

Holographic description of large N gauge theory

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Abstract

Based on the earlier work [S.-S. Lee, Nucl. Rev. B **832**, 567 (2010)], we derive a holographic dual for the D -dimensional $U(N)$ lattice gauge theory from a first principle construction. The resulting theory is a lattice field theory of closed loops, dubbed as *lattice loop field theory* which is defined on a lattice version of the $(D+1)$ -dimensional anti-de Sitter space. The lattice loop field theory is well defined non-perturbatively, and it becomes weakly coupled and local in the large N limit with a large 't Hooft coupling.

I. INTRODUCTION

Although there are many non-trivial evidences for the anti-de Sitter space/conformal field theory (AdS/CFT) correspondence[1–3], a first principle derivation of the conjecture is not available yet. Nonetheless the correspondence has been employed to a wide range of physical systems including condensed matter systems[4–7], in a hope that certain features of strongly coupled quantum field theory can be captured by holographic theories constructed based on phenomenological reasoning. On the one hand, such approaches have produced interesting insights into strongly coupled quantum many-body systems. On the other hand, there exists a clear limitation because it has not been possible to identify precise holographic dual for general quantum field theory that one may want to understand.

Since the 't Hooft's observation that large N gauge theory is related to a weakly interacting string theory[8], it has been suggested that there exist intrinsic connections between the two theories[9]. The fundamental object that bridges between these two theories is Wilson loop which becomes classical in the large N limit. The dual string theory that governs the classical equation of motion of Wilson loop[10, 11] in the loop space is expected to be defined on one higher dimensional space than the space on which the field theory is defined, and the additional dimension corresponds to the energy scale in the renormalization group (RG) sense[9]. There have been many works which made the connection between RG flow of general quantum field theory and holographic theory more precise[12–20].

In this paper, we present a first principle construction of a holographic theory dual to $U(N)$ gauge theory based on the prescription[18] which has been applied to the $O(N)$ vector model[21–25]. We use the lattice regularization for the gauge theory. The derived holographic theory is a lattice field theory of closed loop defined on a $(D + 1)$ -dimensional lattice which can be viewed as a discrete AdS space. The holographic theory becomes classical in the large N limit and locality emerges when the 't Hooft coupling is large. This construction provides a realization of an earlier idea of reformulating large N gauge theory as a classical theory of gauge neutral fields[26, 27].

II. FROM $U(N)$ GAUGE THEORY TO LOOP FIELD THEORY

We start with the $U(N)$ gauge theory defined on the D -dimensional Euclidean hypercubic lattice,

$$Z[\mathcal{J}] = \int dU e^{-S_{\mathcal{J}}[U]} \quad (1)$$

with the action,

$$S[U; \mathcal{J}] = - \sum_{n=1}^{\infty} \sum_{\{C_1, \dots, C_n\}} N^{2-n} \mathcal{J}_{\{C_1, \dots, C_n\}} \prod_{i=1}^n W_{C_i}. \quad (2)$$

Here W_C is Wilson line defined on closed oriented loop C ,

$$W_C = \text{tr} \left[\prod_{\langle ij \rangle \in C} U_{ij} \right], \quad (3)$$

where $U_{ij} = U_{ji}^\dagger$ is $U(N)$ matrix (holonomy) defined on nearest neighbor bond $\langle i, j \rangle$, and $dU \equiv \prod_{\langle i, j \rangle} dU_{ij}$. In Eq. (3) and all products of holonomies hereafter, we assume that the product is path-ordered along the orientation of the curve. $\sum_{\{C_1, \dots, C_n\}}$ is the sum over unordered sets of n closed loops. $\mathcal{J}_{\{C_1\}}$, $\mathcal{J}_{\{C_1, C_2\}}$, ... are coupling constants associated with single-trace, double-trace operators, and so on. For single trace coupling, we will also use the notation $\mathcal{J}_{C_1} \equiv \mathcal{J}_{\{C_1\}}$ interchangeably. The factor of N^{2-n} has been singled out from each coupling. Roughly, the inverse of $\mathcal{J}_{\{C_1, \dots, C_n\}}$ corresponds to the 't Hooft coupling. Throughout the paper, we will focus on the large N limit with fixed $\mathcal{J}_{\{C_1, \dots, C_n\}}$. In this limit, the action is manifestly proportional to N^2 . To guarantee the reality of the action, we impose $\mathcal{J}_{\{\bar{C}_1, \bar{C}_2, \dots\}} = \mathcal{J}_{\{C_1, C_2, \dots\}}^*$ where \bar{C}_i is the inverse loop of C_i , namely the loop with the same trajectory but with the inverse orientation. If one ignores all multi-trace deformations and loops larger than the unit plaquette, one restores the standard lattice gauge theory[28]. Here we consider the most general gauge invariant action. We assume that couplings associated with large loops or multi-trace couplings with loops which are far from each other are exponentially small in the size of large loops or in the separation between loops. If couplings satisfy this condition, we say the theory is local. We note that effective theories obtained by integrating out short distance fluctuations generically contain non-local terms which are exponentially small.

To set the stage for a real-space renormalization, we divide links in the lattice into two sets[29]. The first set X contains links that form a coarse grained lattice, that is, the hypercubic lattice with

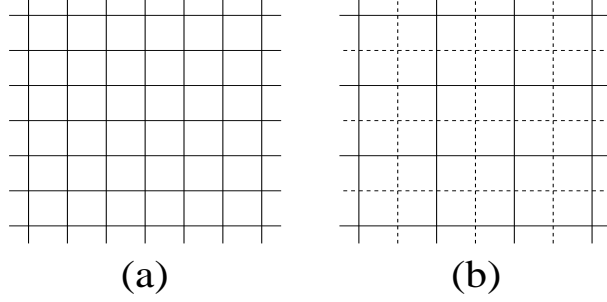


FIG. 1: (a) Two-dimensional illustration of the D -dimensional hypercubic lattice. (b) Links in the hypercubic lattice divided into two sets where solid links belong to the coarse grained lattice X which form a hypercubic lattice with a larger lattice spacing and the dashed links belong to $Y = X^c$.

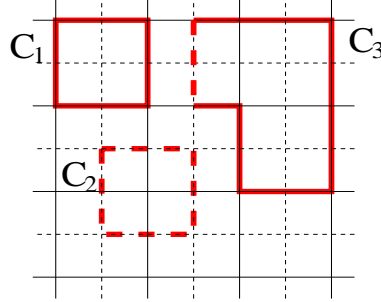


FIG. 2: Three types of loops; Wilson loops for C_1 , C_2 and C_3 are included in S_X , S_Y and S_{XY} , respectively.

the lattice spacing twice larger than the original one. The second set Y contains all other links. This is illustrated in Fig. 1. The action can be divided as

$$S[U, \tilde{U}; \mathcal{J}] = S_X[U] + S_Y[\tilde{U}] + S_{XY}[U, \tilde{U}], \quad (4)$$

where holonomy on links in Y are denoted with tilde : $U_{ij} \rightarrow \tilde{U}_{ij}$ if $\langle i, j \rangle \in Y$. Here S_X and S_Y are the actions which contain Wilson loops only in X and Y , respectively, and S_{XY} includes loops that span across X and Y , as is shown in Fig. 2. Now the partition function is written as

$$Z[\mathcal{J}] = \int dU e^{-S_X[U]} \left\langle e^{-S_{XY}[U, \tilde{U}]} \right\rangle_Y, \quad (5)$$

where $\langle O \rangle_Y = \int d\tilde{U} O e^{-S_Y[\tilde{U}]}$.

We note that $S_{XY}[U, \tilde{U}]$ consists of Wilson loops where parts of the loops are made of holonomy in X and the remaining parts in Y . For example, the loop in Fig. 3 (a) represents a term

$$N \sum_{L_1} \mathcal{J}_{L_1} \text{tr}[H_{L_{1,1}} \tilde{H}_{L_{1,2}}] \quad (6)$$

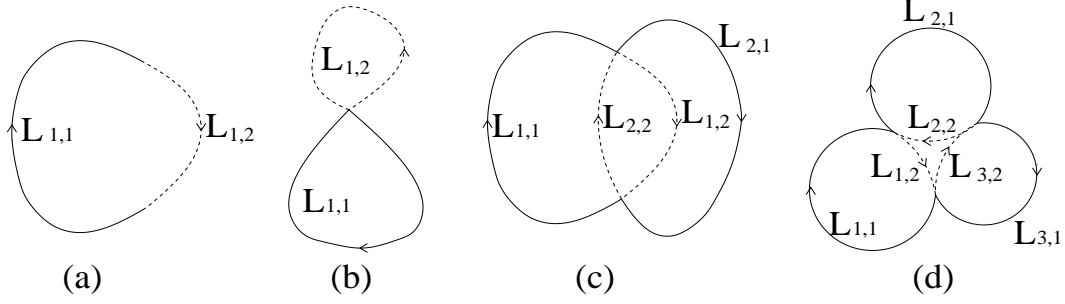


FIG. 3: Examples of Wilson loops included in $e^{-S_{XY}}$. The solid line represents links in X and the dashed line in Y .

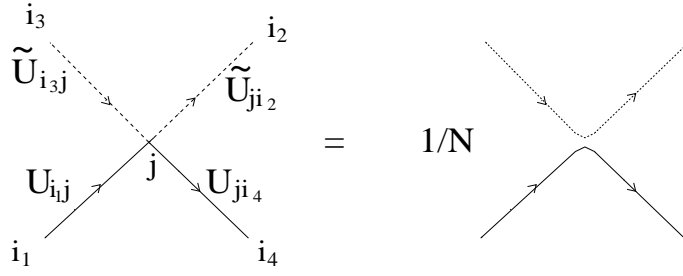


FIG. 4: Parts of two Wilson lines in S_{XY} that cross at site j . The solid (dashed) lines represent links in X (Y). This crossed Wilson lines can be reconnected inside the average of $\langle \dots \rangle_Y$ so that links in X (Y) form a Wilson line by themselves (see the text).

in S_{XY} , where $L_{1,1}$ and $L_{1,2}$ are two segments of the closed loop L_1 which belong to the sublattices X and Y respectively. Hereafter, we will use the notation $L_a = \sum_b L_{a,b}$ to represent that $L_{a,b}$ is the b -th segment of a closed loop L_a , where b increases along the orientation. Accordingly, $H_{L_{a,b}}$ represents the holonomy along the curves $L_{a,b}$. We add tilde as $\tilde{H}_{L_{a,b}}$ to represent holonomies along curves which are in Y . For all diagrams in Fig. 3, we have

$$H_{L_{a,1}} = \prod_{\langle i,j \rangle \in L_{a,1}} U_{ij}, \quad \tilde{H}_{L_{a,2}} = \prod_{\langle i,j \rangle \in L_{a,2}} \tilde{U}_{ij}. \quad (7)$$

Since the action $S_Y[\tilde{U}]$ respects the gauge symmetry in lattice Y , only those configurations where there is no open end in Y survive inside $\langle \dots \rangle_Y$. Therefore, $\langle \text{tr}[H_{L_{1,1}} \tilde{H}_{L_{1,2}}] \rangle_Y$ for Fig. 3 (a) vanishes. Segments in Y may form closed loops by themselves as in Fig. 3 (b), in which case the average does not vanish. Non-vanishing contributions can be written as a product of Wilson loops using the formula [34],

$$\left\langle \text{tr}(AU_{i_1 j} \tilde{U}_{j i_2} \tilde{B}) \text{tr}(\tilde{C} \tilde{U}_{i_3 j} U_{j i_4} D) \right\rangle_Y = \frac{1}{N} \left\langle \text{tr}(AU_{i_1 j} U_{j i_4} D) \text{tr}(\tilde{C} \tilde{U}_{i_3 j} \tilde{U}_{j i_2} \tilde{B}) \right\rangle_Y. \quad (8)$$

This is illustrated in Fig. 4. We emphasize that this identity is valid only inside the average, but not as an operator identity. Using this formula, we obtain

$$\begin{aligned}\langle \text{tr}[H_{L_{1,1}} \tilde{H}_{L_{1,2}}] \rangle_Y &= \frac{1}{N} \langle \text{tr}[H_{L_{1,1}}] \text{tr}[\tilde{H}_{L_{1,2}}] \rangle_Y \\ &= \frac{1}{N} \langle W_{L_{1,1}} \tilde{W}_{L_{1,2}} \rangle_Y\end{aligned}\quad (9)$$

for Fig. 3 (b). Generally, only those diagrams where all open segments in sub-lattice Y form closed loops do not vanish. For Fig. 3 (c) and (d), one obtains

$$\begin{aligned}\langle \text{tr}[H_{L_{1,1}} \tilde{H}_{L_{1,2}}] \text{tr}[H_{L_{2,1}} \tilde{H}_{L_{2,2}}] \rangle_Y &= \frac{1}{N^2} \langle W_{L_{1,1}+L_{2,1}} \tilde{W}_{L_{1,2}+L_{2,2}} \rangle_Y, \\ \langle \text{tr}[H_{L_{1,1}} \tilde{H}_{L_{1,2}}] \text{tr}[H_{L_{2,1}} \tilde{H}_{L_{2,2}}] \text{tr}[H_{L_{3,1}} \tilde{H}_{L_{3,2}}] \rangle_Y &= \frac{1}{N^3} \langle W_{L_{1,1}+L_{2,1}+L_{3,1}} \tilde{W}_{L_{1,2}+L_{3,2}+L_{2,2}} \rangle_Y\end{aligned}\quad (10)$$

Note that if segments in sub-lattice Y form loops, segments in sub-lattice X automatically form loops. Therefore all non-vanishing terms in $\langle e^{-S_{XY}[U, \tilde{U}]} \rangle_Y$ can be expressed as a polynomial of Wilson loops in X and Y ,

$$\langle e^{-S_{XY}[U, \tilde{U}]} \rangle_Y = \left\langle 1 + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{\{C_1, \dots, C_n\} \in X} \sum_{\{\tilde{C}_1, \dots, \tilde{C}_m\} \in Y} F_{\{C_1, \dots, C_n\}; \{\tilde{C}_1, \dots, \tilde{C}_m\}}[\mathcal{J}] \prod_{i=1}^n W_{C_i} \prod_{k=1}^m W_{\tilde{C}_k} \right\rangle_Y, \quad (11)$$

where $F_{\{C_1, \dots, C_n\}; \{\tilde{C}_1, \dots, \tilde{C}_m\}}[\mathcal{J}]$ depends only on \mathcal{J} , $\{C_1, \dots, C_n\}$ and $\{\tilde{C}_1, \dots, \tilde{C}_m\}$, but not on U_{ij} , \tilde{U}_{ij} [35]. This polynomial for W_C and $W_{\tilde{C}}$ can be exponentiated as

$$\langle e^{-S_{XY}[U, \tilde{U}]} \rangle_Y = \langle e^{-S'[W, \tilde{W}]} \rangle_Y, \quad (12)$$

where

$$S'[W, \tilde{W}] = - \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} N^{2-(m+n)} \sum_{\{C_1, \dots, C_n\} \in X} \sum_{\{\tilde{C}_1, \dots, \tilde{C}_m\} \in Y} h_{\{C_1, \dots, C_n\}; \{\tilde{C}_1, \dots, \tilde{C}_m\}}[\mathcal{J}] \prod_{i=1}^n W_{C_i} \prod_{k=1}^m W_{\tilde{C}_k}. \quad (13)$$

Here $N^{2-(m+n)} h_{\{C_1, \dots, C_n\}; \{\tilde{C}_1, \dots, \tilde{C}_m\}}[\mathcal{J}]$ is the cumulant of $F_{\{C_1, \dots, C_n\}; \{\tilde{C}_1, \dots, \tilde{C}_m\}}[\mathcal{J}]$. Since $\ln Z \sim O(N^2)$, $h_{\{C_1, \dots, C_n\}; \{\tilde{C}_1, \dots, \tilde{C}_m\}}[\mathcal{J}]$ is $O(1)$ in the large N limit [36]. $h_{\{C_1, \dots, C_n\}; \{\tilde{C}_1, \dots, \tilde{C}_m\}}[\mathcal{J}]$ can be

computed perturbatively in $\mathcal{J}_{\{C_1, \dots, C_n\}}$ and $1/N$ [37],

$$\begin{aligned}
h_{\{C_1\};\{\tilde{C}_1\}}[\mathcal{J}] &= \sum_{L_1} \mathcal{J}_{L_1} \delta_{C_1+\tilde{C}_1, L_1} + \mathcal{J}_{\{C_1, \tilde{C}_1\}} \\
&+ \frac{1}{2} \sum_{L_1, L_2} \mathcal{J}_{L_1} \mathcal{J}_{L_2} \delta_{L_{1,1}+L_{2,1}, C_1} \delta_{L_{1,2}+L_{2,2}, \tilde{C}_1} \\
&+ \frac{1}{6} \sum_{L_1, L_2, L_3} \mathcal{J}_{L_1} \mathcal{J}_{L_2} \mathcal{J}_{L_3} \delta_{L_{1,1}+L_{2,1}+L_{3,1}, C_1} \delta_{L_{1,2}+L_{3,2}+L_{2,2}, \tilde{C}_1} \\
&+ O(\mathcal{J}^4, 1/N), \tag{14}
\end{aligned}$$

$$\begin{aligned}
h_{\{C_1, C_2\};\{\tilde{C}_1\}}[\mathcal{J}] &= \sum_{L_1} \mathcal{J}_{L_1} \delta_{L_{1,1}, C_1} \delta_{L_{1,2}+L_{1,4}, \tilde{C}_1} \delta_{L_{1,3}, C_2} \\
&+ \sum_{L_1, L_2} \mathcal{J}_{\{L_1, L_2\}} \delta_{L_{1,1}, \tilde{C}_1} \delta_{L_{1,2}, C_1} \delta_{L_2, C_2} \\
&+ \mathcal{J}_{\{C_1, C_2, \tilde{C}_1\}} + O(\mathcal{J}^2, 1/N), \tag{15}
\end{aligned}$$

$$\begin{aligned}
h_{\{C_1\};\{\tilde{C}_1, \tilde{C}_2\}}[\mathcal{J}] &= \sum_{L_1} \mathcal{J}_{L_1} \delta_{L_{1,1}, \tilde{C}_1} \delta_{L_{1,2}+L_{1,4}, C_1} \delta_{L_{1,3}, \tilde{C}_2} \\
&+ \sum_{L_1, L_2} \mathcal{J}_{\{L_1, L_2\}} \delta_{L_{1,1}, C_1} \delta_{L_{1,2}, \tilde{C}_1} \delta_{L_2, \tilde{C}_2} \\
&+ \mathcal{J}_{\{C_1, \tilde{C}_1, \tilde{C}_2\}} + O(\mathcal{J}^2, 1/N), \tag{16}
\end{aligned}$$

and so on. Here δ_{C_1, C_2} is a Kronecker delta function in the space of loops. It is easy to understand physical meaning of each term. The first term in Eq. (14) describes a loop L_1 which has a self-intersecting point as in Fig. 3 (b) decomposes into two loops C_1 and \tilde{C}_1 which are in X and Y , respectively. The second term describes a two-loop state where one is included in X and the other in Y . The third and fourth terms describe the process where two and three loops join to form a loop C_1 in X and a loop \tilde{C}_1 in Y as in Fig. 3 (c) and (d), respectively. The first term in Eq. (15) describes a loop L_1 decomposing into three loops, two of which (C_1, C_2) are in X and one (\tilde{C}_1), in Y through two self intersecting points. The remaining terms can be understood similarly.

We call those Wilson loops that contribute to $h_{\{C_1, \dots, C_n\};\{\tilde{C}_1, \dots, \tilde{C}_m\}}[\mathcal{J}]$ ‘connected Wilson loops’. There are two kinds of them. Connected Wilson loops of the first kind are those that touch each other in space at crossing points where links in X and Y meet. For example, the Wilson loops C_1 and \tilde{C}_1 in the first term of Eq. (14) are connected at a crossing point as in Fig. 3 (b). The second kind includes those Wilson loops that are physically separated but the separated loops originate from multi-trace couplings, such as the second term in Eq. (14), where C_1 and \tilde{C}_1 are in general separated in space but they are ‘connected’ through $\mathcal{J}_{C_1, \tilde{C}_1}$.

Now the partition function can be written as

$$Z[\mathcal{J}] = \int_X dU \int_Y d\tilde{U} e^{-S_Y[\tilde{U}] + \sum_{n=1}^{\infty} \sum_{\{C_1, \dots, C_n\} \in X} N^{2-n} (\mathcal{J}_{\{C_1, \dots, C_n\}} + f_{\{C_1, \dots, C_n\}}[\tilde{W}]) \prod_{i=1}^n W_{C_i}}, \quad (17)$$

where

$$f_{\{C_1, \dots, C_n\}}[\tilde{W}] = \sum_{m=1}^{\infty} N^{-m} \sum_{\{\tilde{C}_1, \dots, \tilde{C}_m\} \in Y} h_{\{C_1, \dots, C_n\}; \{\tilde{C}_1, \dots, \tilde{C}_m\}}[\mathcal{J}] \prod_{k=1}^m W_{\tilde{C}_k}. \quad (18)$$

This theory can be viewed as a theory defined on the lattice X whose sources $\mathcal{J}_{\{C_1, \dots, C_n\}} + f_{\{C_1, \dots, C_n\}}[\tilde{W}]$ are dynamical, where fluctuations of the sources are provided by the dynamical degrees of freedom defined on Y . Since $W_{\tilde{C}_k} \sim O(N)$, the dynamical sources become classical in the large N limit. Therefore it is useful to introduce collective fields for the source fields. We decompose \tilde{W} and W by introducing Hubbard Stratonovich fields[18],

$$Z[\mathcal{J}] = \int dU d\tilde{U} dJ dP e^{-(S_Y[\tilde{U}] + S''[J, P, W, \tilde{W}])}, \quad (19)$$

where $dJ dP \equiv \prod_{n=1}^{\infty} \prod_{\{C_1, \dots, C_n\} \in X} dJ_{\{C_1, \dots, C_n\}} dP_{\{C_1, \dots, C_n\}}$ and

$$\begin{aligned} S''[J, P, W, \tilde{W}] &= iN^2 \sum_{n=1}^{\infty} \sum_{\{C_1, \dots, C_n\} \in X} P_{\{C_1, \dots, C_n\}} \left(J_{\{C_1, \dots, C_n\}} - \mathcal{J}_{\{C_1, \dots, C_n\}} - f_{\{C_1, \dots, C_n\}}[\tilde{W}] \right) \\ &\quad - \sum_{n=1}^{\infty} \sum_{\{C_1, \dots, C_n\} \in X} N^{2-n} J_{\{C_1, \dots, C_n\}} \prod_{i=1}^n W_{C_i}. \end{aligned} \quad (20)$$

$J_{\{C_1, \dots, C_n\}}$'s are fluctuating sources for Wilson loops on X , and $P_{\{C_1, \dots, C_n\}} = P_{\{\tilde{C}_1, \dots, \tilde{C}_n\}}^*$ is a complex Lagrangian multiplier which imposes the constraint $J_{\{C_1, \dots, C_n\}} = \mathcal{J}_{\{C_1, \dots, C_n\}} + f_{\{C_1, \dots, C_n\}}[\tilde{W}]$ and its complex conjugate. With this normalization of $P_{\{C_1, \dots, C_n\}}$, the equation of motion for $J_{\{C_1, \dots, C_n\}}$ implies $i \langle P_{\{C_1, \dots, C_n\}} \rangle = N^{-n} \langle \prod_{i=1}^n W_{C_i} \rangle \sim O(1)$. Physically, $P_{\{C_1, \dots, C_n\}}$ describes fluctuations of the Wilson loop operators. The dynamical action for $P_{\{C_1, \dots, C_n\}}$ is generated once \tilde{U} is integrated over,

$$Z[\mathcal{J}] = \int dU dJ dP e^{-(S_D[\mathcal{J}, J, P] + S_X[U; J])}, \quad (21)$$

where

$$\begin{aligned} S_D[\mathcal{J}, J, P] &= iN^2 \sum_{n=1}^{\infty} \sum_{\{C_1, \dots, C_n\} \in X} P_{\{C_1, \dots, C_n\}} (J_{\{C_1, \dots, C_n\}} - \mathcal{J}_{\{C_1, \dots, C_n\}}) + G[\mathcal{J}, P], \\ G[\mathcal{J}, P] &= -\ln \left\langle e^{iN^2 \sum_{n=1}^{\infty} \sum_{\{C_1, \dots, C_n\} \in X} P_{\{C_1, \dots, C_n\}} f_{\{C_1, \dots, C_n\}}[\tilde{W}]} \right\rangle_Y, \\ S_X[U; J] &= -\sum_{n=1}^{\infty} \sum_{\{C_1, \dots, C_n\} \in X} N^{2-n} J_{\{C_1, \dots, C_n\}} \prod_{i=1}^n W_{C_i}. \end{aligned} \quad (22)$$

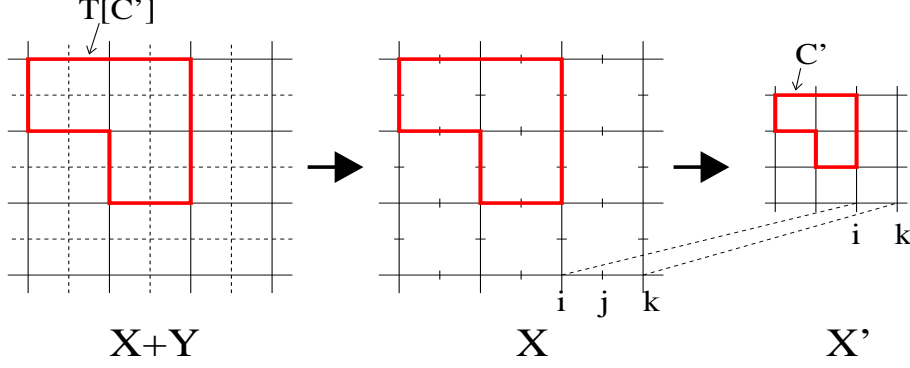


FIG. 5: Procedure of coarse graining. First, the link variables in Y are integrated out. Then, the remaining lattice X is rescaled into a coarse grained lattice X' so that every two consecutive links in X without a branch get merged into one in X' .

In order to repeat the coarse graining procedure, it is convenient to map X into the same form as the original lattice. For this, we merge every two links in X into one link in a new lattice X' as is shown in Fig. 5. Accordingly, two holonomies on merged links $\langle i, j \rangle, \langle j, k \rangle \in X$ are combined to produce one holonomy as $U'_{ik} = U_{ij}U_{jk}$ which is defined on the coarse grained lattice labeled by $\langle i, k \rangle \in X'$. Since the Wilson loops on X depend only on U' , the partition function becomes

$$Z[\mathcal{J}] = \int dU' dJ' dP' e^{-\left(S'_D[\mathcal{J}, J, P] + S_{X'}[U'; J]\right)} \quad (23)$$

upto an unimportant multiplicative constant, where $dJ' dP' \equiv \prod_{n=1}^{\infty} \prod_{\{C'_1, \dots, C'_n\} \in X'} dJ_{\{C'_1, \dots, C'_n\}} dP_{\{C'_1, \dots, C'_n\}}$ and

$$S'_D[\mathcal{J}, J, P] = iN^2 \sum_{n=1}^{\infty} \sum_{\{C'_1, \dots, C'_n\} \in X'} P_{\{C'_1, \dots, C'_n\}} (J_{\{C'_1, \dots, C'_n\}} - \mathcal{J}_{\{T[C'_1], \dots, T[C'_n]\}}) + G'[\mathcal{J}, P], \quad (24)$$

$$G'[\mathcal{J}, P] = -\ln \left\langle e^{iN^2 \sum_{n=1}^{\infty} \sum_{\{C'_1, \dots, C'_n\} \in X'} P_{\{C'_1, \dots, C'_n\}} f_{\{T[C'_1], \dots, T[C'_n]\}}[\tilde{W}]} \right\rangle_Y, \quad (25)$$

$$S_{X'}[U'; J] = -\sum_{n=1}^{\infty} \sum_{\{C'_1, \dots, C'_n\} \in X'} N^{2-n} J_{\{C'_1, \dots, C'_n\}} \prod_{i=1}^n W_{C'_i}. \quad (26)$$

Here C'_i 's are loops on X' . T represents a dilatation map which takes a loop in X' to the original one before rescaling in X as is illustrated in Fig. 5. This is necessary because $J_{\{C'_1, \dots, C'_n\}}$ and $P_{\{C'_1, \dots, C'_n\}}$ are defined on X' while $\mathcal{J}_{\{C_1, \dots, C_n\}}$ and $f_{\{C_1, \dots, C_n\}}[\tilde{W}]$ are defined on $X \subset X + Y$.

This is a theory defined on the coarse grained lattice X' with dynamical coupling fields

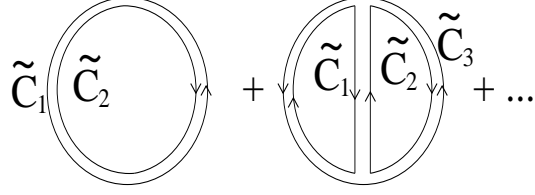


FIG. 6: First two leading order planar diagrams for $G'[\mathcal{J}, P]$. These loops are defined on Y and each loop is associated with $r_{\{\tilde{C}_1, \dots, \tilde{C}_m\}}$.

$J_{\{C'_1, \dots, C'_n\}}, P_{\{C'_1, \dots, C'_n\}}$ with the action S'_D . The second term in Eq. (24) is given by

$$G'[\mathcal{J}, P] = -\ln \int d\tilde{U} e^{-S'_Y[\tilde{U}]}, \quad (27)$$

where

$$S'_Y[\tilde{U}] = -\sum_{m=1}^{\infty} \sum_{\{\tilde{C}_1, \dots, \tilde{C}_m\} \in Y} N^{2-m} r_{\{\tilde{C}_1, \dots, \tilde{C}_m\}} \prod_{i=1}^m W_{\tilde{C}_i} \quad (28)$$

with

$$r_{\{\tilde{C}_1, \dots, \tilde{C}_m\}} = \mathcal{J}_{\{\tilde{C}_1, \dots, \tilde{C}_m\}} + i \sum_{n=1}^{\infty} \sum_{\{C'_1, \dots, C'_n\} \in X'} P_{\{C'_1, \dots, C'_n\}} h_{\{T[C'_1], \dots, T[C'_n]\}; \{\tilde{C}_1, \dots, \tilde{C}_m\}}[\mathcal{J}]. \quad (29)$$

Therefore $G'[\mathcal{J}, P]$ is given by the effective potential for the gauge theory defined on the sublattice Y with the set of couplings, $r_{\{\tilde{C}_1, \dots, \tilde{C}_m\}}$. In the large N limit, planar diagrams give the leading contribution and $G'[\mathcal{J}, P]$ is $O(N^2)$. For large 't Hooft couplings ($\mathcal{J} \ll 1$), it is natural to compute $G'[\mathcal{J}, P]$ as a power series of $r_{\{\tilde{C}_1, \dots, \tilde{C}_m\}}$ in the real space[30],

$$G'[\mathcal{J}, P] = -N^2 \left[\frac{1}{2} \sum_{\tilde{C}_1, \tilde{C}_2 \in Y} r_{\tilde{C}_1} r_{\tilde{C}_2} \delta_{\tilde{C}_1 + \tilde{C}_2, 0} + \frac{1}{6} \sum_{\tilde{C}_1, \tilde{C}_2, \tilde{C}_3 \in Y} r_{\tilde{C}_1} r_{\tilde{C}_2} r_{\tilde{C}_3} \delta_{\tilde{C}_1 + \tilde{C}_2 + \tilde{C}_3, 0} + \dots \right]. \quad (30)$$

This can be easily checked by using $\int d\tilde{U}_{ij} \tilde{U}_{ij; \alpha\beta} \tilde{U}_{ji; \gamma\delta} = \frac{1}{N} \delta_{\alpha, \delta} \delta_{\beta, \gamma} [\int d\tilde{U}_{ij}]$. Due to the Kronecker delta function, the first term survives only when $\tilde{C}_1 = \tilde{C}_2$. Similarly, the second term is nonzero only when the three oriented loops form a connected double-line graph with no unpaired single line. These are illustrated in Fig. 6. General contributions to $G'[\mathcal{J}, P]$ are given by multiple oriented surfaces made of double lines, where each face in double line graphs is associated with $r_{\{\tilde{C}_1, \dots, \tilde{C}_m\}}$. For example, $-N^2 \prod_k r_{\tilde{C}_k} \delta_{\sum_k \tilde{C}_k, 0}$ and $-N^2 r_{\tilde{C}_1, \tilde{C}_2} \prod_k r_{\tilde{C}'_k} \prod_l r_{\tilde{C}''_l} \delta_{\tilde{C}_1 + \sum_k \tilde{C}'_k, 0} \delta_{\tilde{C}_2 + \sum_l \tilde{C}''_l, 0}$ are contributions from one surface and two surfaces, respectively. Note that $G'[\mathcal{J}, P]$ is non-linear in $P_{\{C_1, \dots, C_n\}}$, and fluctuations of $P_{\{C_1, \dots, C_n\}}$ no longer impose a strict delta function for $J_{\{C_1, \dots, C_n\}}$: they become dynamical fields.

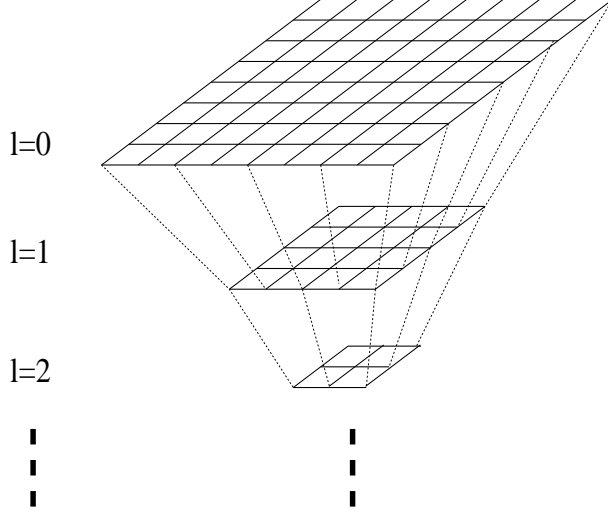


FIG. 7: The ‘lattice AdS’ space. The sequence of D -dimensional coarse grained lattices form a $(D + 1)$ -dimensional space on which the lattice loop field theory is defined.

This completes one step of our RG procedure. If we repeatedly apply this cycle to the theory on X' , and then to the coarse grained lattice of X' and so on, we can write the partition function as a integration over $J^{(l)}$'s and $P^{(l)}$'s[18],

$$\begin{aligned}
 Z[\mathcal{J}] &= \int \prod_{l=1}^{\infty} [dJ^{(l)} dP^{(l)}] e^{-S_{LLFT}}, \\
 S_{LLFT} &= \sum_{l=0}^{\infty} S'_D[J^{(l)}, J^{(l+1)}, P^{(l+1)}],
 \end{aligned} \tag{31}$$

where $J^{(l)}$ and $P^{(l)}$ are Hubbard-Stratonovich fields introduced at the l -th step of coarse graining, and $J^{(0)}$ is fixed by the microscopic couplings,

$$J_{\{C_1, \dots, C_n\}}^{(0)} = \mathcal{J}_{\{C_1, \dots, C_n\}}. \tag{32}$$

Here l plays the role of a discrete coordinate for the new $(D + 1)$ -th dimension that corresponds to the length scale in RG; fields at small (large) l describe UV (IR) physics. This $(D + 1)$ -dimensional theory is a holographic theory for the D -dimensional $U(N)$ lattice gauge theory. We will call it *lattice loop field theory* (LLFT).

Since it is difficult to write down the full theory in a compact form, let us try to understand some general features of the theory from the first few leading terms of the action in the strong

coupling expansion ($J \ll 1$). By plugging Eqs. (14)-(16) into Eqs. (29) and (30), one obtains,

$$S_{LLFT} = N^2 \sum_{l=0}^{\infty} \left[i \sum_{n=1}^{\infty} \sum_{\{C_1, \dots, C_n\}} P_{\{C_1, \dots, C_n\}}^{(l+1)} (J_{\{C_1, \dots, C_n\}}^{(l+1)} - J_{\{T[C_1], \dots, T[C_n]\}}^{(l)}) + \mathcal{H}[J^{(l)}, P^{(l+1)}] \right], \quad (33)$$

where the ‘Hamiltonian’ \mathcal{H} (the reason for this naming will become clear shortly) is given by

$$\begin{aligned} \mathcal{H}[J, P] = & - \left\{ J_{\tilde{C}} + iP_{C_1} \left(J_{L_1} \delta_{L_1, T[C_1] + \tilde{C}} + J_{\{T[C_1], \tilde{C}\}} + \frac{1}{2} J_{L_1} J_{L_2} \delta_{L_{1,1} + L_{2,1}, T[C_1]} \delta_{L_{1,2} + L_{2,2}, \tilde{C}} \right) \right. \\ & + iP_{\{C_1, C_2\}} \left(J_{L_1} \delta_{L_{1,1}, T[C_1]} \delta_{L_{1,2} + L_{1,4}, \tilde{C}} \delta_{L_{1,3}, T[C_2]} + J_{\{L_1, L_2\}} \delta_{L_{1,1}, \tilde{C}} \delta_{L_{1,2}, T[C_1]} \delta_{L_2, T[C_2]} \right. \\ & \left. \left. + J_{\{T[C_1], T[C_2], \tilde{C}\}} \right) + \dots \right\} \times \\ & \left\{ J_{\tilde{C}} + iP_{C_3} \left(J_{L_3} \delta_{L_3, T[C_3] + \tilde{C}} + J_{\{T[C_3], \tilde{C}\}} + \frac{1}{2} J_{L_3} J_{L_4} \delta_{L_{3,1} + L_{4,1}, T[C_3]} \delta_{L_{3,2} + L_{4,2}, \tilde{C}} \right) \right. \\ & + iP_{\{C_3, C_4\}} \left(J_{L_3} \delta_{L_{3,1}, T[C_3]} \delta_{L_{3,2} + L_{3,4}, \tilde{C}} \delta_{L_{3,3}, T[C_4]} + J_{\{L_3, L_4\}} \delta_{L_{3,1}, \tilde{C}} \delta_{L_{3,2}, T[C_3]} \delta_{L_4, T[C_4]} \right. \\ & \left. \left. + J_{\{T[C_3], T[C_4], \tilde{C}\}} \right) + \dots \right\} + \dots, \quad (34) \end{aligned}$$

where ... include terms that involve fields associated with multi-loop states and higher order terms in $J, P, 1/N$. Here indices L_i, C_i and \tilde{C} are understood to be summed over loops in $X + Y, X'$ and Y , respectively. LLFT is defined on the $(D + 1)$ -dimensional lattice shown in Fig. 7. This lattice can be viewed as a discrete version of the anti-de Sitter (AdS) space where the continuous isometry $l = l' + \alpha$, $x^\mu = e^\alpha x^{\mu'}$ of the metric $ds^2 = dl^2 + e^{-2l} \sum_{\mu=1}^D dx^{\mu 2}$ is replaced by a discrete scale invariance, $l = l' + 1$, $x_i = 2x'_i$. This kind of ‘discrete AdS’ space has been the natural setting for the real space renormalization group approach and the multi-scale entanglement renormalization Ansatz (MERA)[31–33]. The degrees of freedom of LLFT are fields of loops $J_{\{C_1, \dots, C_n\}}^{(l)}$ and $P_{\{C_1, \dots, C_n\}}^{(l)}$.

It is convenient to interpret l as a discrete (imaginary) ‘time’. Then we can identify the first term in Eq. (33) as the Berry phase term which dictates that $J_{\{C_1, \dots, C_n\}}^{(l)}$ and $P_{\{C_1, \dots, C_n\}}^{(l)}$ are conjugate to each other : loop fields as operators would satisfy the commutation relation $[J_{\{C_1, \dots, C_n\}}, P_{\{C_1, \dots, C_n\}}] = i/N^2$ if time was continuous. The remaining term \mathcal{H} in Eq. (33) is the ‘Hamiltonian’ that governs the evolution of the loop fields along the discrete time[38]. Formally, $J_{\{C_1, \dots, C_n\}}^{(l)}$ ($iP_{\{C_1, \dots, C_n\}}^{(l)}$) can be viewed as the path integration representation of an operator that annihilates (creates) a set of loops $\{C_1, \dots, C_n\}$ at time l , and $J_{\{C_1, \dots, \tilde{C}_2\}}^{(l)}$ ($iP_{\{C_1, \dots, \tilde{C}_2\}}^{(l)}$) is associated with an annihilation (creation) operator of ‘anti-loops’. It is noted that $J_{\{C_1, \dots, C_n\}}$ and $iP_{\{C_1, \dots, C_n\}}$ are not the usual annihilation and creation operators because they are not Hermitian conjugate to each

other. In the basis given by

$$\begin{aligned} a_{\{C_1, \dots, C_n\}} &= \frac{N}{\sqrt{2}}(J_{\{C_1, \dots, C_n\}} - iP_{\{\bar{C}_1, \dots, \bar{C}_n\}}), \\ b_{\{C_1, \dots, C_n\}} &= \frac{N}{\sqrt{2}}(J_{\{\bar{C}_1, \dots, \bar{C}_n\}} - iP_{\{C_1, \dots, C_n\}}), \end{aligned} \quad (35)$$

$a_{\{C_1, \dots, C_n\}}$ and $a_{\{C_1, \dots, C_n\}}^\dagger$ ($b_{\{C_1, \dots, C_n\}}$ and $b_{\{C_1, \dots, C_n\}}^\dagger$) become the standard annihilation and creation operators of loops $\{C_1, \dots, C_n\}$ (anti-loops $\{\bar{C}_1, \dots, \bar{C}_n\}$) respectively. However, we will use the basis of $J_{\{C_1, \dots, C_n\}}$ and $P_{\{C_1, \dots, C_n\}}$ in the following because it has two merits. First, the Hamiltonian becomes particularly simple in this basis because \mathcal{H} contains only those $P_{\{C_1, \dots, C_n\}}$ with $C_i \in X'$: loops can be created only on those links that survive coarse graining at each step of RG. Second, the equations of motion for $J_{\{C_1, \dots, C_n\}}$ and $J_{\{\bar{C}_1, \dots, \bar{C}_n\}}$ implies $i \langle P_{\{C_1, \dots, C_n\}} \rangle = [i \langle P_{\{\bar{C}_1, \dots, \bar{C}_n\}} \rangle]^*$. Therefore it is convenient to absorb i into $P_{\{C_1, \dots, C_n\}}$ to define $\mathcal{P}_{\{C_1, \dots, C_n\}} = iP_{\{C_1, \dots, C_n\}}$. At the saddle point, the Hamiltonian becomes Hermitian and the Berry phase term for $\mathcal{P}_{\{C_1, \dots, C_n\}}$ and $J_{\{C_1, \dots, C_n\}}$ becomes that of the standard creation and annihilation operators upto the factor of N^2 . Note that the expectation value of creation operator is not in general complex conjugate of that of annihilation operator, i.e. $\langle \mathcal{P}_{\{C_1, \dots, C_n\}} \rangle \neq \langle J_{\{C_1, \dots, C_n\}} \rangle^*$, when the saddle point is ‘time’-dependent.

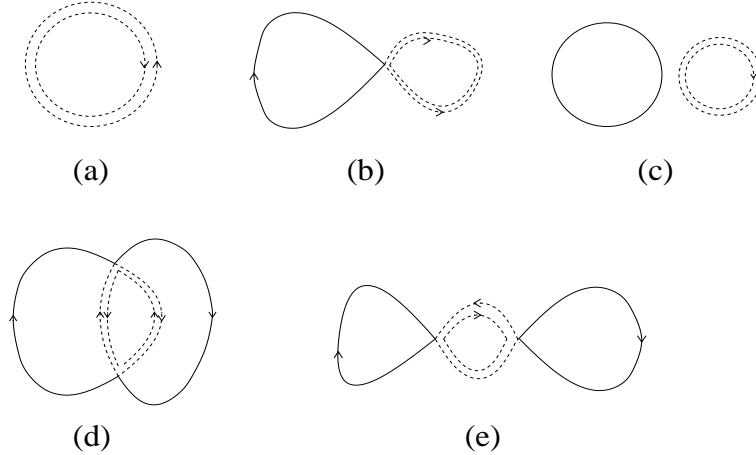


FIG. 8: Diagrams that contribute to the Hamiltonian.

Now let us take a closer look at the Hamiltonian to understand the physical meaning of each term. The quadratic term $J_{\tilde{C}}J_{\bar{\tilde{C}}}$ describes a process where a loop \tilde{C} and its anti-loop $\bar{\tilde{C}}$ are pair-annihilated (Fig. 8 (a)). Higher order terms describe fluctuations and joining/splitting processes of loops: $iP_{C_1}J_{L_1}\delta_{L_1, T[C_1]+\tilde{C}}J_{\bar{\tilde{C}}}$ describes a process where a loop L_1

with a self intersection becomes a smaller loop by combining with a loop \bar{C} (Fig. 8 (b)); $iP_{C_1} J_{\{T[C_1], \bar{C}\}} J_{\bar{C}}$ describes one of the loops in a two-loop state disappearing into vacuum with its anti-loop (Fig. 8 (c)); $iP_{C_1} J_{L_1} J_{L_2} \delta_{L_{1,1}+L_{2,1}, T[C_1]} \delta_{L_{1,2}+L_{2,2}, \bar{C}} J_{\bar{C}}$ describes two loops merging into one loop with a help of an anti-loop which eliminates parts of the two loops (Fig. 8 (d)); $iP_{\{C_1, C_2\}} J_{L_1} \delta_{L_{1,1}, T[C_1]} \delta_{L_{1,2}+L_{1,4}, \bar{C}} \delta_{L_{1,3}, T[C_2]} J_{\bar{C}}$ describes a process where one loop gets split into two (Fig. 8 (e)).

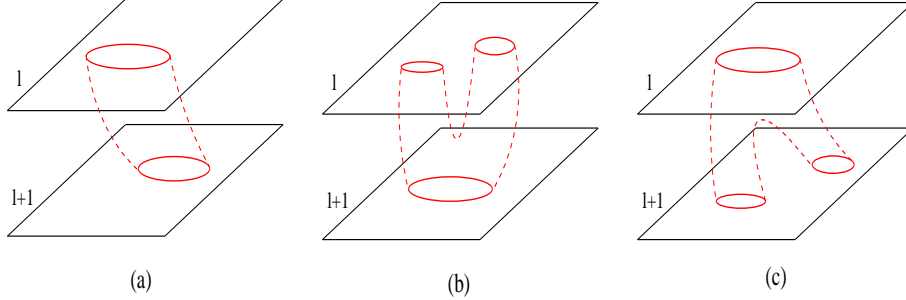


FIG. 9: The quadratic and cubic terms for the propagating loop fields j_C, p_C in the Hamiltonian. The quadratic term (a) describes fluctuation and propagation of loops, and the cubic terms (b), (c) describe joining and splitting processes of loops.

Usually, it is expected that a quadratic kinetic energy term describes propagation of loops in space and time. Actually there is no such term in the Hamiltonian. Instead, fluctuations of loop arise only through cubic and higher order terms in J and P . Here we have to remember that $J_{\{C_1, \dots, C_n\}}^{(l)}$ and $P_{\{C_1, \dots, C_n\}}^{(l)}$ have nonzero expectation value in the vacuum because of the boundary condition $\mathcal{J}_{\{C_1, \dots, C_n\}}^{(0)} \neq 0$. Therefore one has to identify $j_{\{C_1, \dots, C_n\}} = J_{\{C_1, \dots, C_n\}} - \langle J_{\{C_1, \dots, C_n\}} \rangle$ and $p_{\{C_1, \dots, C_n\}} = P_{\{C_1, \dots, C_n\}} - \langle P_{\{C_1, \dots, C_n\}} \rangle$ as the propagating loop fields. For these fields that describe small fluctuations of loop fields, the Hamiltonian includes the quadratic kinetic energy which describes fluctuations and propagation of loops in spacetime,

$$\sum_{C_1, C_2} t_{C_1, C_2} p_{C_1}^{(l+1)} j_{C_2}^{(l)}, \quad (36)$$

where t_{C_1, C_2} is ‘hopping’ probability amplitude for a loop C_2 at time l to change into a loop C_1 at time $l + 1$. The Hamiltonian also contains interactions between propagating loop fields. For

example, the cubic interaction terms,

$$\begin{aligned} & \sum_{C_1, C_2, C_3} V_{C_1, C_2, C_3} p_{C_1}^{(l+1)} j_{C_2}^{(l)} j_{C_3}^{(l)}, \\ & \sum_{C_1, C_2, C_3} V'_{C_1, C_2, C_3} p_{C_1}^{(l+1)} p_{C_2}^{(l+1)} j_{C_3}^{(l)} \end{aligned} \quad (37)$$

describe the processes where two loops join into one loop and one loop splits into two loops, respectively. These are illustrated in Fig. 9. In general, loops involved in hopping and interaction can be far from each other in space. However, such non-local terms will be exponentially suppressed if $\langle J_{\{C_1, \dots, C_n\}} \rangle$ is small, and decreases exponentially as the size of loops $\{C_1, \dots, C_n\}$ increases or as the separation between the loops in multi-trace couplings increases. This is because only connected diagrams contribute to the Hamiltonian. For example, in order for a loop to hop by a large distance, there must be a large number of loops that connect the initial and final states, or there must be a multi-trace coupling that connect them. This suggests that LLFT intrinsically has non-local elements, but a sense of locality emerges if vacuum satisfies certain conditions. In a sense, locality is a property of the vacuum rather than the theory itself. We will discuss more about the condition for locality in the next section.

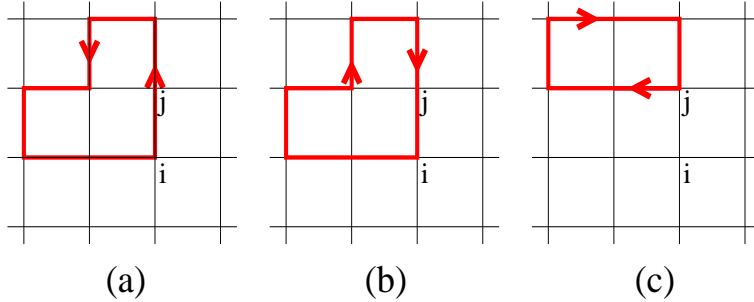


FIG. 10: Three loops that contribute charge (a) +1, (b) -1, (c) 0 to Q^{ij} .

Is there a symmetry underlying this action? The action S_{LLFT} in Eq. (33) has infinitely many conserved charges. Namely, one can define one U(1) charge $Q^{ij} = -Q^{ji}$ for every directed link $\langle i, j \rangle$ so that

$$\begin{aligned} \left[Q^{ij}, J_{\{C_1, \dots, C_n\}}^{(l)} \right] &= Y_{\{T^l[C_1], \dots, T^l[C_n]\}}^{ij} J_{\{C_1, \dots, C_n\}}^{(l)}, \\ \left[Q^{ij}, P_{\{C_1, \dots, C_n\}}^{(l)} \right] &= -Y_{\{T^l[C_1], \dots, T^l[C_n]\}}^{ij} P_{\{C_1, \dots, C_n\}}^{(l)}, \end{aligned} \quad (38)$$

where $\langle i, j \rangle$ is a link on the original lattice (the boundary of the discrete AdS space), $T^l[C_i]$ maps a loop C_i on the l -th coarse grained lattice to a loop on the original lattice by magnifying

the loop by 2^l times ($T^1 = T, T^2 = T \circ T, \dots$), and $Y_{\{C_1, \dots, C_n\}}^{ij}$ is the total number of times that loops $\{C_1, \dots, C_n\}$ pass through the link $\langle i, j \rangle$. If a loop pass the link from i to j (from j to i), it contribute $+1$ (-1) to $Y_{\{C_1, \dots, C_n\}}^{ij}$ as is illustrated in Fig. 10. In general, any term that respects these symmetries arises in S_{LLFT} . For example, there is a term that describes a multi-loop scattering process,

$$\left[\prod_{a=1}^n P_{C_a}^{(l+1)} \right] \left[\prod_{b=1}^m J_{C'_b}^{(l)} \right] \delta_{\sum_a T[C_a], \sum_b C'_b}, \quad (39)$$

where m incoming loops become n outgoing loops. Here the delta function imposes the charge conservation.

Although the action S_{LLFT} is invariant under the local symmetries, the infinite set of symmetries is broken by the boundary condition $J_{\{C_1, \dots, C_n\}}^{(0)} \neq 0$. In general, there remain only D unbroken global symmetries generated by,

$$Q^\mu = \sum_i Q^{ii+\mu}, \quad (40)$$

where $\mu = 1, 2, \dots, D$. Conservation of Q^μ implies that if a loop has a certain number of links along $+\mu$ direction, it should have the same number of links along $-\mu$ to form a closed loop. The conservation of Q^μ means that there are only closed loops. This conserved charge is a lattice version of the charge carried by world sheet of fundamental string which is coupled to the NS-NS two form gauge field in the critical string theory.

III. CLASSICALITY AND LOCALITY

The prefactor N^2 in S_{LLFT} can be identified as the inverse of ‘Planck constant’ that controls quantum fluctuations of $J_{\{C_1, \dots, C_n\}}$ and $P_{\{C_1, \dots, C_n\}}$. Therefore the theory becomes classical in the large N limit. The saddle point occurs along the imaginary axis for $P_{\{C_1, \dots, C_n\}} = -i\mathcal{P}_{\{C_1, \dots, C_n\}}$. The equation of motion takes the form of the Hamilton equation[13] in discrete time,

$$J_{\{C_1, \dots, C_n\}}^{(l+1)} - J_{\{T[C_1], \dots, T[C_n]\}}^{(l)} = -\frac{\partial \mathcal{H}[J^{(l)}, P^{(l+1)}]}{\partial \mathcal{P}_{\{C_1, \dots, C_n\}}^{(l+1)}}, \quad (41)$$

$$\mathcal{P}_{\{T^{-1}[C_1], \dots, T^{-1}[C_n]\}}^{(l+1)} - \mathcal{P}_{\{C_1, \dots, C_n\}}^{(l)} = \frac{\partial \mathcal{H}[J^{(l)}, P^{(l+1)}]}{\partial J_{\{C_1, \dots, C_n\}}^{(l)}}. \quad (42)$$

The second equation needs a further explanation. In general, there is no inverse for the map T because not all loops in step l survives in step $(l+1)$. If there is no inverse for C_i , we simply define $\mathcal{P}_{\{..., T^{-1}[C_i], \dots\}}^{(l+1)} = 0$.

To solve the equations of motion, one needs two boundary conditions for each $\{C_1, \dots, C_n\}$. One condition is given by Eq. (32). The other condition should be implemented dynamically, namely by minimizing the whole action over all possible paths of $J_{\{C_1, \dots, C_n\}}^{(l)}$ and $P_{\{C_1, \dots, C_n\}}^{(l)}$ subject to Eq. (32). This is analogous to the problem of finding the classical trajectory of a particle where the initial position is fixed, but the initial velocity is a variational parameter one uses to minimize the action.

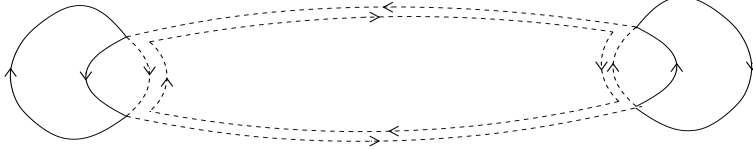


FIG. 11: A non-local coupling between two separated circular small loops mediated by two large elongated loops. The non-local interaction creates dips in the final state of the small loops.

Is the theory local in $(D + 1)$ dimensions? The theory is evidently local along the new dimension l . The locality along the original D dimensions is more tricky. This is because size of loops can be arbitrarily large. Even though one starts with small loops, multi-loop interactions generate large loops. Large loops can, in turn, mediate interactions between loops which are far from each other. Fig. 11 shows an example where two large loops C_1 and C_2 mediate interaction between two small loops C'_1 and C'_2 which are far from each other,

$$P_{\{L_1, L_2\}}^{(l+1)} J_{C_1}^{(l)} J_{C_2}^{(l)} J_{C'_1}^{(l)} J_{C'_2}^{(l)} \delta_{C'_{1,1}+C_{1,1}, T[L_1]} \delta_{C'_{2,1}+C_{1,3}, T[L_2]} \delta_{C_{1,2}+C'_{2,2}+C_{1,4}+C'_{1,2}, C_2}. \quad (43)$$

The non-local coupling between $J_{C'_1}^{(l)}$ and $J_{C'_2}^{(l)}$ is proportional to the amplitude of the large loops, $J_{C_1}^{(l)} J_{C_2}^{(l)}$. This looks bad for locality. However, if the saddle point value of J_C decreases exponentially as the size of the loop increases, the non-local coupling is exponentially small. In this case, locality is still maintained. Therefore, we can choose the initial couplings as

$$\begin{aligned} J_{C_1}^{(0)} &\sim \lambda^{-A_{C_1}}, \\ J_{C_1, C_2}^{(0)} &\sim \lambda^{-(A_{C_1} + A_{C_2} + d_{C_1, C_2})}, \\ &\dots \end{aligned} \quad (44)$$

where λ is the 't Hooft coupling much larger than 1, A_C is the minimum area enclosed by the loop C , and $d_{C, C'}$ is the minimum distance between the two loops. Now we prove that if the saddle point values of $J_{\{C_1, \dots, C_n\}}^{(l)}$ are exponentially small for large loops, those of $J_{\{C_1, \dots, C_n\}}^{(l+1)}$ are

also exponentially small for large loops. Because the Hamiltonian depends on $P_{\{C_1, \dots, C_n\}}$ only through $r_{\{C_1, \dots, C_n\}}$ in Eq. (29), Eq. (41) can be written as

$$\begin{aligned} J_{\{C_1, \dots, C_n\}}^{(l+1)} - J_{\{T[C_1], \dots, T[C_n]\}}^{(l)} &= - \sum_{\{\tilde{C}_1, \dots, \tilde{C}_m\}} \frac{\partial r_{\{\tilde{C}_1, \dots, \tilde{C}_m\}}^{(l)}}{\partial \mathcal{P}_{\{C_1, \dots, C_n\}}^{(l+1)}} \frac{\partial \mathcal{H}}{\partial r_{\{\tilde{C}_1, \dots, \tilde{C}_m\}}^{(l)}} \\ &= - \sum_{\{\tilde{C}_1, \dots, \tilde{C}_m\}} h_{\{T[C_1], \dots, T[C_n]\}; \{\tilde{C}_1, \dots, \tilde{C}_m\}} [J^{(l)}] \frac{\partial \mathcal{H}}{\partial r_{\{\tilde{C}_1, \dots, \tilde{C}_m\}}^{(l)}}, \end{aligned} \quad (45)$$

where we used

$$r_{\{\tilde{C}_1, \dots, \tilde{C}_m\}}^{(l)} = J_{\{\tilde{C}_1, \dots, \tilde{C}_m\}}^{(l)} + \sum_{n=1}^{\infty} \sum_{\{C_1, \dots, C_n\}} \mathcal{P}_{\{C_1, \dots, C_n\}}^{(l+1)} h_{\{T[C_1], \dots, T[C_n]\}; \{\tilde{C}_1, \dots, \tilde{C}_m\}} [J^{(l)}]. \quad (46)$$

Note that $h_{\{T[C_1], \dots, T[C_n]\}; \{\tilde{C}_1, \dots, \tilde{C}_m\}} [J^{(l)}]$ carries Q^{ij} charges for loops, $\{T[C_1], \dots, T[C_n]\}$ and $\{\tilde{C}_1, \dots, \tilde{C}_m\}$. Therefore, if any loop in $\{C_1, \dots, C_n\}$ is large there must be a large number of $J^{(l)}$'s or at least one $J^{(l)}$ with a large loop in $h_{\{T[C_1], \dots, T[C_n]\}; \{\tilde{C}_1, \dots, \tilde{C}_m\}}$ in order to match the charge. Similarly, if there are two loops in $\{C_1, \dots, C_n\}$ which are far from each other, $h_{\{T[C_1], \dots, T[C_n]\}; \{\tilde{C}_1, \dots, \tilde{C}_m\}}$ must include either a large number of loop fields $J^{(l)}$ or at least one multi-loop fields such as $J_{C_1, C_2}^{(l)}$ where the separation between C_1 and C_2 is large which connect the separated two loops. This is because disconnected diagrams do not contribute to $h_{\{T[C_1], \dots, T[C_n]\}; \{\tilde{C}_1, \dots, \tilde{C}_m\}} [J^{(l)}]$. This guarantees that if the condition in Eq. (44) is satisfied for a large $\lambda^{(l)}$ at time l , a similar set of condition will be satisfied with a large $\lambda^{(l+1)} \sim \lambda^{(l)}$ at time $l+1$. Therefore the theory will remain local as far as the theory stays strongly coupled with large 't Hooft couplings along the renormalization group flow. This is certainly true for small l (UV region) if one starts with large 't Hooft couplings.

Whether the amplitudes for large loops remain exponentially small for large l (IR region) will be determined dynamically through the equation of motion. Here we can think of three possibilities. The first possibility is that the energy dependent 't Hooft coupling $\lambda^{(l)}$ increases without a bound as l increases. In this case, the locality becomes better at IR. On the other hand, fluctuations of loop fields become suppressed and loops become very heavy in the IR limit. This is the case for the pure $U(N)$ gauge theory in the strong coupling limit $\lambda \gg 1$. In the low energy limit, there is no light degrees of freedom, which describes the confinement phase. The second possibility is the case where $\lambda^{(l)}$ decreases to a small value in the IR limit. If this happens, non-local couplings between loops become significant below a certain energy scale ($l > l_c$). In this case, the theory becomes non-local in the IR limit. This occurs if the theory flows to a IR fixed point which is not strongly coupled, that is, $\lim_{l \rightarrow \infty} \lambda^{(l)} \leq 1$. The last possibility, for which the

holographic description is most useful, is that the 't Hooft coupling stays at a large but finite value $\lim_{l \rightarrow \infty} \lambda^{(l)} \gg 1$ in the large l limit. This can be realized in a theory which flows to a strongly interacting conformal field theory, such as the $\mathcal{N} = 4$ super Yang-Mills theory at a large 't Hooft coupling. Then the IR dynamics is described by a weakly interacting local theory of closed loops. At the saddle point, loop fields generically have nonzero expectation values. Small fluctuations of loop fields around the classical configuration will be described by a perturbative string theory. However, LLFT is well defined even non-perturbatively, and one can consider non-perturbative objects such as solitons.

IV. DISCUSSIONS

A. Relation between LLFT and string field theory

One may view LLFT as a string field theory put in lattice. In order to obtain a string field theory in the smooth AdS space, it would be better to use a different regularization for the gauge theory. This is because RG steps can not be continuous in the lattice regularized theory. It would be of interest to devise a better regularization scheme for strongly coupled gauge theory which allows one to construct a more cut-off independent holographic theory. However, the lattice construction will be more useful to describe non-critical phases in real lattice models.

B. Comparison with the $O(N)$ vector model

The equation of motion in Eqs. (41) and (42) can be solved only if another boundary condition is provided besides the UV boundary condition in Eq. (32). The extra boundary condition should be imposed dynamically in the IR limit. In the case of the holographic $O(N)$ vector model[18], imposing the IR boundary condition amounts to solving the original field theory. Because the holographic theory for the $O(N)$ model has non-singlet bulk fields which are not classical even in the large N limit, one has to integrate over the non-singlet bulk fields in the IR limit to dynamically impose the second boundary condition. On the other hand, in the $U(N)$ gauge theory only $U(N)$ singlet fields are physical and all bulk degrees of freedom are classical in the large N limit. For this reason, the IR boundary condition can be imposed by minimizing the whole action with respect to the loop fields in the IR limit. Whether this leads to a simple regularity condition as in the standard AdS/CFT correspondence is yet to be understood. Nonetheless, in large N gauge theory,

everything boils down to solving classical problem, which is the major difference from the $O(N)$ model. Therefore, the duality between LLFT and the large N gauge theory is a strong-weak coupling duality, contrary to the case for the $O(N)$ vector model.

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- [34] To prove this, consider $\langle \dots \tilde{U}_{i_3 j; \alpha \beta} \tilde{U}_{j i_2; \gamma \delta} \dots \rangle_Y$, where $\alpha, \beta, \gamma, \delta$ are color indices, while j, i_2, i_3 are site indices, and \dots represents insertions of link fields which do not include the site j . The color index β (γ) carry fundamental (anti-fundamental) charge of $U(N)$ at site j . Because S_Y is gauge invariant, we have $\langle \dots \tilde{U}_{i_3 j; \alpha \beta} \tilde{U}_{j i_2; \gamma \delta} \dots \rangle_Y = V_{\beta' \beta} V_{\gamma' \gamma}^* \langle \dots \tilde{U}_{i_3 j; \alpha \beta'} \tilde{U}_{j i_2; \gamma' \delta} \dots \rangle_Y$ for any $U(N)$ matrix V . The only $U(N)$ singlet tensor that can be constructed from one fundamental and one anti-fundamental indices is the identity. Therefore $\langle \dots \tilde{U}_{i_3 j; \alpha \beta} \tilde{U}_{j i_2; \gamma \delta} \dots \rangle_Y = A_{\alpha \delta} \delta_{\beta \gamma}$ with $A_{\alpha \delta} = \frac{1}{N} \langle \dots \tilde{U}_{i_3 j; \alpha \beta'} \tilde{U}_{j i_2; \beta' \delta} \dots \rangle_Y$. From this, one obtains Eq. (8).
- [35] When more than two Wilson loops cross at a point, one needs more general formula than Eq. (8). However, all non-vanishing contributions can be always written as products of Wilson loops.
- [36] This is why we deliberately singled out the factor of $N^{2-(m+n)}$ out of $h_{\{C_1, \dots, C_n\}; \{\tilde{C}_1, \dots, \tilde{C}_m\}}[\mathcal{J}]$.
- [37] This corresponds to a strong 't Hooft coupling expansion in the large N limit.
- [38] However, the analogy with Hamiltonian is not perfect because \mathcal{H} is not Hermitian.