

The Strange World of Non-amenable Symmetries ¹

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Abstract

Nonlinear sigma models with non-compact target space and non-amenable symmetry group were introduced long ago in the study of disordered electron systems. They also occur in dimensionally reduced quantum gravity; recently they have been considered in the context of the AdS/CFT correspondence. These models show spontaneous symmetry breaking in *any* dimension, even one and two (superficially in contradiction with the Mermin-Wagner theorem) as a consequence of the non-amenability of their symmetry group. The low-dimensional models show other peculiarities: invariant observables remain dependent on boundary conditions in the thermodynamic limit and the Osterwalder-Schrader reconstruction yields a non-separable Hilbert space. The ground state space, however, under quite general conditions, carries a unique unitary and continuous representation. The existence of a continuum limit in 2D is an open question: while the perturbative Renormalization Group suggests triviality, other arguments hint at the existence of a conformally invariant continuum limit at least for suitable observables.

This talk gives an overview of the work done during the last several years in collaboration first of all with Max Niedermaier, some of it also with Peter Weisz and Tony Duncan [1, 2, 3, 4, 5].

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1 Introduction

1.1 What are non-amenable symmetries?

The concept of amenable groups was introduced by J. von Neumann in 1929; it can be described as follows: let $\mathcal{C}(G)$ be the space of continuous bounded functions on G ; then a *mean* m is a positive (hence continuous) linear functional on $\mathcal{C}(G)$ (or on $L^\infty(G)$) with $m(\mathbb{1}) = 1$. Put differently: m is a *state* on the commutative C^* algebra $\mathcal{C}(G)$ (or $L^\infty(G)$). Since the group G has a natural left action on functions, it makes sense to speak about invariance of such a mean.

Definition: The group G is non-amenable if there is no mean on $\mathcal{C}(G)$ which is invariant under G .

A well-known fact is that noncompact semisimple Lie groups are non-amenable [6].

The concept can be generalized to homogeneous spaces G/H by using the algebra $\mathcal{C}(G/H)$ instead of $\mathcal{C}(G)$. One also speaks of (non-)amenable actions of a group G on a general G -space \mathcal{M} : the left action of G induces an action on $\mathcal{C}(\mathcal{M})$ and non-amenable means nonexistence of a mean invariant under G on $\mathcal{C}(\mathcal{M})$.

Bekka [7] has extended the definition to that of amenable unitary representations π on a Hilbert space \mathcal{H} as follows: π is called amenable if there is a state on $\mathcal{B}(\mathcal{H})$ which is invariant under π . In this context the following result is important: if G is simple, noncompact, connected, with finite center, rank > 1 , the trivial representation is the only amenable one.

1.2 Physics motivations

(1) Nonlinear σ models with hyperbolic target space – the prototype of a non-amenable symmetric space – were introduced 1979 by Wegner [8] to describe the conductor-insulator transition in disordered electron systems. Since then there has been a lot of activity, see for instance [9, 10, 11]. Later Efetov [12, 13] and Zirnbauer [14] introduced the supersymmetric version of that model as a better description of the electron system. This line of research was continued more recently in [15, 16, 17].

(2) Some ‘warped’ versions of nonlinear σ models with hyperbolic target

space arise in dimensionally reduced gravity and its quantization [18, 19, 20].

(3) Not surprisingly, these models also appear in the context of string theory; string theorists think of hyperbolic space as ‘Euclidean Anti-de Sitter space’ [21, 22].

2 Quantum Mechanics on hyperbolic spaces

Insight into the peculiarities of non-amenable symmetries is easiest to obtain by studying quantum mechanics on hyperbolic spaces. Hyperbolic space can be described as a hyperboloid imbedded in Minkowski space with the metric induced by the ambient space:

$$\mathbb{H}_N \equiv SO_o(1, N)/SO_o(N) \equiv G/K = \{n \in \mathbb{R}^{N+1} | n \cdot n = 1, n_o > 0\}, \quad (1)$$

where $n \cdot n' = n_o n'_o - \vec{n} \cdot \vec{n}'$.

2.1 One particle

Let Δ denote the Laplace-Beltrami operator on \mathbb{H}_N . The free one particle Hamiltonian, acting on $L^2(\mathbb{H}_N, d\Omega)$ where $d\Omega$ is an invariant measure on \mathbb{H}_N , is then

$$H = -\Delta \geq 0. \quad (2)$$

This Hamiltonian is diagonalized using the Mehler-Fock transformation [23]; it reveals that the spectrum of H is absolutely continuous, covering the interval $[(N-1)^2/4, \infty)$; there is no spectrum in the interval $[0, (N-1)^2/4)$, even though there are bounded eigenfunctions for every value in that interval (the supplementary series). Introducing a spectral parameter ω running from 0 to ∞ , we have the spectral resolution

$$L^2(\mathbb{H}_N, d\Omega) = \int_0^\infty dP(\omega) \mathcal{H}_\omega; \quad H\psi = \int_0^\infty dP(\omega) \left(\frac{1}{4}(N-1)^2 + \omega^2 \right) \psi. \quad (3)$$

Only the principal series appears; the spectrum is infinitely degenerate because all representations in that series are infinite dimensional. Of course there is no normalizable ground state vector; instead we have a ‘ground state

space' corresponding to $\omega = 0$ and spanned by 'generalized ground states' (functions in $\mathcal{C}(\mathcal{M})$ but $\notin L^2$) of the form

$$\mathcal{P}_{-1/2}^{1-N/2}(gn \cdot n^\uparrow), \quad g \in SO_o(1, N), \quad (4)$$

and linear combinations thereof, where $\mathcal{P}_{-1/2}^{1-N/2}$ are Legendre functions.

A different scalar product, produced by the Osterwalder-Schrader (OS) reconstruction makes these ground states normalizable. They then generate a Hilbert space of ground states carrying a special unitary irreducible representation σ_0 . But the main point is this:

**There is no invariant ground state
Spontaneous symmetry breaking (SSB) takes place!**

2.2 ν particles: separation of 'center of mass'

When we consider a ν particle system interacting via translation invariant potentials in Euclidean space, the first step is always to separate out the free center of mass motion. Here we consider ν particles on \mathbb{H}_N with a potential invariant under the symmetry group $G = SO_0(1, N)$, and again we would like to find a way to extract the rigid motions. This requires some tricks. Our Hilbert space is now $\mathcal{H} = L^2(\mathcal{M})$ ($\mathcal{M} = \mathbb{H}_N^\nu$) and the Hamiltonian is

$$H = - \sum_{i=1}^n \Delta_i + \sum_{i < j} V(n_i \cdot n_j) \equiv H_0 + \mathcal{V}. \quad (5)$$

Let $\ell_{\mathcal{M}}$ be the unitary representation of G on \mathcal{H} induced by the left diagonal action of G , representing rigid motions of the particle system. Clearly

$$[H, \ell_{\mathcal{M}}(G)] = 0, \quad (6)$$

We now turn the left diagonal action on \mathcal{M} into a right action on a different manifold $\tilde{\mathcal{M}}_r$ ins such a way that only one 'center of mass' variable is affected. First we define

$$\tilde{\mathcal{M}}_r \equiv G \times \mathcal{H}^{\nu-1} \quad (7)$$

and an injective but not surjective map $\tilde{\phi} : \mathcal{M} \rightarrow \tilde{\mathcal{M}}_r$ given by

$$\tilde{\phi}(n_1, \dots, n_n) = (g_s(n_1)^{-1}, g_s(n_1)^{-1}n_2, \dots, g_s(n_1)^{-1}n_\nu), \quad (8)$$

where g_s is a function (global section) $\mathbb{H}_N \rightarrow G$ such that $n = g_s(n^\dagger)$. g_s is obviously only determined up to $g_s \rightarrow g_s k^{-1}, k \in K$. Let $d_\ell(K)$ be the left diagonal action of K on $\tilde{\mathcal{M}}_r$ and define

$$\mathcal{M}_r = \tilde{\mathcal{M}}_r / d_\ell(K). \quad (9)$$

$\tilde{\phi}$ projects to a well-defined map $\phi : \mathcal{M} \rightarrow \mathcal{M}_r$ and this ϕ does the job of converting the left diagonal action $d_\ell(G)$ on \mathcal{M} into a right action $r(G)$ on \mathcal{M}_r acting only on the first entry:

$$r(g')[(g, n_1, \dots, n_\nu)] = [gg', n_1, \dots, n_\nu], \quad (10)$$

$$\phi \circ d_\ell = r \circ \phi. \quad (11)$$

ϕ induces a unitary map Φ between the corresponding Hilbert spaces $L^2(\mathcal{M}_r)$ and $L^2(\mathcal{M})$; the latter can be viewed as the subspace of $L^2(\tilde{\mathcal{M}}_r)$ invariant under the unitary map induced by $d_\ell(K)$.

The right action of G on the first entry of \mathcal{M}_r induces a unitary representation $\rho(G)$ of the rigid motions:

$$\rho = \Phi^{-1} \circ \ell_{\mathcal{M}} \circ \Phi \quad (12)$$

and $\rho(G)$ commutes with $\ell_r(K)$.

2.3 The ground state representation

The harmonic analysis of ρ is the analogue of the decomposition according to the center of mass momentum in flat space. The Hilbert space $\mathcal{H} = L^2(\mathcal{M}_r)$ decomposes into a direct integral of irreps

$$\mathcal{H} = \int_{\hat{G}_r}^{\oplus} d\nu(\sigma) \mathcal{H}(\sigma), \quad (13)$$

where \hat{G}_r is the restricted dual of G , which is the union of the principal and the discrete series (see [24])

$$\hat{G}_r = \hat{G}_p \cup \hat{G}_d; \quad (14)$$

$d\nu$ arises from the Plancherel measure.

The Hamiltonian on \mathcal{H} is $H_r = \Phi^{-1} \circ H \circ \Phi$; we drop the subscript r from now on. Because H commutes with ρ , it can also be resolved into fiber Hamiltonians

$$H = \int_{\widehat{G}_r}^{\oplus} d\nu(\sigma) h(\sigma), \quad (15)$$

which is analogous to the resolution of a ν particle Hamiltonian according to the c.m. momentum in flat space.

We conjecture that generally $d\nu$ is carried by \widehat{G}_p alone, but we can prove only that

$$\inf_{\sigma} \inf \text{spec } h(\sigma) \notin \widehat{G}_d. \quad (16)$$

This is seen most easily under a certain compactness condition on the interaction \mathcal{V} , namely

$$\text{tr } e^{-t(H_o + \mathcal{V} + \nu_1)} \leq \text{tr } (e^{-tH_o} e^{-t(\mathcal{V} + \nu_1)}) < \infty. \quad (17)$$

This implies that the fiber Hamiltonians $h(\cdot)$ have discrete spectrum; for $\sigma \in \widehat{G}_d$ the ground state of the fiber Hamiltonian would give rise to a (normalizable) eigenfunction of H ; because σ is not the trivial representation, this ground state could not be unique. This leads to a contradiction with the Perron-Frobenius theorem. We can show furthermore that the ground state representation is always σ_0 , the special representation found for the one particle case. Details can be found in [3], where, however, we deal with a discrete time evolution given by a transfer matrix.

The ground state representation σ_0 is universal, and the fact that it is non-trivial means again that there is SSB.

3 Statistical mechanics / lattice quantum field theory

3.1 Action, Gibbs state

We consider configurations of ‘spins’ given by mapping each site $x \in \Lambda \subset \mathbb{Z}^d$ to a $n(x) \in \mathbb{H}_N$. The Gibbs measure is formally given by

$$\exp(-\beta S) \prod_x d\Omega(x), \quad (18)$$

with (for instance)

$$S = \sum_{\langle xy \rangle} n(x) \cdot n(y). \quad (19)$$

To make the Gibbs measure normalizable, ‘gauge fixing’ is needed. The simplest choice is to fix a spin at the boundary of the finite lattice Λ .

3.2 Spontaneous symmetry breaking

If in the thermodynamic limit $\Lambda \nearrow \mathbb{Z}^d$ the Gibbs state is not invariant under G , we speak of SSB. Non-amenability *enforces* SSB, because if there were a symmetric Gibbs measure, it would automatically induce an invariant mean on the functions of a single spin. This holds independent of the dimension d or any other details (type of lattice, action).

SSB is unavoidable!

Note that the Mermin-Wagner theorem is not in conflict with this finding: it forbids SSB only for *compact* symmetry groups in dimensions 1 and 2.

3.3 The hyperbolic spin chain

For $d = 1$ the problem can be solved analytically to a large extent [1]. ‘Gauge fixing’ is done by fixing the spin at the left hand end of the chain, say

$$n(-L) = n^\dagger. \quad (20)$$

As the general considerations require, SSB occurs in the form that the system remembers the orientation of the spin $n(-L)$ even in the limit $L \rightarrow \infty$. A concrete ‘order parameter’ that shows this is

$$T_e(n(0)) := \tanh(n(0) \cdot e), \quad (21)$$

where $e \cdot e = -1$. In [1] it is shown that

$$\lim_{L \rightarrow \infty} \langle T_e(n(0)) \rangle_{bc} = 1 - \frac{2}{\pi} \cos^{-1} \left(\frac{e \cdot n^\dagger}{\sqrt{1 + (e \cdot n^\dagger)^2}} \right). \quad (22)$$

Two facts are might be unexpected:

- (1) even the expectation values of invariant functions, such as $n(0) \cdot n(x)$ remain dependent on b.c. in the thermodynamic limit.
- (2) The OS reconstruction of a Hilbert space from the correlation functions yields a *non-separable* space and discontinuous representations, except for the ground state space. This is related to the fact that the construction always produces a *normalizable* ground state, even though there is none in the original L^2 space, so in some sense the OS reconstruction renormalizes the scalar product.

3.4 Two or more dimensions

Not very much is known rigorously beyond the fact of SSB. The models do not have high temperature expansions; presumably they are massless at all temperatures.

Spencer and Zirnbauer [15] have shown, however, that in dimension 3 or more at low temperature there is a ‘stronger version’ of SSB that presumably is not true in dimensions 1 or 2 or at high temperatures; namely the quadratic fluctuations away from the mean ‘magnetization’ have a finite expectation value.

In [2] we carried out some detailed numerical simulation of the model in $d = 2$ with a different (translation invariant) gauge fixing. We found

- (1) The explicit symmetry violations due to the gauge fixing disappear in the thermodynamic limit; this is seen by verifying Ward identities
- (2) SSB is seen by looking at $\langle T(e) \rangle$ as above.
- (3) The thermodynamic limit for invariant observables seems to exist.

4 Quantum field theoretic considerations

4.1 Peculiarities of the Osterwalder-Schrader reconstruction

As in the hyperbolic spin chain, the OS reconstruction will presumably always lead to a nonseparable Hilbert space. This is a consequence of the fact that

the construction always yields a normalized ground state, even though the spectrum of the transfer matrix is most likely continuous.

This somewhat unphysical feature might be avoided by restricting the space of observables. For instance one might restrict attention to only a certain component of the spins, or maybe a special (horospherical) coordinate and functions of it. In this way one would of course lose the signal of SSB.

4.2 Existence of a continuum limit?

Consideration of the perturbative one loop Renormalization Group [25, 26] yields essentially the Ricci flow, indicating that the model is *infrared asymptotically free*. In the case at hand, however, this is counterintuitive: if the long-distance fluctuations become Gaussian, as infrared asymptotic freedom would predict, this would mean that they don't feel the curvature of the target manifold. But in the infrared the fluctuations necessarily cover the target space over large distances and therefore should become extremely sensitive to the curvature.

But if the conventional wisdom is right, it would suggest that there is no nontrivial continuum limit of 2D nonlinear σ models whose target space has negative curvature; the situation would be similar to the QED₄ or ϕ_4^4 quantum field theories, which are believed to be trivial (i.e. Gaussian) in the continuum limit.

A counterpoint has been provided long ago by Haba [27], who by a formal calculation of the 2D hyperbolic σ model concluded that it corresponded to a conformal quantum field theory with central Virasoro charge $c = 1$, as long as $\beta > 1/3\pi$ (it should be noted, however, that he considered only correlations of a so-called horospherical coordinate on the hyperbolic plane). It would be very interesting to know if this formal calculation can be justified.

4.3 Axiomatic considerations

When considering possible continuum limits, it is worthwhile to pause and think how such limits could possibly look, in agreement with the axiomatic structure of quantum field theory.

One thing becomes clear immediately: it is not possible to have a multiplet of quantum fields ϕ_i transforming under the non-unitary vector representation

of $G = SO(1, N)$ with unbroken symmetry (as one would expect naively if the ϕ_i are continuum fields arising from renormalizing the basic spin components n_i).

The reasoning goes like this: an unbroken symmetry means that there is a unitary representation $U(\cdot)$ of the symmetry group G leaving the vacuum state invariant and transforming the fields according to the vector representation, i.e.

$$U(g)^{-1}\phi_i(x)U(g) = \sum_j (g^{-1})_{ij} \phi_j(x). \quad (23)$$

This leads to a conflict with the positive metric in Hilbert space when considering orbits of $U(\cdot)\psi$: let $\phi_0(f), \phi_1(f)$ be field components smeared with a test function f . Then by the unitarity of $U(\cdot)$

$$\|(\phi_0(f) \operatorname{ch} t + \phi_1(f) \operatorname{sh} t) \Omega\|^2 = \|\phi_0(f) \Omega\|^2 \quad \forall t, \quad (24)$$

which is impossible unless all $\phi_i = 0$.

Possible alternatives are:

- (1) There is SSB, hence no unitary representation $U(\cdot)$ of G (see for instance-blot)
- (2) There is an infinite multiplet of fields, transforming according to a unitary representation of G – this could, however, not correspond to a continuum limit of the lattice model
- (3) A Quantum Field Theory arises only for a subset of fields. The symmetry is then not visible. Haba's computation suggests that this might be the right scenario.

5 Conclusions, open questions

- (1) We have found a certain universal ground state representation both in Quantum Mechanic and Lattice field theory (in a finite spatial volume).
- (2) There is always SSB; the Mermin-Wagner theorem does not apply.
- (3) In a potential continuum limit also Coleman's version of the Mermin-Wagner theorem would not apply because the currents needed for this argument don't have thermodynamic and continuum limits.
- (4) In $D \geq 3$ for large β there is SSB of the conventional kind: with large fluctuations suppressed[15].

- (5) There is probably no mass gap, but a proof is lacking.
- (6) In 2D infrared asymptotic freedom is suggested by perturbation theory, but there is no proof; the existence of a continuum limit remains an unsolved question.

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