 9, 10, 11]

Recently, Delfino and Viti [12] have examined the factorization for three points on an infinite plane for the general Potts model. Here the factorization is exact, following simply from the general form of the three-point function with all three operators the same [13]. However, it is not possible to find a general expression for  $R$  (the OPE or operator product expansion coefficient) using methods specific to minimal CFTs (conformal field theories), such as the result in [14]. In various models, difficulties may occur for a variety of reasons: operators with non-integer Kac indices, coefficients that vanish due to additional symmetries (i.e., in the Ising model, spin reversal symmetry means that  $\langle\sigma\sigma\sigma\rangle = 0$  regardless of cluster properties), or multiple fields with a common weight. By coupling the CFTs to Liouville gravity (LG), Al. Zamolodchikov obtained

OPE coefficient expressions [15] that resolve the issues of non-integer Kac indices (as occurs, e.g., for percolation) and additional symmetries.

In [12] Delfino and Viti used the LG result to find  $R$ . The value they obtain is not identical to the LG three-point coefficient, because the LG analysis assumes a unique operator with each weight, but Delfino and Viti argue that the local selection rules of the Potts disorder operator  $\mu_{\alpha\beta}$  are implemented by two identical weight fields  $\mu$  and  $\bar{\mu}$ . They then suggest that the LG analysis might still apply to a symmetric combination of these fields, which translates to an extra factor of  $\sqrt{2}$  for the degenerate fields. This gives  $R = 1.0220\dots =: C_1$ . In the Appendix, we show that the formula for  $C_1$  of Delfino and Viti may be reduced to a single integral expression, and list numerical values for percolation as well as for other examples of the Potts model with integer  $q$  values, including both low density (FK cluster) and high density (spin cluster) phases. For percolation we indeed find

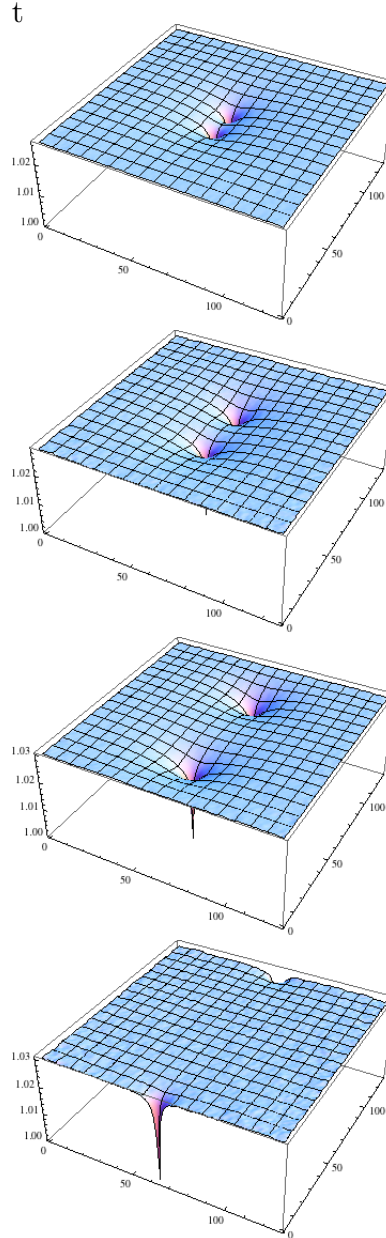
$$C_1 = 1.022013133\dots \quad (4)$$

The value  $C_1 \approx 1.022$  was originally found numerically by the present authors when studying correlations between the two ends of a cylinder and a point  $z_1$  in the interior [3]. When two ends of the cylinder are “wired,” we also found numerically that  $R$  approaches  $C_1$  exponentially as  $\exp(-2\pi x/L)$  where  $L$  is the dimension (circumference) of the end of the cylinder and  $x$  is the distance from the nearest end to  $z_1$ . The correspondence between the cylindrical results and the planar problem follows from the fact that the cylinder can be conformally mapped to the surface of a sphere, so that the cylinder boundaries map onto two circles of equal radius. In the limit of the infinitely long cylinder the radii of the image circles on the sphere shrink to zero; then the problem becomes that of the correlation of three points, and the factorization is everywhere exact, as the sphere is conformally equivalent to the plane.

In this paper, we consider the problem of measuring the three-point correlations on the plane and on the torus. For open boundary conditions, it is not possible to simulate a system large enough to effectively probe the infinite-plane behavior. By using a large periodic system and taking advantage of its translational symmetry, we are able to see the factorization over length scales large compared to the lattice spacing but small compared to the size of the system. We also find interesting behavior of the correlations on the torus itself when points are separated by distances on the order of the size of the torus.

## 2. Simulation method and results

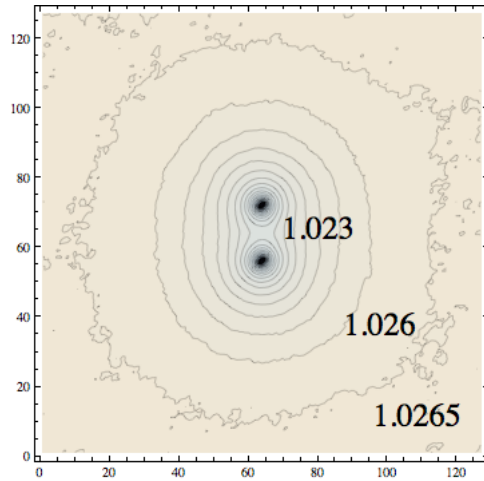
For most of our simulations, we consider bond percolation on square lattices of size  $L \times L$  at the critical point  $p_c = 1/2$ . The number of samples ranged from  $\mathcal{O}(10^5)$  for the largest system, to over  $10^9$  for the smaller ones. We carried out simulations with both open and periodic b.c.



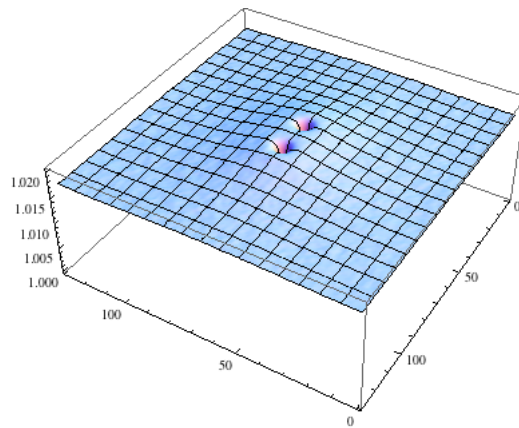
**Figure 1.** (Color online)  $R(z_1, z_2, z_3)$  on a system with open b.c. of  $128 \times 128$  sites, as a function of  $z_3$ , for  $z_1$  and  $z_2$  fixed and separated by  $\Delta = |z_1 - z_2|$  given by (a) 16, (b) 32, (c) 64, and (d) 128 [top to bottom]. At the boundaries,  $R$  is approximately equal to (a) 1.0265, (b) 1.0285, (c) 1.0295, and (d) 1.030.

### 2.1. Open boundary conditions

First we consider open boundary conditions on the square, for relatively small  $L$ . We take  $z_1$  and  $z_2$  fixed about the center of the lattice and separated by  $|z_1 - z_2| = \Delta$ , and determine  $R$  as a function of  $z_3$ , where  $z_3$  can be anywhere on the plane. The simulation technique here is to grow one critical cluster from  $z_1$ , and add 1 to the value of an array  $N_{13}(z_3)$  to every point  $z_3$  that the cluster wets. If the cluster reaches  $z_2$ , then all the wetted-site coordinates  $z_3$  of the cluster are also added to the arrays  $N_{23}(z_3)$  and



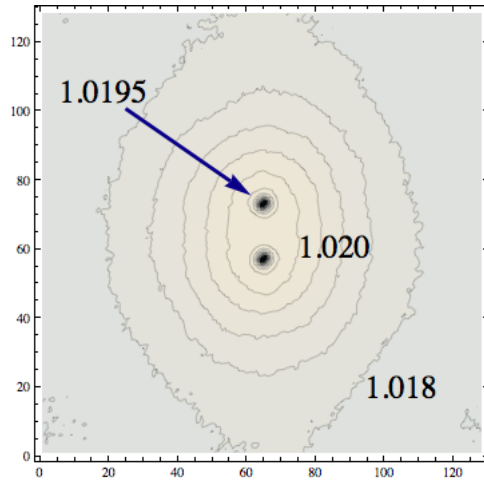
**Figure 2.** (Color online) Contours of  $R$  of Fig. 1(a) (open b.c.) with  $\Delta = 16$  and  $L = 128$ . The first complete contour encircling both points  $z_1$  and  $z_2$  is for  $R = 1.023$ , and  $R$  increases by 0.0005 in each contour going outward. When  $z_3$  goes to  $z_1$  or  $z_2$ ,  $R \rightarrow 1$ .



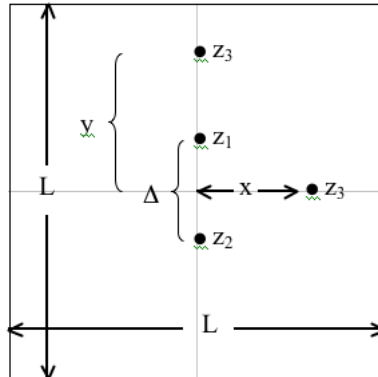
**Figure 3.** (Color online)  $R(z_1, z_2, z_3)$  for a system with periodic b.c. of  $128 \times 128$  sites, with the two points  $z_1$  and  $z_2$  separated by a distance  $\Delta = 16$ , as a function of the third point  $z_3$ . Near  $z_1$  and  $z_2$ ,  $R$  rises to a maximum value of about 1.0205, and drops to a value of about 1.018 far from those points (at the edges in this representation of the torus). Contours are shown in Fig. 4.

$N_{123}(z_3)$ , and to the counter  $N_{12}$  which tells if points 1 and 2 connect. If the cluster does not reach  $z_2$ , then a new cluster is grown from  $z_2$  and all of its sites  $z_3$  are added to the array  $N_{23}(z_3)$ . Finally, we normalize all these quantities by the number of runs to get the probabilities, and calculate  $R(z_1, z_2, z_3)$  according to (2). The results are shown in Fig. 1 for four values of  $\Delta$  and  $L = 128$ .

In all cases there is a downward-pointing spike  $R \rightarrow 1$  around  $z_1$  and  $z_2$ , as  $R = 1$  is the exact value when two points coincide. Note the highly expanded scale in these plots. When  $\Delta = 128$  [Fig. 1(d)], the two points  $z_1$  and  $z_2$  are at the edge, and the results of [1] apply, so  $R$  has the value  $C_0 \approx 1.0299\dots$  of equation (3) at every point in space except for the spikes. The size of the spikes (which decay as a power-law as



**Figure 4.** (Color online) Contours of  $R$  for the periodic system (torus) of Fig. 3 with  $\Delta = 16$  and  $L = 128$ . The first complete contour encircling both fixed points near the center is at  $R = 1.020$ , and  $R$  decreases by  $0.0005$  in each contour going both inward and outward from that contour.



**Figure 5.** Location of the fixed points  $z_1$  and  $z_2$ , and the variable point  $z_3$  in the horizontal and vertical directions, assuming  $(x_0, y_0)$  is centered at the origin.

the separation between  $z_3$  and  $z_1$  or  $z_2$  increases) is controlled by the discreteness of the lattice, and can be understood theoretically [3].

As  $\Delta$  decreases, two things happen: the roughly constant value at the edge decreases (values are given in the caption to Fig. 1) and also  $R$  varies markedly over the whole lattice. While for  $z_3$  near the center  $R$  has value  $\approx 1.022$  predicted by [12] (see Fig. 2), there is no extended region nearby where  $R$  is constant.

We have looked at larger lattices (up to  $L = 512$ ) and find the size of the constant region near the two fixed points increases somewhat, suggesting much larger lattices are needed to observe the infinite-plane behavior. However, the poor statistics of this type of simulation (only one data point for a given triangle  $z_1, z_2, z_3$  is found from each sample) makes going to larger lattices impractical. To overcome this problem, we consider lattices with periodic b.c. (tori).

## 2.2. Periodic boundary conditions

With periodic boundary conditions, every point is equivalent by translational invariance, so it is possible to get  $L^2$  data points on an  $L \times L$  lattice for a given triangle of points  $z_1, z_2$  and  $z_3$ , and results in much better statistics. However, the question of what effect the toroidal geometry imposed by these boundary conditions has, and how to extract the planar result that we are interested in remains. We expect that for a large enough torus, the behavior for the three points separated by distances much less than  $L$  should be the same as for a plane. However, because the density of correlations drops off very slowly according to (1), the influence of the periodic b.c. should remain strong across relatively large systems.

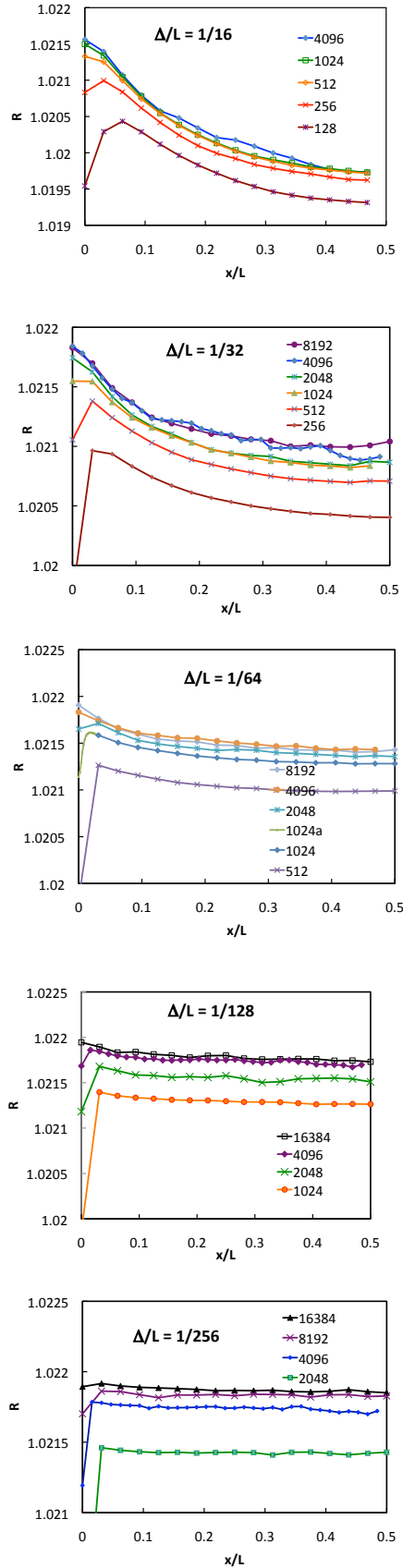
To contrast what happens with periodic vs. free b.c., we first consider a simulation similar to that done for the open b.c. system above, in which  $z_1$  and  $z_2$  are fixed with  $\Delta = 16$ , and  $R$  is determined for all  $z_3$  (thus not making use of the translational invariance). Here (see Figs. 3 and 4) we find an interesting result:  $R$  has a maximum of about 1.02 near the center, but then for large distances drops to  $\approx 1.018$ , which is below the value that would be found on the cylinder or any surface transformed from it. Presumably, this decrease is due to the effects of the periodic b.c. on  $P_2$  and  $P_3$ .

For the rest of our simulations on periodic systems, we make use of the translational invariance by populating the entire lattice, and looking at specific configurations of the three points. We consider every possible location of the two fixed points  $z_1$  and  $z_2$  arranged vertically and separated by a distance  $\Delta$ , and vary  $z_3$  both in the horizontal direction (along the perpendicular to the centerline) and in the vertical direction, as shown in Fig. 5. Specifically, for each point  $(x_0, y_0)$ , we set  $z_1 = (x_0, y_0 + \Delta/2)$ ,  $z_2 = (x_0, y_0 - \Delta/2)$ . To vary in the horizontal direction we considered  $z_3 = (x_0 + x, y_0)$  for a range in values of  $x$ , and for the vertical direction, we considered  $z_3 = (x_0, y_0 + y)$  for a range in values of  $y$ . In the vertical case, we consider both  $|y| < \Delta/2$ , i.e.,  $z_3$  between the two points, and  $|y| > \Delta/2$ , outside the two points.

In these simulations, we create clusters on the entire lattice using the growth algorithm, labeling each cluster with a different index, and then check the indices of the three points in order to calculate  $R$ . If a pair  $i, j$  of the points belong to the same cluster, then we increment an array  $N_{ij}^{(h)}(x)$  or  $N_{ij}^{(v)}(y)$  by 1. If all three points belong to the same clusters, we increment  $N_{123}^{(h)}(x)$  or  $N_{123}^{(v)}(y)$  by 1. We considered  $L = 128, 256, \dots, 16384$ , the latter being the largest size that could easily be simulated in our computer.

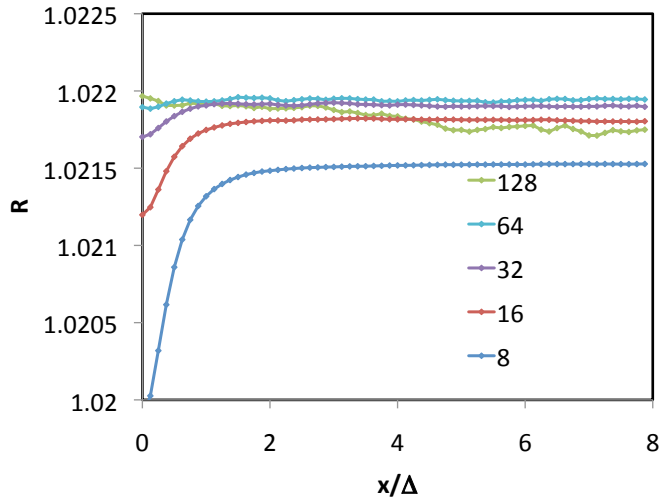
Fig. 6 shows the behavior of  $R$  in the horizontal direction for a series of systems keeping  $\Delta/L = 1/16 \dots 1/256$  constant. From these results we see:

- For small  $\Delta$  ( $\leq 64$ ), there is a decrease in  $R$  as  $x \rightarrow 0$ , as the three points are within the distances in which finite-size lattice effects are significant.
- For large  $x/L$  (and all  $\Delta/L$ ),  $R$  decreases as  $x/L$  increases. However, the amount of decrease becomes less as  $\Delta/L$  decreases, and the curve of  $R$  vs.  $x/L$  becomes nearly horizontal for the smallest  $\Delta/L$  we consider (1/256).



**Figure 6.** (Color online)  $R$  as a function of  $x/L$  on an  $L \times L$  torus, with  $\Delta/L = 1/16$ ,  $1/32$ ,  $1/64$ ,  $1/128$ , and  $1/256$  (top to bottom), where  $x$  is the distance from the center in the horizontal direction for fixed ratios of  $\Delta/L$  and various values of  $L$  as given in





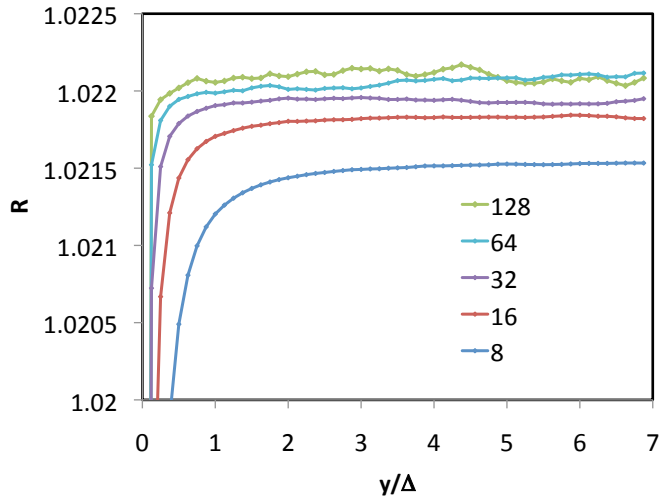
**Figure 7.** (Color online) Values of  $R$  as a function of the scaled distance  $x/\Delta$  to the point  $z_3$  from the center of the pair of fixed points in the horizontal direction, for values of  $\Delta$  given in the legend, on a system with  $L = 16384$ .

- As  $L$  increases (for a fixed value of  $\Delta/L$  and not too small  $x/L$ ), the curves of  $R$  as a function of  $x/L$  approach a limiting form. In fact, except for small  $x/L$ , the effect of increasing  $L$  is simply to move the curves vertically. This can be understood as being due to finite-size effects on  $P_2(z_1, z_2)$ .
- In particular, in the case  $\Delta/L = 1/256$ , for large  $L$ ,  $R$  is nearly independent of  $L$  and is close to the expected value  $R \approx 1.022$ .

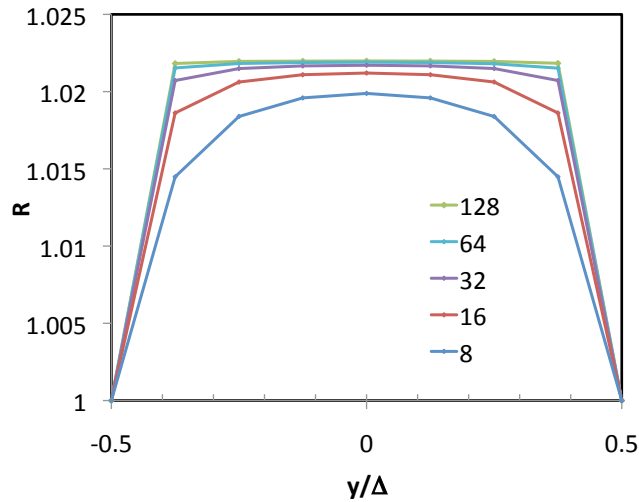
In Figs. 7 and 8 we show the behavior of  $R$  in both the horizontal and vertical directions, for just the largest system  $L = 16384$  and various  $\Delta$ , plotted now as a function of  $x/\Delta$ , so that each value of the abscissa corresponds to a similar triangle of the three points. In general, these curves consider  $x$  at much lower values than in Fig. 6. Here we see the decrease for small  $x/\Delta$  due to the finite-size lattice effects from the points being too close together, and the leveling out to a constant value. The results for larger  $x$  (not shown here) exhibit roughly the same constant values for large  $x$ , for both the horizontal and vertical directions. In particular, the vertical and horizontal results both approach the same value, 1.022. The results along the centerline between the two points are shown in Fig. 9.

### 3. Point in equilateral triangle configurations on the torus

We also considered having the three points configured as an equilateral triangle. To do this, we used a triangular lattice, in which the periodic b.c. were applied on an  $L \times L$  square-lattice representation with diagonal bonds, which has the effect of creating a torus with a half-twisted boundary. Fig. 10 shows  $R$  as a function of the separation



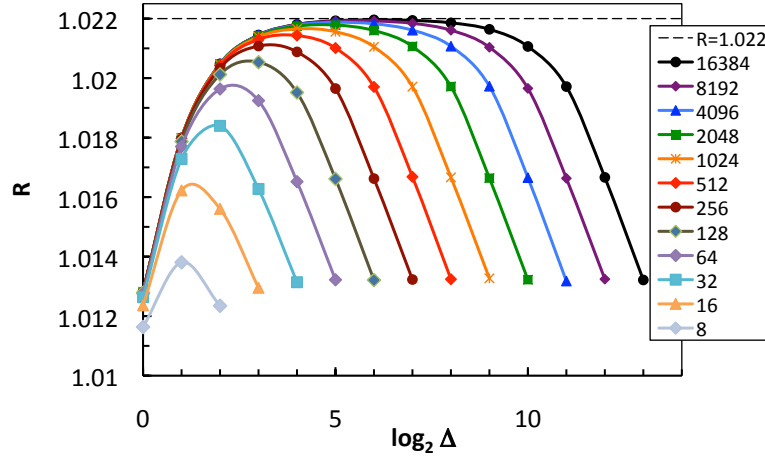
**Figure 8.** (Color online) Similar to Fig. 7, but in the vertical direction. Legend values correspond to  $\Delta$ .



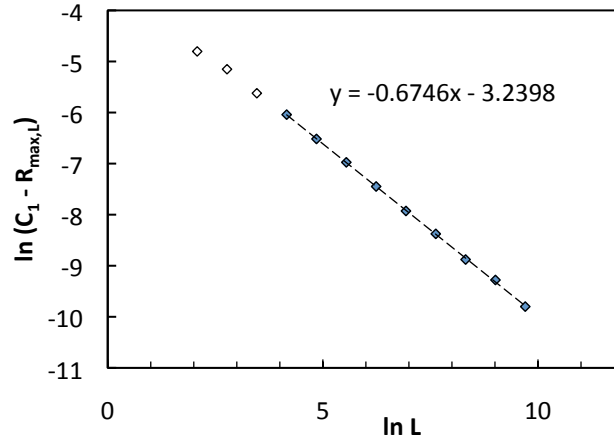
**Figure 9.** (Color online) Values of  $R$  along the centerline connecting the two fixed points  $z_1$  and  $z_2$ , for various  $\Delta$  (see legend) and  $L = 16384$ .

distance  $\Delta = 1, 2, 4, \dots, L/2$  for  $L = 8 \dots 16384$ . At  $\Delta = L/2$ , the three vertices of the triangle are equally spaced around the torus such that each pair is connected by paths of the same distance in two directions. Flattened out and repeated, the points form a kagomé lattice. For this system, we find the following behavior:

- As  $\Delta \rightarrow 0$ ,  $R$  decreases towards 1 as expected, although the value even at  $\Delta = 1$  (one lattice spacing apart) remains at  $\approx 1.0125$ .
- For intermediate values of  $\Delta$  (of the order  $\mathcal{O}(\sqrt{L})$ ),  $R$  approaches 1.022, providing



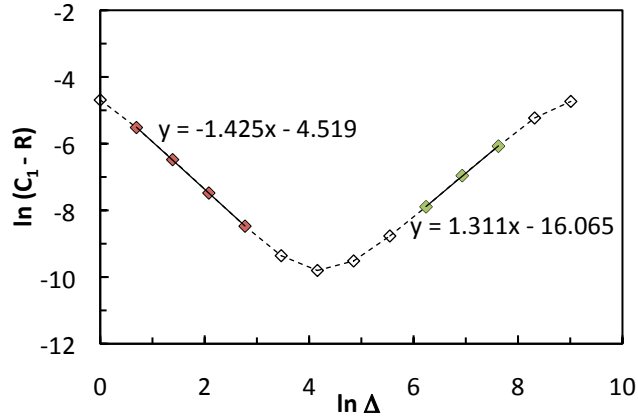
**Figure 10.** (Color online) Values of  $R$  as a function of the side-length  $\Delta$  for an equilateral triangle of points on an  $L \times L$  twisted torus, with  $L$  given in the legend, simulated on a triangular lattice at its bond percolation threshold  $p_c = 2 \sin \pi/18$ . The errors are generally smaller than the size of the symbols. Smoothed curves connecting the data points are drawn for ease of viewing.



**Figure 11.** Plot of  $\ln(C_1 - R_{\max,L})$  vs.  $\ln L$  for the maxima of the curves of Fig. 10 using the theoretical value of  $C_1 \approx 1.022013$ . The linear fit to the shaded points is shown in the plot, with  $x$  representing  $\ln L$  and  $y$  representing  $\ln(C_1 - R_{\max,L})$ .

further evidence for this value of  $R$  for points on an infinite plane. For  $L = 16384$ , the value of  $R$  at the maximum is 1.02196, just 0.00005 below the theoretical value. This maximum corresponds to an equilateral triangle with  $\Delta = 64$ . A plot of  $\ln(C_1 - R_{\max,L})$  vs.  $\ln L$  yields a very good linear fit for  $L \geq 64$  (see Fig. 11) implying  $R_{\max,L} = C_1 - 0.0391L^{-0.674}$ .

- As  $\Delta \rightarrow L/2$ ,  $R$  again decreases to a value of  $\approx 1.0132$ , which is substantially less than the maximum value 1.022 that is found when at least two of the points are close together. For smaller  $L$ ,  $R(\Delta = L/2)$  converges to 1.01323 approximately as



**Figure 12.** (Color online) The data of Fig. 10 for  $L = 16384$ , plotted as  $\ln(C_1 - R)$  vs.  $\ln \Delta$ , using the theoretical value of  $C_1 \approx 1.022013$ . The equations give the linear fits for the points  $\Delta = 2, 4, 8$ , and  $16$  (left) and  $\Delta = 512, 1024$ , and  $2048$  (right). In the linear formulas,  $x$  represents  $\ln \Delta$  and  $y$  represents  $\ln(C_1 - R)$ .

$L^{-1.5}$ . For  $\Delta > L/2$ ,  $R$  again increases due to the wraparound, so  $\Delta = L/2$  is evidently a minimum point for  $R$ .

To study the approach to  $C_1 \approx 1.022$ , in Fig. 12 we plot  $\ln(C_1 - R)$  vs.  $\ln \Delta$  using the theoretical value of  $C_1$ . For small  $x$  we expect a power-law, and fitting the behavior in the linear regime we find a slope of about  $-1.43$ . Note that in [3], we found numerically for the cylinder that  $C$  behaves as  $\exp(-2\pi x/L)$  where  $L$  is the circumference and  $x$  the distance to the end. Transforming to the annulus this implies a decay of  $R$  with two points separated by  $\Delta$  (and the third far away) as  $\Delta^{-1}$ . Here we find that when all three points are separated by  $\Delta$ , the decay behaves as  $\approx \Delta^{-1.43}$ .

For large  $\Delta$ , we again seem to find that  $R$  behaves as a power law, here decaying with exponent  $\approx 1.3$ . All the curves for large  $L$  show a similar behavior. We have no explanation of why  $R$  drops to this lower value as  $\Delta \rightarrow L/2$ .

While the curves in Fig. 10 appear to be nearly symmetric, this symmetry is in fact an artifact of the particular system used (bond percolation). We also considered site percolation on the triangular lattice, where  $p_c = 1/2$ . For site percolation, Eq. (2) must be modified by dividing by  $\sqrt{p_c}$  to account for factors of  $p_c$  in the probabilities  $P_2$  and  $P_3$  so that they represent conditional probabilities that the sites are occupied, and this insures that  $R \rightarrow 1$  when  $z_1 = z_2 = z_3$ . For large  $\Delta$ , the behavior is identical to that of bond percolation as seen in Fig. 10, but for small  $\Delta$ , the behavior is much different: while  $R$  is exactly 1 at  $\Delta = 1$  (nearest neighboring occupied sites always connect in site percolation), at  $\Delta = 2$  it jumps to  $\approx 1.0243$  and then drops monotonically as  $\Delta$  increases, leveling at  $R \approx 1.022$  in the intermediate range  $\Delta = \mathcal{O}(\sqrt{L})$ .

#### 4. Conclusions

We have shown that the behavior of  $R$  on a plane can be effectively studied in simulations on tori of very large size, by keeping the three points far enough apart to minimize finite-size effects, but also keeping the separations of at least one pair of the points much smaller than the system size  $L$ . We have confirmed the result of Delfino and Viti [12] that  $R$  goes to the value 1.0220..., the same as found on a cylinder far from the two endpoints [3]. We verified this value moving  $z_3$  in both the vertical and horizontal directions. This can be seen in Figs. 7 and 8 for larger  $\Delta$  ( $\geq 64$ ) and  $x/L$  or  $y/L$  greater than 2. We also verified it for  $z_3$  along the centerline between  $z_1$  and  $z_1$  (when all three points are well separated) as seen in Fig 9.

We found the interesting result that  $R \approx 1.022$  also when two of the points are very close together (though far apart compared to the lattice spacing), and the third anywhere on the torus. This behavior is consistent with conformal field theory, because in this limit  $R$  only depends on the OPE coefficient, which is the same on the torus and on the plane.

We also considered the three points in an equilateral triangle configuration, on an effectively twisted torus. For intermediate separations,  $R$  goes to 1.022, but when the three points are far apart,  $R$  drops to 1.0132, which is the lowest value of  $R$  that we have found (other than for  $z_1 \rightarrow z_2$  where  $R \rightarrow 1$ ). We have verified this behavior on the triangular lattice using both site and bond percolation.

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#### 6. Appendix. Evaluation of $C_1$

Here we give an expression for the constant  $C_1$ , which takes the value  $C_1 \approx 1.022$  for percolation, that follows from the work of Delfino and Viti [12] and Zamolodchikov [15].

Specializing Zamolodchikov's result for the three-point OPE coefficient, Eq. (49) of Ref. [15] for  $\alpha_1 = \alpha_2 = \alpha_3 = 1/(4\beta) - \beta/2$ , where  $\beta = \sqrt{4/\kappa}$ , with  $\kappa$  the SLE parameter, and including a multiplicative factor of  $\sqrt{2}$ , Delfino and Viti find

$$C_1 = \frac{\beta^{\beta^{-2}-\beta^2-1} \sqrt{2\gamma(\beta^2)\gamma(\beta^{-2}-1)} \Upsilon_\beta\left(\frac{\beta}{2} - \frac{1}{4\beta}\right) \Upsilon_\beta\left(\frac{\beta}{2} + \frac{1}{4\beta}\right)^3}{\Upsilon_\beta(\beta) \Upsilon_\beta\left(\frac{1}{2\beta}\right)^{3/2} \Upsilon_\beta\left(\beta - \frac{1}{2\beta}\right)^{3/2}} \quad (5)$$

where  $\gamma(x) := \Gamma(x)/\Gamma(1-x)$  and

$$\Upsilon_\beta(x) := \exp \int_0^\infty \frac{dt}{t} \left\{ \left(\frac{Q}{2} - x\right)^2 e^{-t} - \frac{\sinh^2 \left[ \left(\frac{Q}{2} - x\right) \frac{t}{2} \right]}{\sinh \frac{\beta t}{2} \sinh \frac{t}{2\beta}} \right\} \quad (6)$$

with  $Q := \beta + \beta^{-1}$ . Using the following identities

$$\begin{aligned}
 \Upsilon_\beta(x) &= \Upsilon_{1/\beta}(x) = \Upsilon_\beta(Q - x) \\
 \Upsilon_\beta(x + \beta) &= \beta^{1-2\beta x} \gamma(\beta x) \Upsilon_\beta(x) \\
 \Upsilon_\beta(Q/2) &= 1 \\
 \Upsilon_\beta(x + 1/\beta) &= \beta^{2x/\beta-1} \gamma(x/\beta) \Upsilon_\beta(x)
 \end{aligned} \tag{7}$$

we can reduce (5) to a single integral expression

$$C_1 = \beta^{1/2-\beta^2} \sqrt{\frac{2\gamma(\beta^2)\gamma(\frac{1}{2} + \frac{1}{4}\beta^{-2})^2}{\gamma(2 - \beta^{-2})\gamma(\frac{1}{2}\beta^{-2})^3}} \exp(I_1) \tag{8}$$

where

$$I_1 = \int_0^\infty \frac{dt}{t} \left\{ \frac{(1 - 2\beta^2)e^{-t} - 1}{2} - \frac{-4 \cosh \frac{t}{4\beta} + 3 \cosh \frac{\beta t}{2} + \cosh \frac{\beta t}{2} \cosh \frac{t}{2\beta}}{2 \sinh \frac{\beta t}{2} \sinh \frac{t}{2\beta}} \right\} \tag{9}$$

Table 1 shows  $C_1$  for various values of the Potts model parameter  $q$ , with  $\kappa = 4\pi/[\pi - \cos^{-1}(\sqrt{q}/2)]$  in the low-density (FK-cluster) phase, and  $\kappa' = 16/\kappa$  for the high-density (spin cluster) phase. These values were found numerically using the Mathematica function `NIntegrate[]`, increasing the working precision to 25 digits and higher to verify the 16 digits shown here. These values agree with those given in [12] which were quoted to just four truncated digits past the decimal point. Note that for  $\beta = 1$  ( $\kappa = 4$ ), the coefficient of (8) is undefined, but taking the limit  $\beta \rightarrow 1$ , it converges to  $\sqrt{2}\Gamma(3/4)/\Gamma(1/4) = 0.477988\dots$ . For  $\beta = \sqrt{3/2}$  ( $\kappa = 8/3$ ), we can rearrange (8) using the  $\Upsilon_\beta$  identities to show that  $C_1 = \sqrt{2}$ . This corresponds to Zamolodchikov's coefficient being exactly 1, which is natural because  $h_{1/2,0} = 0$  for  $\kappa = 8/3$  so that the LG analysis does not distinguish between  $\phi_{1/2,0}$  and the identity operator. Using this exact result to test the accuracy of our integral expression, we indeed find  $C_1 = \sqrt{2}$  to all digits of the working precision of the `NIntegrate[]` function.

$q$	$\kappa$	$\beta$	$C_1$
1	6	$\sqrt{2/3}$	1.0220131331461556
2	16/3	$\sqrt{3/4}$	1.0524474717449139
3	24/5	$\sqrt{5/6}$	1.0923552364945137
4	4	1	1.1892071150027211
3	10/3	$\sqrt{6/5}$	1.3107927060993472
2	3	$\sqrt{4/3}$	1.3767325887917331
—	8/3	$\sqrt{3/2}$	$\sqrt{2}$

**Table 1.** Values of  $C_1$  for the  $q$ -state Potts model in the low-density ( $\beta < 1$ ) and high-density ( $\beta > 1$ ) phases;  $q = 1$  corresponds to percolation.

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