Period differential equations for families of $K3$ surfaces derived from 3 dimensional reflexive polytopes with 5 vertices

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December 2, 2010

Abstract

In this article we study the families of $K3$ surfaces derived from 3 dimensional 5 verticed reflexive polytopes with at most terminal singularity. We determine the lattice structures, the period differential equations and the projective monodromy groups for these families.

Introduction

A $K3$ surface S is characterized by the condition $K_S = 0$ and simply connectedness. It means that S is a 2-dimensional Calabi-Yau manifold. V. V. Batyrev [Ba] introduced the notion of the reflexive polytope for the study of Calabi-Yau manifolds.

In this article we use the 3-dimensional reflexive polytopes with at most terminal singularities. Such a polytope P is defined by the intersection of several half spaces

$$a_j x + b_j y + c_j z \le 1, \quad (a_j, b_j, c_j) \in \mathbb{Z}^3 \ (j = 1, \cdots, s)$$

in $\mathbb{R}^3$ with the conditions

(i) every vertex is a lattice point,

(ii) the origin is the unique inner lattice point,

(iii) only the vertices are the lattice points on the boundary.

Moreover, if a reflexive polytope satisfies the condition

(iv) every face is triangle and its 3 vertices generate the lattice,

it is called a Fano polytope.

All 3-dimensional 5-verticed reflexive polytopes with at most terminal singularity are listed up (see [KS] or [O]):

$$P_1 = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & -1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & -1 & -1 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \end{pmatrix},$$

$$P_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & -1 \end{pmatrix}, \quad P_5 = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & -2 \end{pmatrix}.$$

Among them, P_2, P_3, P_4 and P_5 are Fano polytopes.

We can find a family of $K3$ surfaces for each polytope by a natural method. In this article we study the polytopes P_1, P_2 and P_3. Namely, we determine the lattice structure, the period differential equation and the projective monodromy group for each of them.

Keywords: $K3$ surfaces ; period differential equations ; toric varieties
Mathematics Subject Classification 2010: 14J28, 14K99, 33C70, 52B10

T. Ishige [I] has made a detailed research on the family of $K3$ surfaces coming from the polytope P_4. Especially he noticed characterization of the corresponding monodromy group by a numerical approach.

Inspired by Ishige's discovery, we have studied families of $K3$ surfaces derived from the other polytopes P_1, P_2, P_3 and P_5. We have made an intensive study on the polytope P_5 in our previous article [Na]. There, we have studied the period map for a family, saying $\mathcal{F} = \{S(\lambda, \mu)\}$, of $K3$ surfaces, where

$$S(\lambda, \mu) : xyz^2(x + y + z + 1) + \lambda xyz + \mu.$$

Namely, we have determined the lattice structure of a generic member of the family $\mathcal{F}$, the period differential equation and the projective monodromy group using the Torelli type theorem for polarized $K3$ surfaces and the lattice theory. Furthermore, we have shown that our differential equation coincides with the uniformizing differential equation of the Hilbert modular orbifold for $\mathbb{Q}(\sqrt{5})$ studied by Sasaki-Yoshida [SY] and T. Sato [Sa].

Here, we study the remaining cases P_1, P_2 and P_3. Namely we investigate corresponding families $\mathcal{F}_j$ $(j = 1, 2, 3)$ of $K3$ surfaces.

In Section 1, we show explicit defining equations for the families $\mathcal{F}_j = \{S_j(\lambda, \mu)\}$ $(j = 1, 2, 3)$ (see (1.1), (1.2) and (1.3)) and we introduce elliptic fibrations for these families. The singular fibres of each elliptic fibration are described in Table 1.

In Section 2, we determine the lattice structure for a generic member of each family $\mathcal{F}_j$ $(j = 1, 2, 3)$. Namely, we obtain the Néron-Severi lattice $\mathrm{NS}(S_j(\lambda, \mu))$ $(j = 1, 2, 3)$ as in Table 2. Note that in the case P_5 we could determine $NS(S(\lambda, \mu))$ for $S(\lambda, \mu) \in \mathcal{F}$ by a naive method (see [Na]). In this article we need more advanced theory of the Mordell-Weil lattice due to T. Shioda [Sh1].

For 95 weighted projective $K3$ surfaces, there is a result of S. M. Belcastro [Be]. And for $K3$ surfaces with non-symplectic involution, there is a result of V. V. Nikulin [Ni]. Our case is not contained in these results. Furthermore, we note that the result of K. Koike [Koi] and our result in this article support the mirror symmetry conjecture (see Remark 2.1).

In Section 3, we determine the period differential equations (Theorem 3.2). Furthermore, we obtain their monodromy groups (Theorem 3.3).

1 Families of $K3$ surfaces and elliptic fibrations

We obtain a family of algebraic surfaces by the following canonical procedure from P_j $(j = 1, 2, 3)$:

(i) Make a toric 3-fold X_j from the reflexive polytope P_j. This is a rational variety.

(ii) Take a divisor D on X_j that is linearly equivalent to $-K_{X_j}$.

(iii) Generically D is represented by a $K3$ surface.

We obtain the corresponding families of $K3$ surfaces $\mathcal{F}_j = \{S_j(\lambda, \mu)\}$ for P_j $(j = 1, 2, 3)$ given by

$$S_1(\lambda, \mu) : xyz(x + y + z + 1) + \lambda x + \mu y = 0, \quad (1.1)$$

$$S_2(\lambda, \mu) : xyz(x + y + z + 1) + \lambda x + \mu = 0, \quad (1.2)$$

$$S_3(\lambda, \mu) : xyz(x + y + z + 1) + \lambda z + \mu xy = 0. \quad (1.3)$$

We can find an elliptic fibration for every surface of our family $\mathcal{F}_j$ $(j = 1, 2, 3)$. Moreover we can describe these surfaces in the form

$$y^2 = x^3 - g_2(z)x_3 - g_3(z),$$

where g_2 (g_3, resp.) is a polynomial of z with $5 \leq \deg(g_2) \leq 8$ ($7 \leq \deg(g_3) \leq 12$, resp.). In this paper we call it the Kodaira normal form. From the Kodaira normal form we can obtain singular fibres of elliptic fibration. Corresponding singular fibres of our elliptic fibration of $\mathcal{F}_j$ $(j = 1, 2, 3)$ are shown in Table 1.

Family	$\mathcal{F}_1$	$\mathcal{F}_2$	$\mathcal{F}_3$
Singular Fibres	$I_9 + I_3^* + 6I_1$	$I_1^* + I_{11} + 6I_1$	$I_9 + I_9 + 6I_1$

Table 1.

1.1 $\mathcal{F}_1$

Proposition 1.1. (1) *The surface* $S_1(\lambda,\mu)$ *is birationally equivalent to the surface defined by the equation*

$$z_1^2 = y_1^3 + (\mu^2 + 2\mu x_1 + x_1^2 - 4x_1^3)y_1^2 + (-8\lambda\mu x_1^3 - 8\lambda x_1^4)y_1 + 16\lambda^2 x_1^6. \tag{1.4}$$

This equation gives an elliptic fibration of $S_1(\lambda,\mu)$.

(2) *The elliptic surface given by* (1.4) *has the holomorphic section*

$$P : x_1 \mapsto (x_1, y_1, z_1) = (x_1, 0, 4\lambda x_1^3). \tag{1.5}$$

Proof. (1) By the birational transformation

$$x = -\frac{2x_1^2 y_1}{-4\lambda x_1^3 + \mu y_1 + x_1 y_1 + z_1}, y = \frac{y_1^2}{2x_1(-4\lambda x_1^3 + \mu y_1 + x_1 y_1 + z_1)}, z = -\frac{-4\lambda x_1^3 + \mu y_1 + x_1 y_1 + z_1}{2x_1 y_1},$$

(1.1) is transformed to (1.4).

(2) This is apparent. □

(1.4) gives an elliptic fibration for the surface $S_1(\lambda,\mu)$. Set

$$\Lambda_1 = \{(\lambda,\mu) \in \mathbb{C}^2 | \lambda\mu(729\lambda^2 - 54\lambda(27\mu - 1) + (1 + 27\mu)^2 \neq 0)\}. \tag{1.6}$$

Proposition 1.2. *Suppose* $(\lambda,\mu) \in \Lambda_1$. *The elliptic surface given by* (1.4) *has the singular fibres of type* I_9 *over* $x_1 = 0$, *of type* I_3^* *over* $x_1 = \infty$, *and other six fibres of type* I_1.

Proof. (1.4) is described in the Kodaira normal form

$$z_1^2 = y_2^3 - g_2(x_1)y_2 - g_3(x_1), \quad x_1 \neq \infty,$$

with

$$\begin{cases} g_2(x_1) = -\Big(-\dfrac{\mu}{3} - \dfrac{4\mu^3 x_1}{3} - 2\mu^2 x_1^2 - \dfrac{4\mu x_1^3}{3} - 8\lambda\mu x_1^3 + \dfrac{x_1^4}{3} - 8\lambda x_1^4 + \dfrac{16\mu x_1^4}{3} + \dfrac{8x_1^5}{3} - \dfrac{16x_1^6}{3}\Big), \\ g_3(x_1) = -\Big(\dfrac{2\mu^6}{27} + \dfrac{4\mu^5 x_1}{9} + \dfrac{10\mu^4 x_1^2}{9} + \dfrac{40\mu^3 x_1^3}{9} + \dfrac{8\lambda\mu^3 x_1^3}{3} - \dfrac{8\mu^4 x_1^3}{9} + \dfrac{10\mu^2 x_1^4}{9} \\ \qquad +8\lambda\mu^2 x_1^4 - \dfrac{32\mu^3 x_1^4}{9} + \dfrac{4\mu x_1^5}{9} + 8\lambda\mu x_1^5 - \dfrac{16\mu^2 x_1^5}{3} + \dfrac{2x_1^6}{27} + \dfrac{8\lambda x_1^6}{3} + 16\lambda^2 x_1^6 - \dfrac{32\mu x_1^6}{9} \\ \qquad -\dfrac{32\lambda\mu x_1^6}{3} + \dfrac{32\mu^2 x_1^6}{9} - \dfrac{8x_1^7}{9} - \dfrac{32\lambda x_1^7}{3} + \dfrac{64\mu x_1^7}{9} + \dfrac{32x_1^8}{9} - \dfrac{128x_1^9}{27}\Big), \end{cases}$$

and

$$z_2^2 = y_3^3 - h_2(x_2)y_3 - h_3(x_2), \quad x_2 \neq \infty,$$

with

$$\begin{cases} h_2(x_2) = -\Big(-\dfrac{16x_2^2}{3} + \dfrac{8x_2^3}{3} - \dfrac{x_2^4}{3} - 8\lambda x_2^4 + \dfrac{16\mu x_2^4}{3} - \dfrac{4\mu x_2^5}{3} - 8\lambda\mu x_2^5 + \dfrac{8\mu^2 x_2^5}{3} - 2\mu^2 x_2^6 - \dfrac{4\mu^3 x_2^7}{3} - \dfrac{\mu^4}{3}\Big), \\ h_3(x_2) = -\Big(-\dfrac{128x_2^3}{27} + \dfrac{32x_2^4}{9} - \dfrac{8x_2^5}{9} - \dfrac{32\lambda x_2^5}{3} + \dfrac{64\mu x_2^5}{9} + \dfrac{2x_2^6}{27} + \dfrac{8\lambda x_2^6}{3} + 16\lambda^2 x_2^6 - \dfrac{32\mu x_2^6}{9} \\ \qquad -\dfrac{32\lambda\mu x_2^6}{3} + \dfrac{32\mu^2 x_2^6}{9} + \dfrac{4\mu x_2^7}{9} + 8\lambda\mu x_2^7 - \dfrac{16\mu^2 x_2^7}{3} + \dfrac{10\mu^2 x_2^8}{9} + 8\lambda\mu^2 x_2^8 - \dfrac{32\mu^4 x_2^8}{9} \\ \qquad +\dfrac{40\mu^5 x_2^{11}}{27} + \dfrac{8\lambda\mu^3 x_2^9}{3} - \dfrac{8\mu^4 x_2^9}{9} + \dfrac{10\mu^4 x_2^{10}}{9} + \dfrac{4\mu^5 x_2^{11}}{9} + \dfrac{2\mu^6 x_2^{12}}{27}\Big), \end{cases}$$

where $x_1 = 1/x_2$. We have the discriminant of the right hand side for $y_1 (y_2$, resp.):

$$\begin{cases} D_0 = 256\lambda^2 x_1^9(\lambda\mu^3 - \mu^4 + 3\lambda\mu^2 x_1 - 4\mu^3 x_1 + 3\lambda\mu x_1^2 - 6\mu^2 x_1^2 + \lambda x_1^3 + 27\lambda^2 x_1^3 \\ \qquad -4\mu x_1^3 - 36\lambda\mu x_1^3 + 8\mu^2 x_1^3 - x_1^4 - 36\lambda x_1^4 + 16\mu x_1^4 + 8x_1^5 - 16x_1^6), \\ D_\infty = 256\lambda^2 x_2^9(-16 + 8x_2 - x_2^2 - 36\lambda x_2^2 + 16\mu x_2^2 + \lambda x_2^3 + 27\lambda^2 x_2^3 - 4\mu x_2^3 - 36\lambda\mu x_2^3 \\ \qquad 8\mu^2 x_2^3 + 3\lambda\mu x_2^4 - 6\mu^2 x_2^4 + 3\lambda\mu^2 x_2^5 - 4\mu^3 x_2^5 + \lambda\mu^3 x_2^6 - \mu^4 x_2^6). \end{cases}$$

From these deta, we obtain the required statement (see [Kod]). □

1.2 $\mathcal{F}_2$

Proposition 1.3. (1) *The surface $S_2(\lambda,\mu)$ is birationally equivalent to the surface defined by the equation*

$$z_1^2 = x_1^3 + (-4\lambda y + y^2 + 2y^3 + y^4)x_1^2 + (-8\mu y^3 - 8\mu y^4)x_1 + 16\mu^2 y^4. \tag{1.7}$$

This equation gives an elliptic fibration of $S_2(\lambda,\mu)$.

(2) *The elliptic surface given by* (1.7) *has the holomorphic section*

$$P : y \mapsto (x_1, y, z_1) = (0, y, 4\mu y^2) \tag{1.8}$$

Proof. (1) By the birational transformation

$$x = \frac{x_1^2}{2y(x_1 y - 4\mu y^2 + x_1 y + z_1)}, z = -\frac{x_1 y - 4\mu y^2 + x_1 y + z_1}{2x_1 y},$$

(1.2) is transformed to (1.7).

(2) This is apparent. □

(1.7) gives an elliptic fibration for $S_2(\lambda,\mu)$. Set

$$\Lambda_2 = \{(\lambda,\mu) \in \mathbb{C}^2 | \lambda\mu(\lambda^2(1+27\lambda)^2 - 2\lambda\mu(1+189\lambda) + (1+576\lambda)\mu^2 - 256\mu^3) \neq 0\}. \tag{1.9}$$

Proposition 1.4. *Suppose $(\lambda,\mu) \in \Lambda_2$. The elliptic surface given by* (1.7) *has the singular fibres of type I_1^* over $y = 0$, of type I_{11} over $y = \infty$, and other six fibres of type I_1.*

Proof. (1.7) is described in the Kodaira normal form

$$z_1^2 = x_2^3 - g_2(y)x_2 - g_3(y), \quad y \neq \infty,$$

with

$$\begin{cases} g_2(y) = -\Big(-\frac{16\lambda^2 y^2}{3} + \frac{8\lambda y^3}{3} - 8\mu y^3 - \frac{y^4}{3} + \frac{16\lambda y^4}{3} - 8\mu y^4 - \frac{4y^5}{3} + \frac{8\lambda y^5}{3} - 2y^6 - \frac{4y^7}{3} - \frac{y^8}{3}\Big), \\ g_3(y) = -\Big(-\frac{128\lambda^3 y^3}{27} + \frac{32\lambda^2 y^4}{9} - \frac{32\lambda\mu y^4}{3} + 16\mu^2 y^4 - \frac{8\lambda y^5}{9} + \frac{64\lambda^2 y^5}{9} + \frac{8\mu y^5}{3} - \frac{32\lambda\mu y^5}{3} + \frac{2y^6}{27} - \frac{32\lambda^2 y^6}{9} \\ \qquad +8\mu y^6 + \frac{4y^7}{9} - \frac{16\lambda y^7}{3} + 8\mu y^7 + \frac{10y^8}{9} - \frac{32\lambda y^8}{9} + \frac{8\mu y^8}{3} + \frac{40y^9}{27} - \frac{8\lambda y^9}{9} + \frac{10y^{10}}{9} + \frac{4y^{11}}{9} + \frac{2y^{12}}{27}\Big), \end{cases}$$

and

$$z_2^2 = x_3^3 - h_2(y_1)x_3 - h_3(y_1), \quad y_1 \neq \infty,$$

with

$$\begin{cases} h_2(y_1) = -\Big(-\frac{1}{3} - \frac{4y_1}{3} - 2y_1^2 - \frac{4y_1^2}{3} + \frac{8\lambda y_1^3}{3} - \frac{y_1^4}{3} + \frac{116\lambda y_1^4}{3} - 8\mu y_1^4 + \frac{8\lambda y_1^5}{3} - 8\mu y_1^5 - \frac{16\lambda^2 y_1^6}{3}\Big), \\ h_3(y_1) = -\Big(\frac{2}{27} + \frac{4y_1}{9} + \frac{10y_1^2}{9} + \frac{40y_1^3}{27} - \frac{8\lambda y_1^3}{9} + \frac{10y_1^4}{9} - \frac{32\lambda y_1^4}{9} + \frac{8\mu y_1^4}{3} + \frac{4y_1^5}{9} - \frac{16\lambda y_1^5}{3} + 8\mu y_1^5 \\ \qquad +\frac{2y_1^6}{27} - \frac{32\lambda y_1^6}{9} + \frac{32\lambda^2 y_1^6}{9} + 8\mu y_1^6 - \frac{8\lambda y_1^7}{9} + \frac{64\lambda^2 y_1^7}{9} + \frac{8\mu y_1^9}{3} \\ \qquad -\frac{32\lambda\mu y_1^7}{3} + \frac{32\lambda^2 y_1^8}{9} - \frac{32\lambda\mu y_1^8}{3} + 16\mu^2 y_1^8 - \frac{128\lambda^3 y_1^9}{27}\Big), \end{cases}$$

where $y = 1/y_1$. We have the discriminant of the right hand side for x_2(x_3, resp.):

$$\begin{cases} D_0 = -256\mu^2 y^7(16\lambda^3 - 8\lambda^2 y + 36\lambda\mu y - 27\mu^2 y + \lambda y^2 - 16\lambda^2 y^2 - \mu y^2 + 36\lambda\mu y^2 + 4\lambda y^3 \\ \qquad -8\lambda^2 y^3 - 3\mu y^3 + 6\lambda y^4 - 3\mu y^4 + 4\lambda y^5 - \mu y^5 + \lambda y^6), \\ D_\infty = -256\mu^2 y_1^{11}(\lambda + 4\lambda y_1 - \mu y_1 + 6\lambda y_1^2 - 3\mu y_1^2 + 4\lambda y_1^3 - 8\lambda^2 y_1^3 - 3\mu y_1^3 + \lambda y_1^4 - 16\lambda^2 y_1^4 \\ \qquad -\mu y_1^4 + 36\lambda\mu y_1^4 - 8\lambda^2 y_1^5 + 36\lambda\mu y_1^5 - 27\mu^2 y_1^5 + 16\lambda^3 y_1^6). \end{cases}$$

From these data, we obtain the required statement. □

1.3 $\mathcal{F}_3$

Proposition 1.5. (1) *The surface $S_3(\lambda,\mu)$ is birationally equivalent to the surface defined by the equation*

$$y_1^2 = z_1^3 + (\lambda^2 + 2\lambda x_1 + x_1^2 - 4\mu x_1^2 - 4x_1^3)z_1^2 + 16\mu x_1^5. \tag{1.10}$$

This equation gives an elliptic fibration of $S_3(\lambda,\mu)$.

(2) *The elliptic surface given by* (1.10) *has the holomorphic sections*

$$\begin{cases} P : z_1 \mapsto (x_1, y_1, z_1) = (x_1, 4\mu x_1^2(x_1+\lambda), 4x_1^2\mu, \\ O' : z_1 \mapsto (x_1, y_1, z_1) = (x_1, 0, 0). \end{cases} \tag{1.11}$$

The section O' satisfies $2O' = O$.

Proof. (1) By the birational transformation

$$x = \frac{2x_1^2(4\mu x_1^2 - z_1)}{y_1 + \lambda z_1 + x_1 z_2}, y = \frac{y_1 + \lambda z_1 + x_1 z_1}{2x_1(4\mu x_1^2 - z_1)}, z = -\frac{z_1(4\mu x_1^2 - z_1)}{2x_1(y_1 + \lambda z_1 + x_1 z_1)},$$

(1.3) is transformed to (1.10).

(2) This is apparent. □

(1.10) gives an elliptic fibration for $S_3(\lambda,\mu)$. Set

$$\Lambda_3 = \{(\lambda,\mu) \in \mathbb{C}^2 | \lambda\mu(729\lambda^2 - (4\mu-1)^3 + 54\lambda(1+12\mu)) \neq 0\}. \tag{1.12}$$

Proposition 1.6. *Suppose $(\lambda,\mu) \in \Lambda_3$. The elliptic surface given by* (1.10) *has the singular fibres of type I_{10} over $z = 0$, of type I_2^* over $z = \infty$, and other six fibres of type I_1.*

Proof. (1.10) is described in the Kodaira normal form

$$y_1^2 = z_2^3 - g_2(x_1)z_2 - g_3(x_1), \quad x_1 \neq \infty,$$

with

$$\begin{cases} g_2(x_1) = -\Big(-\frac{\lambda^4}{3} - \frac{4\lambda^3 x_1}{3} - 2\lambda^2 x_1^2 + \frac{8\lambda^2\mu x_1^2}{3} - \frac{4\lambda x_1^3}{3} + \frac{8\lambda^2 x_1^3}{3} + \frac{16\lambda\mu x_1^3}{3} \\ \qquad -\frac{x_1^4}{3} + \frac{16\lambda x_1^4}{3} + \frac{8\mu x_1^4}{3} - \frac{16\mu^2 x_1^4}{3} + \frac{8x_1^5}{3} + \frac{16\mu x_1^5}{3} - \frac{16x_1^6}{3}\Big), \\ g_3(x_1) = -\Big(\frac{2\lambda^6}{27} + \frac{4\lambda^5 x_1}{9} + \frac{10\lambda^4 x_1^2}{9} - \frac{8\lambda^4\mu x_1^2}{9} + \frac{40\lambda^3 x_1^3}{27} - \frac{8\lambda^4 x_1^3}{9} - \frac{32\lambda^3\mu x_1^3}{9} + \frac{10\lambda^2 x_1^4}{9} - \frac{32\lambda^3 x_1^4}{9} \\ \quad -\frac{16\lambda^2 x_1^4}{3} + \frac{32\lambda^2\mu^2 x_1^4}{9} + \frac{4\lambda x_1^5}{9} - \frac{16\lambda^2 x_1^5}{3} - \frac{32\lambda\mu x_1^5}{9} + \frac{16\lambda^2\mu x_1^5}{9} + \frac{64\lambda\mu x_1^5}{9} + \frac{2x_1^6}{27} - \frac{32\lambda x_1^6}{9} + \frac{32\lambda^2 x_1^6}{9} \\ \quad -\frac{8\mu x_1^6}{9} + \frac{32\lambda\mu x_1^6}{9} + \frac{32\mu^2 x_1^6}{9} - \frac{128\mu^3 x_1^6}{27} - \frac{8x_1^7}{9} + \frac{64\lambda x_1^7}{9} + \frac{16\mu x_1^7}{9} + \frac{64\mu^2 x_1^7}{9} + \frac{32x_1^8}{9} + \frac{64\mu x_1^8}{9} - \frac{128x_1^9}{27}\Big), \end{cases}$$

and

$$y_2^2 = z_3^3 - h_2(x_2)z_3 - h_3(x_2), \quad x_2 \neq \infty,$$

with

$$\begin{cases} h_2(x_2) = -\Big(-\dfrac{16x_2^2}{3} + \dfrac{8x_2^3}{3} + \dfrac{16\mu x_2^3}{3} - \dfrac{x_2^4}{3} + \dfrac{16\lambda x_2^4}{3} + \dfrac{8\mu x_2^4}{3} - \dfrac{16\mu^2 x_2^4}{3} - \dfrac{4\lambda x_2^5}{3} \\ \qquad + \dfrac{8\lambda^2 x_2^5}{3} + \dfrac{\lambda\mu x_2^5}{3} - 2\lambda^2 x_2^6 + \dfrac{8\lambda^2\mu x_2^6}{3} - \dfrac{4\lambda^3 x_2^7}{3} - \dfrac{\lambda^4 x_2^8}{3}, \Big), \\ h_3(x_2) = -\Big(-\dfrac{128x_2^3}{27} + \dfrac{32x_2^4}{9} + \dfrac{64\mu x_2^4}{9} - \dfrac{8x_2^5}{9} + \dfrac{64\lambda x_2^5}{9} + \dfrac{2x_2^6}{27} - \dfrac{32\lambda x_2^6}{9} + \dfrac{32\lambda^2 x_2^6}{9} - \dfrac{8\mu x_2^6}{9} \\ \qquad + \dfrac{32\lambda\mu x_2^6}{9} + \dfrac{32\mu^2 x_2^6}{9} - \dfrac{128\mu^3 x_2^6}{27} + \dfrac{4\lambda x_2^7}{9} - \dfrac{16\lambda^2 x_2^7}{3} - \dfrac{32\lambda^3 x_2^8}{9} - \dfrac{16\lambda^2\mu x_2^8}{9} - \dfrac{32\lambda^2\mu^2 x_2^8}{9} \\ \qquad + \dfrac{40\lambda^3 x_2^9}{27} - \dfrac{8\lambda^4 x_2^9}{9} - \dfrac{32\lambda^3\mu x_2^9}{9} + \dfrac{10\lambda^4 x_2^{10}}{9} - \dfrac{8\lambda^4\mu x_2^{10}}{9} + \dfrac{4\lambda^5 x_2^{11}}{9} + \dfrac{2\lambda^6 x_2^{12}}{9} \Big), \end{cases}$$

where $x_1 = 1/x_2$. We have the discriminant of the right hand side for z_2 (z_3 resp):

$$\begin{cases} D_0 = -256\mu^3 x_1^{10}(\lambda^4 + 4\lambda^3 x_1 + 6\lambda^2 x_1^2 - 8\lambda^2\mu x_1^2 + 4\lambda x_1^3 - 8\lambda^2 x_1^3 - 16\lambda\mu x_1^3 \\ \qquad + x_1^4 - 16\lambda x_1^4 - 8\mu x_1^4 + 16\mu^2 x_1^4 - 8x_1^5 - 32\mu x_1^5 + 16x_1^6), \\ D_\infty = -256\mu^2 x_2^8(16 - 8x_2 - 32\mu x_2 + x_2^2 - 16\lambda x_2^2 - 8\mu x_2^2 + 16\mu^2 x_2^2 \\ \qquad + 4\lambda x_2^3 - 8\lambda^2 x_2^3 - 16\lambda\mu x_2^3 + 6\lambda^2 x_2^4 - 8\lambda^2\mu x_2^4 + 4\lambda^3 x_2^5 + \lambda^4 x_2^6). \end{cases}$$

From these data, we obtain the required statement. □

We need another elliptic fibration.

Proposition 1.7. (1) *The surface $S_3(\lambda,\mu)$ is birationally equivalent to the surface defined by the equation*

$$y_1'^2 = x_1'^3 + (\mu^2 + 2\mu z + z^2 + 2\mu z^2 + 2z^3 + z^4)x_1'^2 + (-8\lambda\mu z^3 - 8\lambda z^4 - 8\lambda z^5)x_1' + 16\lambda^2 z^6. \qquad (1.13)$$

This equation gives an elliptic fibration of $S_3(\lambda,\mu)$.

(2) *The elliptic surface given by* (1.13) *has the holomorphic sections*

$$\begin{cases} P : z \mapsto (x_1', y_1', z) = (0, 4\lambda z^3, z), \\ Q : z \mapsto (x_1', y_1', z) = (0, -4\lambda z^3, z). \end{cases} \qquad (1.14)$$

Proof. (1) By the birational transformation

$$x = -\frac{4\lambda z^2}{x_1'}, y = \frac{-\mu x_1' - y_1' - x_1' z - x_1' z^2 + 4\lambda z^3}{2x_1' z},$$

(1.3) is transformed to (1.13).

(2) This is apparent. □

Proposition 1.8. *Suppose $(\lambda,\mu) \in \Lambda_3$. The elliptic surface given by* (1.13) *has the singular fibres of type I_9 over $z = 0$, of type I_9 over $z = \infty$, and other six fibres of type I_1.*

Proof. (1.13) is described in the Kodaira normal form

$$y'^2_1 = x'^3_2 - g_2(z)x'_2 - g_3(z), \quad z \neq \infty,$$

with

$$\begin{cases} g_2(z) = -\Big(-\dfrac{\mu^4}{3} - \dfrac{4\mu^3 z}{3} - 2\mu^2 z^2 - \dfrac{4\mu^3 z^2}{3} - \dfrac{4\mu^3 z^3}{3} - 8\lambda\mu z^3 - 4\mu^2 z^3 - \dfrac{\mu^4}{3} - 8\lambda z^4 \\ \qquad -4\mu z^4 - 2\mu^2 z^4 - \dfrac{4z^5}{3} - 8\lambda z^5 - 4\mu z^5 - 2z^6 - \dfrac{4\mu z^6}{3} - \dfrac{4z^7}{3} - \dfrac{z^8}{3} \Big), \\ g_3(z) = -\Big(\dfrac{2\mu^6}{27} + \dfrac{4\mu^5 z}{9} + \dfrac{10\mu^4 z^2}{9} + \dfrac{4\mu^5 z^2}{9} + \dfrac{40\mu^3 z^3}{27} + \dfrac{8\lambda\mu^3 z^3}{3} + \dfrac{20\mu^4 z^3}{9} + \dfrac{10\mu^2 z^4}{9} + 8\lambda\mu^2 z^4 + \dfrac{40\mu^3 z^4}{9} \\ \quad + \dfrac{10\mu^4 z^4}{9} + \dfrac{4\mu z^5}{9} + 8\lambda\mu z^5 + \dfrac{40\mu^2 z^5}{9} + 8\lambda\mu^2 z^5 + \dfrac{40\mu^3 z^5}{9} + \dfrac{2z^6}{27} + \dfrac{8\lambda z^6}{3} + 16\lambda^2 z^6 + \dfrac{20\mu z^6}{9} + 16\lambda\mu z^6 \\ \quad + \dfrac{20\mu^2 z^6}{3} + \dfrac{40\mu^3 z^6}{27} + \dfrac{4z^7}{9} + 8\lambda z^7 + \dfrac{40\mu z^8}{9} + \dfrac{10\mu^2 z^8}{9} + \dfrac{40z^9}{27} + \dfrac{8\lambda z^9}{3} + \dfrac{20\mu z^9}{9} + \dfrac{10z^{10}}{9} + \dfrac{4z^{11}}{9} + \dfrac{2z^{12}}{27} \Big), \end{cases}$$

and

$$y'^2_2 = x'^3_3 - h_2(z_1)x'_3 - h_3(z_1), \quad z_1 \neq \infty,$$

with

$$\begin{cases} h_2(z_1) = -\Big(-\dfrac{\mu^4}{3} - \dfrac{4\mu^3 z}{3} - 2\mu^2 z^2 - \dfrac{4\mu^3 z^2}{3} - \dfrac{4\mu^3 z^3}{3} - 8\lambda\mu z^3 - 4\mu^2 z^3 - \dfrac{\mu^4}{3} - 8\lambda z^4 \\ \qquad -4\mu z_1^4 - 2\mu^2 z_1^2 - \dfrac{4\mu z_1^5}{3} - 8\lambda\mu z_1^5 - 4\mu^2 z_1^5 - 2\mu^2 z_1^6 - \dfrac{4\mu^3 z_1^6}{3} - \dfrac{4\mu^3 z_1^7}{3} - \dfrac{\mu^4 z_1^8}{3}\Big), \\ h_3(z_1) = -\Big(\dfrac{2}{27} + \dfrac{4z_1}{9} + \dfrac{10z_1^2}{9} + \dfrac{4\mu z_1^2}{9} + \dfrac{40z_1^3}{27} + \dfrac{8\lambda z_1^3}{3} + \dfrac{20\mu z_1^3}{9} + \dfrac{10z_1^4}{9} + 8\lambda z_1^4 + \dfrac{40\mu z_1^4}{9} + \dfrac{10\mu^2 z_1^4}{9} \\ \quad + \dfrac{4z_1^5}{9} + 8\lambda z_1^5 + \dfrac{40\mu z_1^5}{9} + 8\lambda\mu z_1^5 + \dfrac{40\mu^2 z_1^5}{9} + \dfrac{2z_1^6}{27} + \dfrac{4\mu z_1^7}{9} + 8\lambda\mu^2 z_1^7 + \dfrac{40\mu^3 z_1^7}{9} + \dfrac{10\mu^2 z_1^8}{9} \\ \quad +8\lambda\mu^2 z_1^8 + \dfrac{40\mu^3 z_1^8}{9} + \dfrac{40\mu^3 z_1^9}{27} + \dfrac{8\lambda\mu^3 z_1^9}{3} + \dfrac{20\mu^4 z_1^9}{9} + \dfrac{10\mu^4 z_1^{10}}{9} + \dfrac{4\mu^5 z_1^{10}}{9} + \dfrac{4\mu^5 z_1^{11}}{9} + \dfrac{2\mu^6 z_1^{12}}{27}\Big), \end{cases}$$

where $z = 1/z_1$ We have the discriminant of the right hand side for $x'_2(x'_3$, resp.):

$$\begin{cases} D_0 = 256\lambda^3 z^9(\mu^3 + 3\mu^2 z + 3\mu z^2 + 3\mu^2 z^2 + z^3 + 27\lambda z^3 + 6\mu z^3 + 3z^4 + 3\mu z^4 + 3z^5 + z^6), \\ D_\infty = 256\lambda^3 z_1^9(1 + 3z_1 + 3z_1^2 + 3\mu z_1^2 + z_1^3 + 27\lambda z_1^3 + 6\mu z_1^3 + 3\mu z_1^4 + 3\mu^2 z_1^4 + 3\mu^2 z_1^5 + \mu^3 z_1^6). \end{cases}$$

From these data, we obtain the required statement. □

2 Lattices for $\mathcal{F}_j$

In this section, we determine the lattice structure of a generic member of $\mathcal{F}_j$ $(j = 1, 2, 3)$.

For a general $K3$ surface S, $H_2(S,\mathbb{Z})$ is a free $\mathbb{Z}$-module of rank 22. The intersection form of $H_2(S,\mathbb{Z})$ is given by

$$E_8(-1) \oplus E_8(-1) \oplus U \oplus U \oplus U,$$

where

$$E_8(-1) = \begin{pmatrix} -2 & 1 & & & & & & \\ 1 & -2 & 1 & & & & & \\ & 1 & -2 & 1 & & & & \\ & & 1 & -2 & 1 & & & \\ & & & 1 & -2 & 1 & 1 & \\ & & & & 1 & -2 & 0 & \\ & & & & 1 & 0 & -2 & 1 \\ & & & & & & 1 & -2 \end{pmatrix},$$

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let NS(S) denote the sublattice in $H_2(S,\mathbb{Z})$ generated by the divisors on S. It is called the Néron-Severi lattice. The rank of $H_2(S,\mathbb{Z})$ is called the Picard number. We call the orthogonal complement of NS(S) in $H_2(S,\mathbb{Z})$ the transcendental lattice. We note that the Picard number is equal to $\dim_{\mathbb{Q}}(\mathrm{NS}(S) \otimes_{\mathbb{Z}} \mathbb{Q})$.

Theorem 2.1. *Let $j \in \{1, 2, 3\}$. The Picard number of a generic member of the family $\mathcal{F}_j$ is equal to 18.*

As a principle, we obtain the above theorem by the method exposed in the section 2 of the article [Na]. Because we shall have the lattice L_1 $(L_2, L_3$, resp.) in (2.3) ((2.10),(2.15), resp.) for $\mathcal{F}_1$ $(\mathcal{F}_2, \mathcal{F}_3, resp.)$, we have rank$(NS(S_j(\lambda,\mu))) \geq 18$. Let $j \in \{1, 2, 3\}$. Take $(\lambda_0, \mu_0) \in \Lambda_j$. Take a small neighborhood δ of (λ_0, μ_0) in Λ_j so that we have a local trivialization

$$\tau : \{S_j(\lambda,\mu) | (\lambda,\mu) \in \delta\} \to S_j(\lambda_0,\mu_0) \times \delta.$$

We note that τ may preserves the lattice L_j. Let $\omega_j(\lambda,\mu)$ be the unique holomorphic 2-form on the $K3$ surface $S_j(\lambda,\mu)$ up to a constant factor. By using the pairing

$$\langle\cdot,\cdot\rangle : H^2(S_j(\lambda_0,\mu_0),\mathbb{C})\times H_2(S_j(\lambda_0,\mu_0))\to\mathbb{C},$$

we define a period $\Phi(\lambda_0,\mu_0)\in\mathbb{P}^{21}(\mathbb{C})$ of $S_j(\lambda_0,\mu_0)$ given by $\langle\omega_j(\lambda_0,\mu_0),\gamma_k\rangle$ $(k=1,\cdots,22)$ for a fixed basis $\{\gamma_1\cdots,\gamma_{22}\}$ of $H_2(S_j(\lambda_0,\mu_0),\mathbb{Z})$. We have a natural extension $\Phi(\lambda,\mu)$ for $(\lambda,\mu)\in\delta$ by using $\langle\omega(\lambda,\mu),\tau_*^{-1}(\gamma_k)\rangle$ $(k=1,\cdots,22)$. Then we can define a local period map

$$\Phi_\delta:\delta\to\mathbb{P}^{21}(\mathbb{C}).$$

It is sufficient to have that Φ_δ is injective on δ to prove $\mathrm{rank}(NS(S_j(\lambda,\mu)))=18$ for generic $(\lambda,\mu)\in\Lambda_j$. In this situation, we have $\dim(\Phi_\delta(\delta))=2$. It implies $\mathrm{rank}(\mathrm{NS}(S_j(\lambda,\mu))^\perp)=4$ for generic $(\lambda,\mu)\in\Lambda_j$. But, to assure this assertion, we need a delicate observation exposed in the argument to obtain Theorem 2.2 in [Na].

We have the following fact for the elliptic fibration of $S_j(\lambda,\mu)$ stated in Section 1 by the same argument to prove Lemma 1.1 in [Na].

Fact 2.1. *Let $j\in\{1,2,3\}$ and $(\lambda_1,\mu_1),(\lambda_2,\mu_2)\in\Lambda_j$. Two elliptic surfaces $(S_j(\lambda_1,\pi_1,\mathbb{P}^1(\mathbb{C})))$ and $(S_j(\lambda_2,\pi_2,\mathbb{P}^1(\mathbb{C})))$ are isomorphic as elliptic surfaces if and only if $(\lambda_1,\mu_1)=(\lambda_2,\mu_2)$.*

Also, we have the following fact.

Fact 2.2. ([Na] Lemma 2.1) *Let S be a $K3$ surface with elliptic fibration $\pi:S\to\mathbb{P}^1(\mathbb{C})$, and let F be a fixed general fibre. Then π is the unique elliptic fibration up to* $\mathrm{Aut}(\mathbb{P}^1(\mathbb{C}))$ *which has F as a general fibre.*

Because we established Fact 2.1 and Fact 2.2, by the same argument to obtain Proposition 2.1 in [Na], we have the same marked $K3$ surfaces $S_j(\lambda_1,\mu_1)$ and $S_j(\lambda_2,\mu_2)$ if and only if $(\lambda_1,\mu_1)=(\lambda_2,\mu_2)$. According to the Torelli type theorem for $K3$ surfaces, we obtain that Φ_δ is injective.

We need explicit lattice structures of the Néron-Severi lattices and the transcendental lattices for further study.

In the article [Na] we could determine the explicit Néron-Severi lattice for the polytope P_5 in a naive way, for we have $\det(\mathrm{NS}(S(\lambda,\mu)))=-5$ which does not contain any square factor.

However, it is much more difficult to determine the explicit Néron-Severi lattice for the polytopes P_j $(j=1,2,3)$, for $\det(\mathrm{NS}(S_j(\lambda,\mu)))=-9=-3^2$. In this section, we prove the following theorem.

Theorem 2.2. *For a generic point $(\lambda,\mu)\in\Lambda_j$ $(j=1,2,3)$, we have the intersection matrices of Néron-Severi lattices NS and the transcendental lattices Tr as in* Table 2.

Family	$\mathcal{F}_1$	$\mathcal{F}_2$	$\mathcal{F}_3$
NS	$E_8(-1)\oplus E_8(-1)\oplus\begin{pmatrix}0&3\\3&0\end{pmatrix}$	$E_8(-1)\oplus E_8(-1)\oplus\begin{pmatrix}0&3\\3&2\end{pmatrix}$	$E_8(-1)\oplus E_8(-1)\oplus\begin{pmatrix}0&3\\3&-2\end{pmatrix}$
Tr	$A_1:=U\oplus\begin{pmatrix}0&3\\3&0\end{pmatrix}$	$A_2:=U\oplus\begin{pmatrix}0&3\\3&-2\end{pmatrix}$	$A_3:=U\oplus\begin{pmatrix}0&3\\3&2\end{pmatrix}$

Table 2.

Remark 2.1. *K. Koike* [Koi] *has made a research on the families of $K3$ surfaces derived from the dual polytopes of* 3 *dimensional Fano polytopes. The polytopes P_2 and P_3 in our notation are the Fano polytopes. Due to Koike we have Néron-Severi lattices for the dual polytopes P_2° and P_3° (given by* Table 3*).*

Dual Polytope	P_2°	P_3°
Néron-Severi lattice	$\begin{pmatrix} 0 & 3 \\ 3 & -2 \end{pmatrix}$	$\begin{pmatrix} 0 & 3 \\ 3 & 2 \end{pmatrix}$

Table 3.

By comparing Table 3 *and* Table 2*, we can assure the mirror symmetry conjecture for the reflexive polytopes P_2 and P_3.*

2.1 The Mordell-Weil lattices

Let us recall the theory of Mordell-Weil lattices due to T. Shioda. For detail, see [Sh1] and [Sh2].

Let S be a compact complex surface and C be a algebraic curve. Let $\pi : S \to C$ be an elliptic fibration with sections. For generic $v \in C$, the fibre $\pi^{-1}(v)$ is an elliptic curve. In the following we assume that the elliptic fibre $\pi : S \to C$ has singular fibres. $\mathbb{C}(C)$ denotes the algebraic function field on C. If $C = \mathbb{P}^1(\mathbb{C})$, the field $\mathbb{C}(C)$ is isomorphic to the rational function field $\mathbb{C}(t)$.

In this article, $(\cdot)$ denotes the intersection number and $E(\mathbb{C}(C))$ denotes the Mordell-Weil group of sections of $\pi : S \to C$. For all $P \in E(\mathbb{C}(C))$ and $v \in C$, we have $(P \cdot \pi^{-1}(v)) = 1$. Note that the section P intersects an irreducible component with multiplicity 1 of every fibre $\pi^{-1}(v)$. Let O be the zero of the group $E(\mathbb{C}(C))$. The section O is given by the set of the points at infinity on every generic fibre.

Set

$$R = \{v \in C | \pi^{-1}(C) \text{ is a singular fibre of } \pi\}.$$

For all $v \in R$ we have

$$\pi^{-1}(v) = \Theta_{v,0} + \sum_{j=1}^{m_v - 1} \mu_{v,j} \Theta_{v,j}, \tag{2.1}$$

where m_v is the number of irreducible components of $\pi^{-1}(v)$, $\Theta_{v,j}$ $(j = 0, \cdots, m_v - 1)$ are irreducible components with multiplicity $\mu_{v,j}$ of $\pi^{-1}(v)$, and $\Theta_{v,0}$ is the component with $\Theta_{v,0} \cap O \neq \phi$.

Let F be a generic fibre of π. Set

$$T = \langle F, O, \Theta_{v,j} | v \in R, 1 \leq j \leq m_v - 1 \rangle_{\mathbb{Z}} \subset \mathrm{NS}(S).$$

We call T the trivial lattice for π. For $P \in E(\mathbb{C}(C))$, $(P) \in \mathrm{NS}(S)$ denotes the corresponding element.

Theorem 2.3. (T. Shioda [Sh1] (see also [Sh2] Theorem (3·10))) (1) *The Mordell-Weil group $E(\mathbb{C}(C))$ is a finitely generated Abelian group.*
(2) *The Néron-Severi group $\mathrm{NS}(S)$ is a finitely generated Abelian group and torsion free.*
(3) *We have the isomorphism of groups $E(\mathbb{C}(C)) \simeq \mathrm{NS}(S)/T$ given by*

$$P \mapsto (P) \mod T.$$

We set $\hat{T} = (T \otimes_{\mathbb{Z}} \mathbb{Q}) \cap \mathrm{NS}(S)$ for the trivial lattice T.

Corollary 2.1. ([Sh1], see also [Sh2] Proposition (3·11)) (1) *We have*

$$\mathrm{rank}(E(\mathbb{C}(C))) = \mathrm{rank}(\mathrm{NS}(S)) - 2 - \sum_{v \in R} (m_v - 1).$$

(2) *We have*

$$E(\mathbb{C}(C))_{tor} \simeq \hat{T}/T,$$

where $E(\mathbb{C}(C))_{tor}$ is the torsion part of $E(\mathbb{C}(C))$.

Set

$$E(\mathbb{C}(C))^0 = \{P \in E(\mathbb{C}(C)) | P \cap \Theta_{v,0} \neq \phi \text{ for all } v \in R\}.$$

We have

$$E(\mathbb{C}(C))^0 \subset E(\mathbb{C}(C))/E(\mathbb{C}(C))_{tor} \tag{2.2}$$

(see [Sh1], see also [Sh2] Section 5).

Let $v \in R$. Under the notation (2.1), we set

$$(\pi^{-1}(v))^\sharp = \bigcup_{0 \leq j \leq m_v - 1,\, \mu_{v,j}=1} \Theta^\sharp_{v,j},$$

where $\Theta^\sharp_{v,j} = \Theta_{v,j} - \{\text{singular points of } \pi^{-1}(v)\}$. Set $m_v^{(1)} = \sharp\{j | 0 \leq j \leq m_v - 1,\ \mu_{v,j} = 1\}$.

Theorem 2.4. ([Ne], [Kod], see also [Sh2] Section 7) *Let $v \in R$. The set $(\pi^{-1}(v))^\sharp$ has a canonical group structure.*

Remark 2.2. *Especially, for the singular fibre $\pi^{-1}(v)$ of type I_b $(b \geq 1)$,*

$$(\pi^{-1}(v))^\sharp \simeq \mathbb{C}^\times \times (\mathbb{Z}/b\mathbb{Z}).$$

For the singular fibre $\pi^{-1}(v)$ of type I_b^* $(b \geq 0)$,

$$(\pi^{-1}(v))^\sharp \simeq \begin{cases} \mathbb{C} \times (\mathbb{Z}/4\mathbb{Z}) & (b \in 2\mathbb{Z}+1), \\ \mathbb{C} \times (\mathbb{Z}/2\mathbb{Z})^2 & (b \in 2\mathbb{Z}). \end{cases}$$

For each $v \in C$, we introduce the map

$$sp_v : E(\mathbb{C}(C)) \to (\pi^{-1}(v))^\sharp : P \mapsto P \cap \pi^{-1}(v).$$

Note that

$$P \cap \pi^{-1}(v) = (x, a) \in \begin{pmatrix} \mathbb{C}^\times \\ \mathbb{C} \end{pmatrix} \times \{\text{finite group}\}$$

(see [Sh2] Section 7). We call sp_v the specialization map.

Theorem 2.5. ([Sh2] Section 7) *For all $v \in C$, the specialization map*

$$sp_v : P \mapsto (x, a) \in \begin{pmatrix} \mathbb{C}^\times \\ \mathbb{C} \end{pmatrix} \times \{\text{finite group}\}$$

is a homomorphism of groups.

Remark 2.3. *Especially for the singular fibre $\pi^{-1}(v)$ of type I_b (I_b^*, resp.), the projection of sp_v*

$$E(\mathbb{C}(C)) \to (\mathbb{Z}/b\mathbb{Z}) \quad ((\mathbb{Z}/4\mathbb{Z}) \text{ or } (\mathbb{Z}/2\mathbb{Z})^2, \text{ resp.})$$

is a homomorphism of groups.

2.2 $\mathcal{F}_1$

The elliptic fibration given by (1.4) is described in Figure 1.

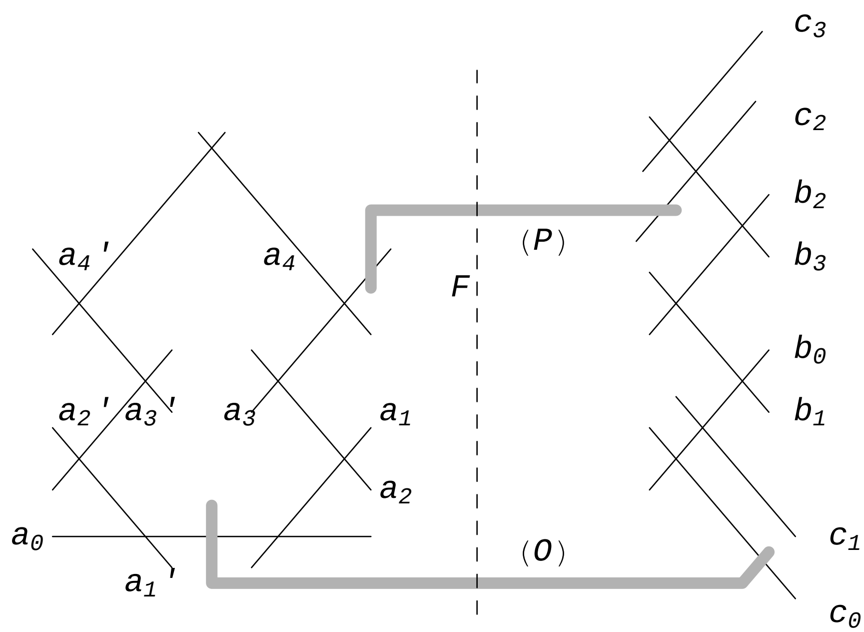

Figure 1.

The trivial lattice for this fibration is

$$T_1 = \langle a_1, a_2, a_3, a_4, a_4', a_3', a_2', a_1', c_1, b_1, b_2, b_3, c_2, c_3, O, F\rangle_{\mathbb{Z}}.$$

Let P be the section in (1.5). $P \cap a_3 \neq \phi$ at $x_1 = 0$ and $P \cap c_2 \neq \phi$ at $x_1 = \infty$. Set

$$L_1 = \langle P, T_1\rangle_{\mathbb{Z}}. \tag{2.3}$$

This is a subgroup of $\mathrm{NS}(S_1(\lambda,\mu))$. We have $\det(L_1) = -9$. According to Theorem 2.1 and Theorem 2.3 (3), we obtain $\mathrm{NS}(S_1(\lambda,\mu)) \otimes_{\mathbb{Z}} \mathbb{Q} = L_1 \otimes_{\mathbb{Z}} \mathbb{Q}$, and we obtain also

$$\mathrm{NS}(S_1(\lambda,\mu)) = (\langle P\rangle_{\mathbb{Q}} \cap \mathrm{NS}(S_1(\lambda,\mu))) + \hat{T}_1 \tag{2.4}$$

for generic $(\lambda,\mu) \in \Lambda_1$. We have

$$[\mathrm{NS}(S_1(\lambda,\mu)) : L_1] = 1 \quad \text{or} \quad [\mathrm{NS}(S_1(\lambda,\mu)) : L_1] = 3. \tag{2.5}$$

In the following, we prove

$$[\mathrm{NS}(S_1(\lambda,\mu)) : L_1] = 1.$$

Lemma 2.1. *For generic* $(\lambda,\mu) \in \Lambda_1$*, we have* $\hat{T}_1 = T_1$.

Proof. From (2.4) and (2.5) it is necessary that $\hat{T}_1 = T_1$ or $[\hat{T}_1 : T_1] = 3$. We assume $[\hat{T}_1 : T_1] = 3$. Then, according to Corollary 2.1 (2),

$$E(\mathbb{C}(x_1))_{tor} \simeq \hat{T}_1/T_1 \simeq \mathbb{Z}/3\mathbb{Z}. \tag{2.6}$$

Therefore there exists $S_0 \in E(\mathbb{C}(x_1))_{tor}$ such that $3S_0 = O$. By Remark 2.3 and (2.2), we can assume that $S_0 \cap a_3 \neq \phi$ at $x_1 = 0$ and $S_0 \cap c_0 \neq \phi$ at $x_1 = \infty$. Put $(S_0 \cdot O) = k \in \mathbb{Z}$. Set $\tilde{T}_1 = \langle T_1, S_0\rangle_{\mathbb{Z}}$. By calculating the intersection matrix, we have

$$\det(\tilde{T}_1) = -72(1 + k + k^2) \neq 0. \tag{2.7}$$

On the other hand, due to (2.6), we have $\mathrm{rank}(\tilde{T}_1) = 17$. So it follows $\det(\tilde{T}_1) = 0$. This contradicts (2.7). □

By the above lamma, we have

$$\mathrm{NS}(S_1(\lambda,\mu)) = (\langle P\rangle_{\mathbb{Q}} \cap \mathrm{NS}(S_1(\lambda,\mu))) + T_1. \tag{2.8}$$

Lemma 2.2. *For generic* $(\lambda,\mu) \in \Lambda_1$*, we have* $\mathrm{NS}(S_1(\lambda,\mu)) = L_1$*.*

Proof. It is sufficient to prove $[\mathrm{NS}(S_1(\lambda,\mu)) : L_1] = 1$. We assume $[\mathrm{NS}(S_1(\lambda,\mu)) : L_1] = 3$. By (2.8) there exists $R \in E(\mathbb{C}(x_1))$ such that $3R = P$. According to Remark 2.3,

$$(R\cdot c_3) = 1, \quad \text{at } x_1 = \infty$$

and

$$\begin{cases} (R\cdot a_1) = 1, \\ \text{or} \\ (R\cdot a_4) = 1, & \text{at } x_1 = 0. \\ \text{or} \\ (R\cdot a_7) = 1, \end{cases}$$

We can assume $(R\cdot O) = 0$, for P in (1.5) does not intersect O. By the addition theorem for elliptic curves, we have $2P$ and we can check $2P$ does not intersect O. So, we assume $(R\cdot P) = 0$ also. Set $\tilde{L_1} = \langle L_1, R\rangle_{\mathbb{Z}}$. By calculating the intersection matrix, we have

$$\det(\tilde{L_1}) = \begin{cases} 12 & (\text{if } (R\cdot a_1) = 1), \\ -30 & (\text{if } (R\cdot a_4) = 1), \\ 6 & (\text{if } (R\cdot a_7) = 1). \end{cases} \tag{2.9}$$

On the other hand, we have $\mathrm{rank}(\tilde{L_1}) = 18$ from Theorem 2.1. Hence, we obtain $\det(\tilde{L_1}) = 0$. This contradicts (2.9). Therefore, we have $[\mathrm{NS}(S_1(\lambda,\mu)) : L_1] = 1$. □

Lemma 2.3. *The lattice* L_1 *is isomorphic to the lattice given by the intersection matrix*

$$\begin{pmatrix} E_8(-1) & & & \\ & E_8(-1) & & \\ & & 0 & 3 \\ & & 3 & 0 \end{pmatrix},$$

and its orthogonal complement is given by the intersection matrix

$$A_1 = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 0 & 3 \\ & & 3 & 0 \end{pmatrix}.$$

Proof. We obtain the corresponding intersection matrix M_1 for the lattice L_1:

$$
\begin{pmatrix}
-2 & 1 & & & & & & & & & & & & & & & & \\
1 & -2 & 1 & & & & & & & & & & & & & & & \\
 & 1 & -2 & 1 & & & & & & & & & & & & & 1 & \\
 & & 1 & -2 & 1 & & & & & & & & & & & & & \\
 & & & 1 & -2 & 1 & & & & & & & & & & & & \\
 & & & & 1 & -2 & 1 & & & & & & & & & & & \\
 & & & & & 1 & -2 & 1 & & & & & & & & & & \\
 & & & & & & 1 & -2 & & & & & & & & & & \\
 & & & & & & & & -2 & 1 & & & & & & & & \\
 & & & & & & & & 1 & -2 & 1 & & & & & & & \\
 & & & & & & & & & 1 & -2 & 1 & & & & & & \\
 & & & & & & & & & & 1 & -2 & 1 & & & & & \\
 & & & & & & & & & & & 1 & -2 & 1 & 1 & & & \\
 & & & & & & & & & & & & 1 & -2 & 0 & & 1 & \\
 & & & & & & & & & & & & 1 & 0 & -2 & & & \\
 & & & & & & & & & & & & & & & -2 & 0 & 1 \\
 & & 1 & & & & & & & & & & & 1 & & 0 & -2 & 1 \\
 & & & & & & & & & & & & & & & 1 & 1 & 0
\end{pmatrix}.
$$

Let U_1 be the unimodular matrix

$$
\begin{pmatrix}
 & & & & & & & 1 & & & & & & & & 11 & 8 & 91 \\
 & & & & & & 1 & & & & & & & & & 22 & 16 & 182 \\
 & & & & 1 & & & & & & & & & & & 33 & 24 & 273 \\
 & & & 1 & & & & & & & & & & & & 26 & 19 & 214 \\
 & & 1 & & & & & & & & & & & & & 19 & 14 & 155 \\
 & 1 & & & & & & & & & & & & & & 12 & 9 & 96 \\
1 & & & & & & & & & & & & & & & 5 & 4 & 37 \\
 & & & & & & & & & & & & & & & -2 & -1 & -22 \\
 & & & & & & & & 1 & & & & & & & 2 & 2 & 18 \\
 & & & & & & & & & 1 & & & & & & 4 & 4 & 36 \\
 & & & & & & & & & & 1 & & & & & 6 & 6 & 54 \\
 & & & & & & & & & & & 1 & & & & 8 & 8 & 72 \\
 & & & & & & & & & & & & 1 & & & 10 & 10 & 90 \\
 & & & & & & & & & & & & & & 1 & 7 & 7 & 63 \\
 & & & & & & & & & & & & & 1 & & 5 & 5 & 45 \\
 & & & & & & & & & & & & & & -1 & 1 & -1 & 0 \\
 & & & & & 1 & & & & & & & & & & 18 & 13 & 150 \\
 & & & & & & & & & & & & & & -1 & -4 & -5 & -36
\end{pmatrix}.
$$

We have

$$
{}^tU_1 M_1 U_1 = E_8(-1) \oplus E_8(-1) \oplus \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}.
$$

□

Therefore we obtain Theorem 2.2 for $\mathcal{F}_1$.

2.3 $\mathcal{F}_2$

The elliptic fibration given by (1.7) is described in Figure 2.

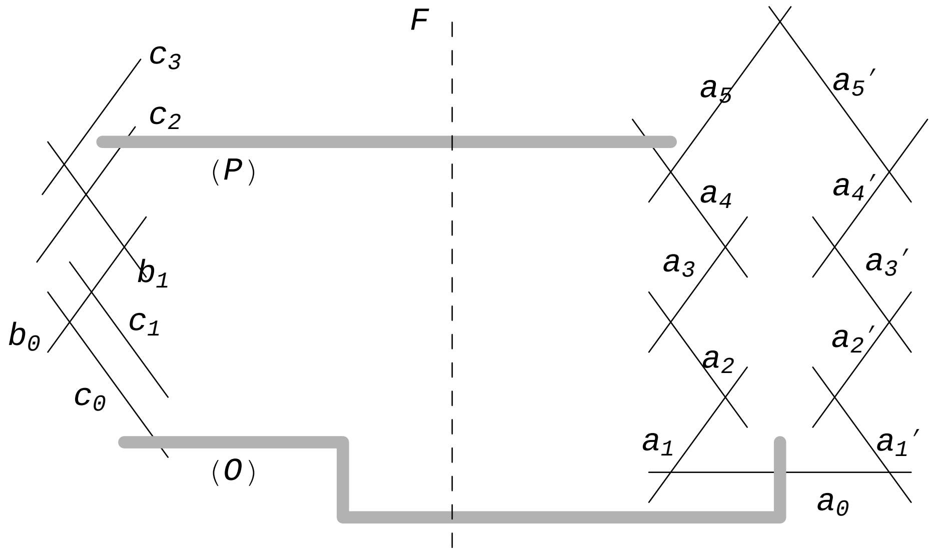

Figure 2.

The trivial lattice for this fibration is

$$T_2 = \langle a_1, a_2, a_3, a_4, a_5, a_5', a_4', a_3', a_2', a_1', c_1, b_0, b_1, c_2, c_3, O, F\rangle_{\mathbb{Z}}.$$

Let P be the section in (1.8). Note $P \cap a_4 \neq \phi$ and $P \cap c_2 \neq \phi$. Set

$$L_2 = \langle P, T_2\rangle_{\mathbb{Z}}. \tag{2.10}$$

This is a subgroup of $\mathrm{NS}(S_2(\lambda,\mu))$. We have $\det(L_2) = -9$. As in the case $\mathcal{F}_1$, so we obtain

$$\mathrm{NS}(S_2(\lambda,\mu)) = (\langle P\rangle_{\mathbb{Q}} \cap \mathrm{NS}(S_2(\lambda,\mu))) + \hat{T_2}$$

for generic $(\lambda,\mu) \in \Lambda_2$. We have

$$[\mathrm{NS}(S_2(\lambda,\mu)) : L_2] = 1 \text{ or } [\mathrm{NS}(S_2(\lambda,\mu)) : L_2] = 3. \tag{2.11}$$

In the following, we prove $[\mathrm{NS}(S_2(\lambda,\mu)) : L_2] = 1$.

Lemma 2.4. *For generic $(\lambda,\mu) \in \Lambda_2$, we have $\hat{T_2} = T_2$.*

Proof. By a direct calculation, we have $\det(T_2) = -44$. From (2.11), we have $\hat{T_2} = T_2$. □

Therefore we obtain

$$\mathrm{NS}(S_2(\lambda,\mu)) = (\langle P\rangle_{\mathbb{Q}} \cap \mathrm{NS}(S_2(\lambda,\mu))) + T_2. \tag{2.12}$$

Lemma 2.5. *For generic $(\lambda,\mu) \in \Lambda_2$, we have $\mathrm{NS}(S_2(\lambda,\mu)) = L_2$.*

Proof. We assume $[\mathrm{NS}(S_2(\lambda,\mu)) : L_2] = 3$. From (2.12) there exists $R \in E(\mathbb{C}(y))$ such that $3R = P$. According to Remark 2.3, we obtain $(R \cdot a_4) = 1$ and $(R \cdot c_3) = 1$. Because the section P in (1.8) and the section $2P$ do not intersect O, we have $(R \cdot O) = 0$ and $(R \cdot P) = 0$. Set $\tilde{L_2} = \langle L_2, R\rangle_{\mathbb{Z}}$. Calculating its intersection matrix, we have $\det(\tilde{L_2}) = -38$. As in the proof of Lemma 2.2, this contradicts to Theorem 2.1. □

Lemma 2.6. *The lattice L_2 is isomorphic to the lattice given by the following intersection matrix*

$$\begin{pmatrix} E_8(-1) & & & \\ & E_8(-1) & & \\ & & 0 & 3 \\ & & 3 & 2 \end{pmatrix},$$

and its orthogonal complement is given by the intersection matrix

$$A_2 = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 0 & 3 \\ & & 3 & -2 \end{pmatrix}.$$

Proof. We obtain the corresponding intersection matrix M_2 for L_2:

$$\begin{pmatrix}
-2 & 1 & & & & & & & & & & & & & & & & \\
1 & -2 & 1 & & & & & & & & & & & & & & & \\
 & 1 & -2 & 1 & & & & & & & & & & & & & & \\
 & & 1 & -2 & 1 & & & & & & & & & & & & 1 & \\
 & & & 1 & -2 & 1 & & & & & & & & & & & & \\
 & & & & 1 & -2 & 1 & & & & & & & & & & & \\
 & & & & & 1 & -2 & 1 & & & & & & & & & & \\
 & & & & & & 1 & -2 & 1 & & & & & & & & & \\
 & & & & & & & 1 & -2 & 1 & & & & & & & & \\
 & & & & & & & & 1 & -2 & & & & & & & & \\
 & & & & & & & & & & -2 & 1 & & & & & & \\
 & & & & & & & & & & 1 & -2 & 1 & & & & & \\
 & & & & & & & & & & & 1 & -2 & 1 & 1 & & & \\
 & & & & & & & & & & & & 1 & -2 & 0 & & 1 & \\
 & & & & & & & & & & & & 1 & 0 & -2 & & & \\
 & & & & & & & & & & & & & & & -2 & 0 & 1 \\
 & & & 1 & & & & & & & & & & 1 & & 0 & -2 & 1 \\
 & & & & & & & & & & & & & & & 1 & 1 & 0
\end{pmatrix}.$$

Let U_2 be the unimodular matrix

$$\begin{pmatrix}
 & & & & & & & & & & & & & 5 & 1 & 0 & 56 & 27 \\
 & & & & & & & & & & & & & 4 & -2 & 0 & 13 & 6 \\
1 & & & & & & & & & & & & & 15 & 3 & 0 & 162 & 80 \\
 & 1 & & & & & & & & & & & & 26 & 8 & 0 & 311 & 154 \\
 & & & & & & & & & & & & & 13 & 1 & 0 & 120 & 60 \\
 & & & & & & & & & & & & 1 & 10 & 0 & 0 & 100 & 50 \\
 & & & & & & & & & & & 1 & & 8 & 0 & 0 & 80 & 40 \\
 & & & & & & & & & & 1 & & & 6 & 0 & 0 & 60 & 30 \\
 & & & & & & & & & 1 & & & & 4 & 0 & 0 & 40 & 20 \\
 & & & & & & & & 1 & & & & & 2 & 0 & 0 & 20 & 10 \\
 & & & & & & & 1 & & & & & & 12 & 6 & 0 & 170 & 84 \\
 & & & & & & 1 & & & & & & & 24 & 12 & 0 & 340 & 168 \\
 & & & & 1 & & & & & & & & & 36 & 18 & 0 & 510 & 252 \\
 & & & 1 & & & & & & & & & & 30 & 15 & 0 & 425 & 210 \\
 & & & & & 1 & & & & & & & & 18 & 9 & 0 & 255 & 126 \\
 & & & & & & & & & & & & & -4 & 0 & 1 & -28 & -14 \\
 & & 1 & & & & & & & & & & & 24 & 12 & 0 & 340 & 168 \\
 & & & & & & & & & & & & & -8 & 1 & 0 & -56 & -28
\end{pmatrix}.$$

We have

$$ {}^tU_2 M_2 U_2 = E_8(-1) \oplus E_8(-1) \oplus \begin{pmatrix} 0 & 3 \\ 3 & 2 \end{pmatrix}.$$

□

Therefore, we obtain Theorem 2.2 for $\mathcal{F}_2$.

2.4 $\mathcal{F}_3$

The elliptic fibration given by (1.10) is described in Figure 3.

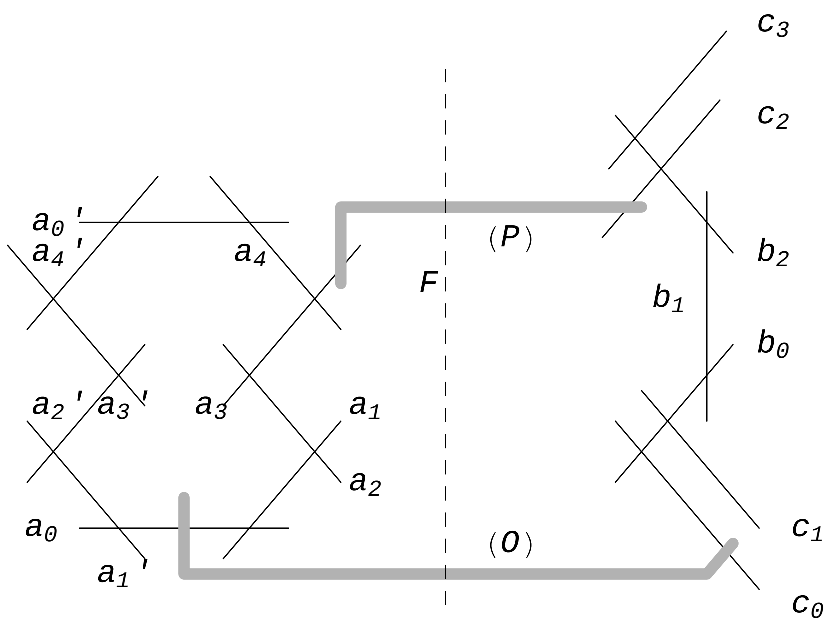

Figure 3.

The trivial lattice for this fibration is

$$T_3 = \langle a_1, a_2, a_3, a_4, a_0', a_4', a_3', a_2', a_1', c_1, b_0, b_1, b_2, c_2, c_3, O, F \rangle_{\mathbb{Z}}.$$

Let P be the section in (1.11). Set

$$L_3' = \langle P, T_3 \rangle_{\mathbb{Z}}.$$

This is a subgroup of $\mathrm{NS}(S_3(\lambda,\mu))$ and we have $\det(L_3') = -36$. Moreover, the section O' in (1.11) is a 2-torsion section for this elliptic fibretion. Due to Corollary 2.1, $[\hat{T_3} : T_3]$ is divided by 2. Hence, we have

$$[\mathrm{NS}(S_3(\lambda,\mu)) : L_3'] = 2 \ \text{ or } \ [\mathrm{NS}(S_3(\lambda,\mu)) : L_3'] = 6. \tag{2.13}$$

Lemma 2.7. *For generic $(\lambda,\mu) \in \Lambda_3$, we have $[\hat{T_3} : T_3] = 2$.*

Proof. We have $\det(T_3) = -40$. From (2.13), we obtain $[\hat{T_3} : T_3] = 2$. □

Lemma 2.8. *For generic $(\lambda,\mu) \in \Lambda_3$, we have $[\mathrm{NS}(S_3(\lambda,\mu)) : L_3'] = 2$.*

Proof. We shall show that $[\mathrm{NS}(S_3(\lambda,\mu)) : L_3'] = 2$. We assume $[\mathrm{NS}(S_3(\lambda,\mu)) : L_3'] = 6$. From Lemma 2.7, there exists $R \in E(\mathbb{C}(x_1))$ such that $3R = P$. According to Remark 2.3, it is necessary that $(R \cdot c_2) = 1$ and $(R \cdot a_4) = 1$. Also we have $(R \cdot O) = 0$, for P in (1.11) does not intersect O. Moreover we can assume that $(R \cdot P) = 0$ or 1, for the section $2P$ does not intersect O at $x_1 \neq \infty$. Set $\tilde{L_3'} = \langle L_3', R \rangle_{\mathbb{Z}}$. Calculating the intersection matrix, we have

$$\det(\tilde{L_3'}) = \begin{cases} -16 & (\text{if } \ (R \cdot P) = 0) \\ -112 & (\text{if } \ (R \cdot P) = 1) \end{cases}. \tag{2.14}$$

On the other hand, Theorem 2.1 implies that $\mathrm{rank}(\tilde{L_3}) = 18$ and $\det(\tilde{L_3}) = 0$. This is a contradiction to (2.14). □

Due to the above lemma, we have

$$|\det(\mathrm{NS}(S_3(\lambda,\mu)))| = 9$$

for generic $(\lambda,\mu) \in \Lambda_3$.

To determine the explicit lattice structure for $\mathcal{F}_3$ we use another elliptic fibration defined by (1.13). This fibration is described in Figure 4.

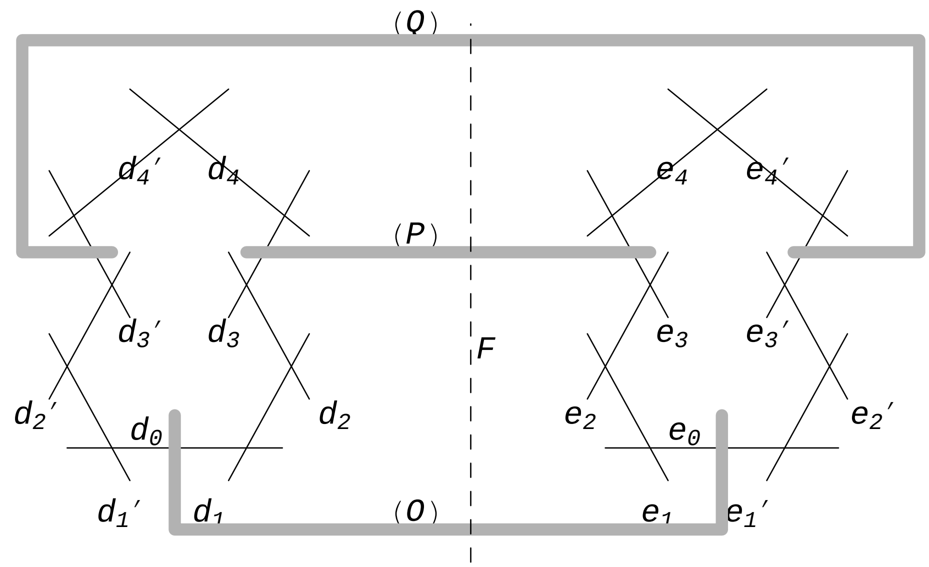

Figure 4.

Let P_0 and Q_0 be the sections in (1.14) for this elliptic fibration.

Set

$$L_3 = \langle d_1, d_2, d_3, d_4, d_4', d_3', d_2', d_1', e_1, e_2, e_3, e_4, e_3', e_2', , P_0, Q_0, O, F\rangle_{\mathbb{Z}}. \tag{2.15}$$

We have $L_3 \otimes_{\mathbb{Z}} \mathbb{Q} = \mathrm{NS}(S_3(\lambda,\mu)) \otimes_{\mathbb{Z}} \mathbb{Q}$ for generic $(\lambda,\mu) \in \Lambda_3$ and $\det(L_3') = -9$. Therefore we have

$$L_3 = \mathrm{NS}(S_3(\lambda,\mu))$$

for generic $(\lambda,\mu) \in \Lambda_3$.

Lemma 2.9. *The lattice L_3 is isomorphic to the lattice given by the intersection matrix*

$$\begin{pmatrix} E_8(-1) & & & \\ & E_8(-1) & & \\ & & 0 & 3 \\ & & 3 & -2 \end{pmatrix},$$

and its orthogonal complement is given by the intersection matrix

$$A_3 = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 0 & 3 \\ & & 3 & 2 \end{pmatrix}.$$

Proof. We obtain the corresponding intersection matrix M_3 for the lattice L_3:

$$
\begin{pmatrix}
-2 & 1 & & & & & & & & & & & & & & & & \\
1 & -2 & 1 & & & & & & & & & & & & & & & \\
 & 1 & -2 & 1 & & & & & & & & & & & & 1 & & \\
 & & 1 & -2 & 1 & & & & & & & & & & & & & \\
 & & & 1 & -2 & 1 & & & & & & & & & & & & \\
 & & & & 1 & -2 & 1 & & & & & & & & & & 1 & \\
 & & & & & 1 & -2 & 1 & & & & & & & & & & \\
 & & & & & & 1 & -2 & & & & & & & & & & \\
 & & & & & & & & -2 & 1 & & & & & & & & \\
 & & & & & & & & 1 & -2 & 1 & & & & & & & \\
 & & & & & & & & & 1 & -2 & 1 & & & & 1 & & \\
 & & & & & & & & & & 1 & -2 & 0 & & & & & \\
 & & & & & & & & & & & 0 & -2 & 1 & & & 1 & \\
 & & & & & & & & & & & & 1 & -2 & 0 & & & \\
 & & & & & & & & & & & & & 0 & -2 & 0 & & 1 \\
 & & 1 & & & & & & & & 1 & & & & 0 & -2 & & 1 \\
 & & & & & 1 & & & & & & & 1 & & & & -2 & 1 \\
 & & & & & & & & & & & & & & 1 & 1 & 1 & 0
\end{pmatrix}.
$$

Let U_3 be the unimodular matrix

$$
\begin{pmatrix}
1 & & & & & & & & 28 & & & & & & & & 5 & 468 \\
 & 1 & & & & & & & 56 & & & & & & & & 10 & 936 \\
 & & 1 & & & & & & 84 & & & & & & & & 15 & 1404 \\
 & & & & & & & & 27 & & & & & & & & 5 & 432 \\
 & & & & & & & & 21 & & & & & 1 & & & 4 & 378 \\
 & & & & & & & & 15 & & & & 1 & & & & 3 & 324 \\
 & & & & & & & & 10 & & & & & & 1 & & 2 & 216 \\
 & & & & & & & & 5 & & & & & & & 1 & 1 & 108 \\
 & & & & & & & 1 & 34 & & & & & & & & 6 & 576 \\
 & & & & & & 1 & & 68 & & & & & & & & 12 & 1152 \\
 & & & & 1 & & & & 102 & & & & & & & & 18 & 1728 \\
 & & & & & 1 & & & 51 & & & & & & & & 9 & 864 \\
 & & & & & & & & -1 & & 1 & & & & & & & 36 \\
 & & & & & & & & -1 & 1 & & & & & & & 0 & 18 \\
 & & & & & & & & 1 & & & & & & & & 0 & 35 \\
 & & & 1 & & & & & 85 & & & & & & & & 15 & 1440 \\
 & & & & & & & & -1 & & & 1 & & & & & 0 & 54 \\
 & & & & & & & & -16 & & & & & & & & -3 & -252
\end{pmatrix}.
$$

We have

$$
{}^tU_3 M_3 U_3 = E_8(-1) \oplus E_8(-1) \oplus \begin{pmatrix} 0 & 3 \\ 3 & -2 \end{pmatrix}.
$$

□

Therefore, we obtain Theorem 2.2 for $\mathcal{F}_3$.

3 Period differential equations

In this section, we determine the system of period differential equations and its projective monodromy group for the family $\mathcal{F}_j$ $(j = 1, 2, 3)$.

Set

$$
\begin{cases}
F_1(x,y,z) = xyz(x+y+z+1) + \lambda x + \mu y, & (\lambda, \mu) \in \Lambda_1, \\
F_2(x,y,z) = xyz(x+y+z+1) + \lambda x + \mu, & (\lambda, \mu) \in \Lambda_2, \\
F_3(x,y,z) = xyz(x+y+z+1) + \lambda z + \mu xy, & (\lambda, \mu) \in \Lambda_3.
\end{cases}
$$

The unique holomorphic 2-form on the $K3$ surface $S_j(\lambda,\mu)\in\Lambda_j$ $(j=1,2,3)$ is given by

$$\omega_j=\frac{dz\wedge dx}{\partial F_j/\partial y}$$

up to a constant factor.

First, we consider a period of $S_j(\lambda,\mu)$ $(j=1,2,3)$.

Theorem 3.1. *We can find a 2-cycle Γ_j $(j=1,2,3)$ so that we have the following power series expansion of the period $\iint_{\Gamma_j}\omega_j$ which is valid in a sufficiently small neighborhood of $(\lambda,\mu)=(0,0)$.*

(1) *(A period for $\mathcal{F}_1$) We have a period of $S_1(\lambda,\mu)$:*

$$\eta_1(\lambda,\mu)=\iint_{\Gamma_1}\omega_1=(2\pi i)^2\sum\frac{(3m+3n)!}{(n!)^2(m!)^2(m+n)!}\lambda^n\mu^m. \tag{3.1}$$

(2) *(A period for $\mathcal{F}_2$) We have a period of $S_2(\lambda,\mu)$:*

$$\eta_2(\lambda,\mu)=\iint_{\Gamma_2}\omega_2=(2\pi i)^2\sum_{n,m=0}^{\infty}(-1)^n\frac{(4m+3n)!}{(m!)^2n!((m+n)!)^2}\lambda^n\mu^m. \tag{3.2}$$

(3) *(A period for $\mathcal{F}_3$) We have a period of $S_3(\lambda,\mu)$:*

$$\eta_3(\lambda,\mu)=\iint_{\Gamma_3}\omega_3=(2\pi i)^2\sum_{n,m=0}^{\infty}(-1)^n\frac{(3m+2n)!}{(m!)^2(n!)^3}\lambda^n\mu^m. \tag{3.3}$$

Proof. Let $j\in\{1,2,3\}$. By the same argument in the proof of Theorem 3.1 of the article [Na], we can choose a certain 2-cycle Γ_j on $S_j(\lambda,\mu)$ so that the period integral $\iint_{\Gamma_j}\omega_j$ is given by a power series of (λ,μ). □

Remark 3.1. *In the case P_1, our period is reduced to the Appell F_4(see* [Koi] *):*

$$\eta_1(\lambda,\mu)=F_4\Big(\frac{1}{3},\frac{2}{3},1,1;27\lambda,27\mu\Big)=F\Big(\frac{1}{3},\frac{2}{3},1;x\Big)F\Big(\frac{1}{3},\frac{2}{3},1,;y\Big),$$

where F is the Gauss hypergeometric function and $x(1-y)=27\lambda, y(1-x)=27\mu$.

Secondary, we apply the theory of the GKZ hypergeometric functions to obtain the system of differential equations whose solution is the period integral in Theorem 3.1. In the following, set

$$\theta_\lambda=\lambda\frac{\partial}{\partial\lambda},\quad \theta_\mu=\mu\frac{\partial}{\partial\mu}.$$

Proposition 3.1. (1) *(The GKZ system of equations for $\mathcal{F}_1$) Set*

$$\begin{cases}L_1^{(1)}=\lambda\theta_\mu^2-\mu\theta_\lambda^2,\\ L_2^{(1)}=\lambda(3\theta_\lambda+3\theta_\mu)(3\theta_\lambda+3\theta_\mu-1)(3\theta_\lambda+3\theta_\mu-2).\end{cases} \tag{3.4}$$

It holds

$$L_1^{(1)}\eta_1(\lambda,\mu)=L_2^{(1)}\eta_1(\lambda,\mu)=0.$$

(2) *(The GKZ system of equations for $\mathcal{F}_2$) Set*

$$\begin{cases}L_1^{(2)} &=\lambda\theta_\mu^2+\mu\theta_\lambda(3\theta_\lambda+4\theta_\mu+1),\\ L_2^{(2)} &=\theta_\lambda(\theta_\lambda+\theta_\mu)^2+\lambda(3\theta_\lambda+4\theta_\mu+1)(3\theta_\lambda+4\theta_\mu+2)(3\theta_\lambda+4\theta_\mu+3).\end{cases} \tag{3.5}$$

It holds

$$L_1^{(2)}\eta_2(\lambda,\mu)=L_2^{(2)}\eta_2(\lambda,\mu)=0.$$

(3) *(The GKZ system of equations for $\mathcal{F}_3$) Set*

$$\begin{cases} L_1^{(3)} & = \theta_\lambda^2 - \mu(3\theta_\lambda + 2\theta_\mu + 1)(3\theta_\lambda + 2\theta_\mu + 2), \\ L_2^{(3)} & = \theta_\lambda^3 + \lambda(3\theta_\lambda + 2\theta_\mu + 1)(3\theta_\lambda + 2\theta_\mu + 2)(3\theta_\lambda + 2\theta_\mu + 3). \end{cases} \tag{3.6}$$

It holds

$$L_1^{(3)} \eta_3(\lambda, \mu) = L_2^{(3)} \eta_3(\lambda, \mu) = 0.$$

Proof. Set

$$A_1 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & -1 \end{pmatrix}, \ A_2 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & -1 & -1 \end{pmatrix}, \ A_3 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{pmatrix},$$

and

$$\beta = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Let $j \in \{1, 2, 3\}$. From the matrix A_j and the vector β, we have the system of the GKZ system of equations concerned with the period $\eta_j(\lambda, \mu)$ in Theorem 3.1. For detail, see the proof of Proposition 3.1 in [Na]. □

Each system in the above proposition has the 6-dimensional space of solutions. On the other hand, Theorem 2.1 says that the rank of transcendental lattice for $\mathcal{F}_j$ is 4. It implies that there are the system of period differential equations for the family $\mathcal{F}_j$ $(j = 1, 2, 3)$ with the 4-dimensional space of solutions.

Theorem 3.2. (1) *(The period differential equation for $\mathcal{F}_1$) Set*

$$\begin{cases} L_1^{(1)} & = \lambda\theta_\mu^2 + \mu\theta_\lambda(3\theta_\lambda + 4\theta_\mu + 1), \\ L_3^{(1)} & = \lambda\theta_\lambda(3\theta_\lambda + 2\theta_\mu) + \mu\theta_\lambda(1 - \theta_\lambda) + 9\lambda^2(3\theta_\lambda + 4\theta_\mu + 1)(3\theta_\lambda + 4\theta_\mu + 2). \end{cases} \tag{3.7}$$

It holds

$$L_1^{(1)} \eta_1(\lambda, \mu) = L_3^{(1)} \eta_1(\lambda, \mu) = 0.$$

The space of solutions of the system $L_1^{(1)} u = L_3^{(1)} u = 0$ is 4-dimensional.

(2) *(The period differential equation for $\mathcal{F}_2$) Set*

$$\begin{cases} L_1^{(2)} & = \lambda\theta_\mu^2 + \mu\theta_\lambda(3\theta_\lambda + 4\theta_\mu + 1), \\ L_3^{(2)} & = \lambda\theta_\lambda(3\theta_\lambda + 2\theta_\mu) + \mu\theta_\lambda(1 - \theta_\lambda) + 9\lambda^2(3\theta_\lambda + 4\theta_\mu + 1)(3\theta_\lambda + 4\theta_\mu + 2). \end{cases} \tag{3.8}$$

It holds

$$L_1^{(2)} \eta_2(\lambda, \mu) = L_3^{(2)} \eta_2(\lambda, \mu) = 0.$$

The space of solutions of the system $L_1^{(2)} u = L_3^{(2)} u = 0$ is 4-dimensional.

(3) *(The period differential equation for $\mathcal{F}_3$) Set*

$$\begin{cases} L_1^{(3)} & = \theta_\lambda^2 - \mu(3\theta_\lambda + 2\theta_\mu + 1)(3\theta_\lambda + 2\theta_\mu + 2), \\ L_3^{(3)} & = \theta_\lambda(3\theta_\lambda - 2\theta_\mu) + 9\lambda(3\theta_\lambda + 2\theta_\mu + 1)(3\theta_\lambda + 2\theta_\mu + 2) + 4\mu\theta_\lambda(3\theta_\lambda + 2\theta_\mu + 1). \end{cases} \tag{3.9}$$

It holds

$$L_1^{(3)} \eta_3(\lambda, \mu) = L_3^{(3)} \eta_3(\lambda, \mu) = 0.$$

The space of solutions of the system $L_1^{(3)} u = L_3^{(3)} u = 0$ is 4-dimensional.

Proof. We determine these systems by the method of indeterminate coefficients. For detail, see the proof of Theorem 3.2 in [Na].

In the following we prove that those spaces of solutions is 4-dimensional.

(1) Set $\varphi =^t (1, \theta_\lambda, \theta_\mu, \theta_\lambda^2)$. We obtain the corresponding Pfaffian system $\Omega_1 = A_1 d\lambda + B_1 d\mu$ with $d\varphi = \Omega_1 \varphi$ by the following way. Setting

$$t_1 = 729\lambda^2 - 54\lambda(27\mu - 1) + (1 + 27\mu)^2,$$

we have

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1/9 & -1/2 & -1/2 & -(1+27\lambda+27\mu)/(54\lambda) \\ a_{11}/t_1 & a_{12}/(2t_1) & a_{23}/(2t_1) & a_{24}/(2t_1) \end{pmatrix},$$

with

$$\begin{cases} a_{11} = 3\lambda(1 - 27\lambda + 27\mu), \\ a_{12} = 3\lambda(5 - 351\lambda + 135\mu), \\ a_{13} = 27\lambda(1 - 3\lambda + 27\mu), \\ a_{14} = 3(-729\lambda^2 + (1 + 27\mu)^2), \end{cases}$$

and

$$B_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ -1/9 & -1/2 & -1/2 & -(1+27\lambda+27\mu)/(54\lambda) \\ 0 & 0 & 0 & \mu/\lambda \\ b_{11}/t_1 & b_{12}/(2t_1) & b_{13}/(2t_1) & b_{14}/(2t_1) \end{pmatrix},$$

with

$$\begin{cases} b_{11} = 3\lambda(1 + 27\lambda - 27\mu), \\ b_{12} = 27\lambda(1 + 27\lambda - 3\mu), \\ b_{13} = 3\lambda(5 + 135\lambda - 351\mu), \\ b_{14} = (1 + 27\lambda)^2 + 108(27\lambda - 1)\mu - 3645\mu^2. \end{cases}$$

We have $d\Omega_1 = \Omega_1 \wedge \Omega_1$. Therefore the system $L_1^{(1)}u = L_3^{(1)}u = 0$ has the 4-dimensional space of solutions.

(2) Set $\varphi =^t (1, \theta_\lambda, \theta_\mu, \theta_\lambda^2)$. We obtain the corresponding Pfaffian system $\Omega_2 = A_2 d\lambda + B_2 d\mu$ with $d\varphi = \Omega_2 \varphi$ as the following way. Setting

$$\begin{cases} t_2 = \lambda^2(1 + 27\lambda)^2 - 2\lambda\mu(1 + 189\lambda) + (1 + 576\lambda)\mu^2 - 256\mu^3, \\ s_2 = 1 + 108\lambda - 288\mu, \end{cases}$$

we have

$$A_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ a_{11}/s_2 & a_{12}/(2\lambda s_2) & a_{13}/(s_2) & a_{14}/(2\lambda s_2) \\ a_{21}/(t_2 s_2) & a_{22}/(t_2 s_2) & a_{23}/(t_2 s_2) & a_{24}/(t_2 s_2) \end{pmatrix},$$

with

$$\begin{cases} a_{11} = -9\lambda, \\ a_{12} = -(81\lambda^2 + \mu - 144\lambda\mu), \\ a_{13} = -54\lambda, \\ a_{14} = -3\lambda(1 + 27\lambda - 144\mu) + \mu, \\ a_{21} = -6\lambda^3(1 + 1458\lambda^2 - 2592\lambda\mu + 6\mu(-55 + 4608\mu)), \\ a_{22} = -3\lambda^2(11 + 54\lambda(5 + 351\lambda)) + \lambda(1 + 4\lambda(61 + 810\lambda(5 + 72\lambda)))\mu + 64(17 + 2808\lambda)\mu^3 \\ \qquad -147456\mu^4 - 2(1 + 9\lambda(53 + 32\lambda(131 + 864\lambda)))\mu^2, \\ a_{23} = -8\lambda^3((2 - 27\lambda)^2 + 9(-133 + 2160\lambda)\mu + 82944\mu^2), \\ a_{24} = 3r_2 s_2 + 162\lambda r_2 - 3\lambda s_2(\lambda + 81\lambda^2 + 1458\lambda^3 - 378\lambda\mu + \mu(-1 + 288\mu)), \end{cases}$$

and

$$B_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ b_{11}/s_2 & b_{12}/(2\lambda s_2) & b_{13}/s_2 & b_{14}/(2\lambda s_2) \\ b_{21}/(s_2) & b_{22}/(\lambda^2 s_2) & b_{23}/s_2 & b_{24}/(\lambda^2 s_2) \\ b_{31}/(t_2 s_2) & b_{32}/(2\lambda t_2 s_2) & b_{33}/(t_2 s_2) & b_{34}/(2\lambda t_2 s_2) \end{pmatrix},$$

with

$$\begin{cases} b_{11} = -9\lambda, \\ b_{12} = -(81\lambda^2 + \mu - 144\lambda\mu), \\ b_{13} = -54\lambda, \\ b_{14} = -3\lambda(1 + 27\lambda - 144\mu) + \mu, \\ b_{21} = 36\mu, \\ b_{22} = \mu(\lambda(-1 + 54\lambda) + 2\mu), \\ b_{23} = 216\mu, \\ b_{24} = (3(1 - 54\lambda)\lambda - 2\mu)\mu, \\ b_{31} = 3\lambda(81\lambda^3(1 + 27\lambda) + \lambda(-1 + 36\lambda)(-5 + 108\lambda)\mu + 3(-1 + 32\lambda)(1 + 432\lambda)\mu^2 + 768\mu^3, \\ b_{32} = 2187\lambda^5(1 + 27\lambda) - (1 + 192\lambda(11 + 1164\lambda))\mu^3 + 256(1 + 864\lambda)\mu^4 \\ \qquad -\lambda^2(2 + 27\lambda(4 + 9\lambda(77 + 864\lambda)))\mu + \lambda(5 + \lambda(1279 + 864\lambda(85 + 864\lambda)))\mu^2, \\ b_{33} = 2\lambda(3\lambda^2(1 + 27\lambda)(-1 + 135\lambda) + 2\lambda(23 + 54\lambda(-11 + 972\lambda))\mu \\ \qquad +9(-3 + 64\lambda)(1 + 432\lambda)\mu^2 + 6912\mu^3, \\ b_{34} = -(-81\lambda^4(1 + 27\lambda)^2 + \lambda^2(-7 + 9\lambda(-58 + 27\lambda(-125 + 3456\lambda)))\mu \\ \qquad +\lambda(8 + 9\lambda(425 + 24192\lambda))\mu^2 - (1 + 3456\lambda(1 + 162\lambda))\mu^3 + 256(1 + 1440\lambda)\mu^4. \end{cases}$$

We see $d\Omega_2 = \Omega_2 \wedge \Omega_2$. Therefore the system $L_1 u = L_3 u = 0$ has the 4-dimensional solution space.

(3) Set $\varphi = {}^t(1, \theta_\lambda, \theta_\mu, \theta_\lambda^2)$. We obtain the corresponding Pfaffian system $\Omega_3 = A_3 d\lambda + B_3 d\mu$ with $d\varphi = \Omega_3\varphi$ as the following way. Setting

$$\begin{cases} t_3 = 729\lambda^2 - (4\mu - 1)^3 + 54\lambda(1 + 12\mu), \\ s_3 = -54\lambda + (1 - 4\mu)^2, \end{cases}$$

we have

$$A_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ a_{11}/s_3 & a_{12}/(2s_3) & a_{13}/s_3 & a_{14}/(2s_3) \\ a_{21}/(t_3 s_3) & a_{22}/(t_3 s_3) & a_{23}/(t_3 s_3) & a_{24}/(t_3 s_3) \end{pmatrix},$$

with

$$\begin{cases} a_{11} = 9\lambda, \\ a_{12} = 81\lambda + 4(1 - 4\mu)\mu, \\ a_{13} = 27\lambda, \\ a_{14} = 3 + 81\lambda - 48\mu^2, \\ a_{21} = -2\lambda(-2187\lambda^2 + 27\lambda(4\mu - 9)(4\mu - 1) - (-1 + 4\mu)^3(3 + 8\mu)), \\ a_{22} = 3\lambda(9477\lambda^2 + (1 - 4\mu)^2(-11 + 4\mu(-9 + 16\mu)) - 27\lambda(25 + 4\mu(-31 + 40\mu))), \\ a_{23} = 2\lambda(729\lambda^2 + (-1 + 4\mu)^3(11 + 16\mu) + 27\lambda(-1 + 4\mu)(19 + 20\mu)), \\ a_{24} = 81\lambda(-2 + 27\lambda + 8\mu)(1 + 27\lambda - 16\mu^2), \end{cases}$$

and

$$B_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ b_{11}/s_3 & b_{12}/(2s_3) & b_{13}/s_3 & b_{14}/(2s_3) \\ b_{21}/s_3 & b_{22}/s_3 & b_{23}/s_3 & b_{24}/s_3 \\ b_{31}/(t_3 s_3) & b_{32}/(2t_3 s_3) & b_{33}/(t_3 s_3) & b_{34}/(2t_3 s_3) \end{pmatrix},$$

with

$$\begin{cases} b_{11} = 9\lambda, \\ b_{12} = 81\lambda + 4(1-4\mu)\mu, \\ b_{13} = 27\lambda, \\ b_{14} = 3 + 81\lambda - 48\mu^2, \\ b_{21} = -2\mu(-1+4\mu), \\ b_{22} = -3\mu(-3+4\mu), \\ b_{23} = -6\mu(-1+4\mu), \\ b_{24} = 9\mu(3+4\mu), \\ b_{31} = -3\lambda(2187\lambda^2 + 32(1-4\mu)^2\mu(1+\mu) + 27\lambda(3+16\mu(2+\mu))) \\ b_{32} = -9\lambda(6561\lambda^2 - 81\lambda(-3+4\mu)(1+8\mu) + 4\mu(-1+4\mu)(-33+4\mu(-3+16\mu))), \\ b_{33} = -3\lambda(3645\lambda^2 + 2(1-4\mu)^2(1+16\mu(3+2\mu)) + 27\lambda(7+16\mu(5+9\mu))), \\ b_{34} = -r_3 s_3 + r_3(-8+351\lambda+32\mu) + s_3(9(729\lambda^2 + (1-4\mu)^2 + 54\lambda(1+8\mu)). \end{cases}$$

We have $d\Omega_3 = \Omega_3 \wedge \Omega_3$. So the system $L_1^{(3)} u = L_3^{(3)} u = 0$ has the 4-dimensional space of solutions. □

Remark 3.2. *From the Puffian systems in the above proof, we obtain the singular locus of the system* (3.7)*:*

$$\lambda = 0, \quad \mu = 0, \quad 729\lambda^2 - 54\lambda(27\mu - 1) + (1+27\mu)^2 = 0,$$

the singular locus of the system (3.8)*:*

$$\lambda = 0, \quad \mu = 0, \quad \lambda^2(1+27\lambda)^2 - 2\lambda\mu(1+189\lambda) + (1+576\lambda)\mu^2 - 256\mu^3 = 0,$$

and the singular locus of the system (3.9)*:*

$$\lambda = 0, \quad \mu = 0, \quad 729\lambda^2 - (4\mu - 1)^3 + 54\lambda(1+12\mu) = 0.$$

Omitting these locus from $\mathbb{C}^2$ *we have the domain* Λ_j $(j = 1, 2, 3)$ *in* (1.6), (1.9) *and* (1.12).

Finally, we determine the projective monodromy groups.

Let $j \in \{1, 2, 3\}$. For generic $(\lambda, \mu) \in \Lambda_j$, we can take a basis $\{\gamma_5, \cdots, \gamma_{22}\}$ of $\mathrm{NS}(S_j(\lambda, \mu))$ such that the intersection matrix $(\gamma_k \cdot \gamma_l)_{5 \le k, l \le 22}$ is equal to the matrix in Theorem 2.2. This basis is extended to a basis $\{\gamma_1, \cdots, \gamma_4, \gamma_5, \cdots, \gamma_{22}\}$ of $H_2(S_j(\lambda, \mu))$. Let $\{\gamma_1^*, \cdots, \gamma_{22}^*\}$ be its dual basis (namely $(\gamma_k \cdot \gamma_j^*) = \delta_{k,l}$). By Theorem 2.2, we have $(\gamma_k^* \cdot \gamma_l^*) = A_j$.

Using this basis $\{\gamma_1, \cdots, \gamma_{22}\}$, we define the local period map as in the beginning of Section 2. Moreover, we define the multivalued period map

$$\Phi_j : \Lambda_j \to \mathbb{P}^3(\mathbb{C})$$

by the analytic continuation of the local period map along any arc in Λ_j.

Set

$$\mathcal{D}_j = \{\xi \in \mathbb{P}^3(\mathbb{C}) | \xi A_j {}^t\xi = 0, \xi A_j {}^t\bar{\xi} > 0\}.$$

By the Riemann-Hodge relation, we have $\Phi_j(\Lambda_j) \subset \mathcal{D}_j$.

The fundamental group $\pi_1(\Lambda_j, *)$ acts on $\Phi_j(\Lambda_j)$ by the analytic continuation of the local period map. This action induces a group of projective linear transformations which is a subgroup of $PGL(4, \mathbb{Z})$. We call it the projective monodromy group of the multivalued period map Φ_j.

Note that $\mathcal{D}_j$ is composed of two connected components: $\mathcal{D}_j = \mathcal{D}_j^+ \cup \mathcal{D}_j^-$. Set $PO(A_j, \mathbb{Z}) = \{g \in GL(4, \mathbb{Z}) | g A_j {}^t g = A_j\}$. It acts on $\mathcal{D}_j$ by ${}^t\xi \mapsto g {}^t\xi \quad (\xi \in \mathcal{D}_j, g \in PO(A_j, \mathbb{Z}))$. Let $PO^+(A_j, \mathbb{Z})$ be the subgroup of $PO(A_j, \mathbb{Z})$ given by $\{g \in PO(A_j, \mathbb{Z}) | g(\mathcal{D}_j^+) = \mathcal{D}_j^+\}$.

Theorem 3.3. *Let $j \in \{1, 2, 3\}$. The projective monodromy group of the period differential equation for the family $\mathcal{F}_j$ is equal to $PO^+(A_j, \mathbb{Z})$.*

Proof. Because the projective monodromy group G_j of the multivalued period map Φ_j is equal to that of the period differential equation for $\mathcal{F}_j$, we determine G_j. It is obvious $G_j \subset PO^+(A_j, \mathbb{Z})$. However, we need a delicate observation to prove the converse $PO^+(A_j, \mathbb{Z}) \subset G_j$. For precise argument, see Section 4 in [Na]. □

Acknowledgment

The author would like to express his gratitude to Professor Hironori Shiga for various advices and suggestions. He is also thankful to Professor Kimio Ueno for encouragements and useful comments.

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