VARIATION FORMULAS FOR PRINCIPAL FUNCTIONS (II) APPLICATIONS TO VARIATION FOR HARMONIC SPANS

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Abstract. For a domain D in $\mathbb{C}_z$ with smooth boundary and for $a, b \in D, a \neq b$, we have the circular (radial) slit mapping $P(z)(Q(z))$ on D such that $P(z) - \frac{1}{z-a}$ $(Q(z) - \frac{1}{z-a})$ is regular at a and $P(b)(Q(b)) = 0$, and we call $p(z) = \log |P(z)|$ $(q(z) = \log |Q(z)|)$ the L_1-(L_0-)principal function; $\alpha = \log |P'(b)|$ $(\beta = \log |Q'(b)|)$ the L_1-(L_0-)constant, and $s = \alpha - \beta$ the harmonic span, for D. S. Hamano in [8] showed the variation formula of the second order for the L_1-const. $\alpha(t)$ for the moving domain $D(t)$ in $\mathbb{C}_z$ with $t \in B := \{t \in \mathbb{C} : |t| < \rho\}$. We show the corresponding formula for the L_0-const. $\beta(t)$ for $D(t)$, and combine these formulas to obtain, if the total space $\mathcal{D} = \cup_{t\in B}(t, D(t))$ is pseudoconvex in $B \times \mathbb{C}_z$, then $s(t)$ is subharmonic on B. Since the geometric meaning of $s(t)$ is showed, this fact gives one of the relations between the conformal mappings on each fiber $D(t), t \in B$ and the pseudoconvexity of $\mathcal{D}$. As a simple application we obtain the subharmonicity of $\log \cosh d(t)$ on B, where $d(t)$ is the Poincaré distance between a and b.

1. Introduction

Let R be a bordered Riemann surface with boundary $\partial R = C_1 + \cdots + C_\nu$ in a larger Riemann surface $\widetilde{R}$, where $C_j, j = 1, \dots, \nu$ is a C^ω smooth contour in $\widetilde{R}$. Fix two points a, b with local coordinates $|z| < \rho$ and $|z - \xi| < \rho$ where $a\,(b)$ corresponds to $0\,(\xi)$. Among all harmonic functions u on $R \setminus \{0, \xi\}$ with logarithmic singularity $\log \frac{1}{|z|}$ at 0 and $\log |z - \xi|$ at ξ normalized $\lim_{z\to 0}(u(z) - \log \frac{1}{|z|}) = 0$, we uniquely have two special ones p and q with the following boundary conditions: for each C_j, p satisfies $p(z) =$ const. c_j on C_j and $\int_{C_j} \frac{\partial p(z)}{\partial n_z} ds_z = 0$ (where $\frac{\partial}{\partial n_z}$ is the outer normal derivative and ds_z is the arc length element at z of C_j), while q does $\frac{\partial q(z)}{\partial n_z} = 0$ on C_j. We call p and q the L_1- and the L_0-principal function for $(R, 0, \xi)$, respectively. The constant terms $\alpha := \lim_{z\to\xi}(p(z) - \log |z - \xi|)$ and $\beta := \lim_{z\to\xi}(q(z) - \log |z - \xi|)$ are called the L_1- and the L_0-constant for $(R, 0, \xi)$ (see [1] and [15]).

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Now let $B = \{t \in \mathbb{C} : |t| < \rho\}$ and let $\mathcal{R} : t \in B \to R(t) \Subset \widetilde{R}$ be a variation of Riemann surface $R(t)$ with $t \in B$ such that each $R(t), t \in B$ contains the origin 0; $\partial R(t) = C_1(t) + \cdots + C_\nu(t)$ is C^ω smooth in $\widetilde{R}$, and $\partial R(t)$ varies C^ω smoothly on $\widetilde{R}$ with $t \in B$. Let $\xi(t) \in R(t),\ t \in B$ vary holomorphically in $\widetilde{R}$ with $t \in B$. Then each $R(t), t \in B$ admits the L_1-(L_0-)principal function $p(t,z)(q(t,z))$ and L_1-(L_0-)constant $\alpha(t)(\beta(t))$ for $(R(t), 0, \xi(t))$. S. Hamano [8] established the variation formula of the second order for $\alpha(t)$ (see Lemma 2.1 in this paper), which implied that, if the total space $\mathcal{R} = \cup_{t \in B}(t, R(t))$ is a pseudoconvex domain in $B \times \widetilde{R}$, then $\alpha(t)$ is subharmonic on B. Continuing on [8] we show the variation formula for $\beta(t)$ (Lemma 2.2) in this paper, which continues on [10]. To prove the formula for $\beta(t)$ we add a new idea to her proof for $\alpha(t)$. In fact, the formula for $\alpha(t)$ does not concern the genus of $R(t)$ but that for $\beta(t)$ does concern it. The formula for $\beta(t)$ implies that, if $\mathcal{R}$ is pseudoconvex in $B \times \widetilde{R}$ and if $R(t), t \in B$ is planar, then $\beta(t)$ is superharmonic on B. This contrast between the subharmincity of $\alpha(t)$ and the superharmonicity of $\beta(t)$ are unified with the notion of the harmonic span $s(t) := \alpha(t) - \beta(t)$ for $(R(t), 0, \xi(t))$ introduced by M. Nakai (see (3.1) in §3): *if $\mathcal{R}$ is pseudoconvex in $B \times \widetilde{R}$ and $R(t), t \in B$ is planar, then $s(t)$ is subharmonic on B.* This implies (Corollary 4.2): assume moreover that each $R(t), t \in$ is simply connected. Let $\xi_i := \cup_{t \in B}(t, \xi_i(t)), i = 1, 2$ be two holomorphic sections of $\mathcal{R}$ over B and let $d(t)$ denote the Poincaré distance between $\xi_1(t)$ and $\xi_2(t)$ on $R(t)$. Then $\delta(t) := \log\cosh d(t)$ is subharmonic on B. Further, $\delta(t)$ is harmonic on B iff $\mathcal{R}$ is biholomorphic to the product $B \times R(0)$.

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2. Variation formulas for L_0-principal functions

Let $B = \{t \in \mathbb{C} : |t| < \rho\}$ and let $\widetilde{\mathcal{R}}$ be an unramified (Riemann) domain over $B \times \mathbb{C}_z$. We write $\widetilde{\mathcal{R}} = \cup_{t \in B}(t, \widetilde{R}(t))$, where $\widetilde{R}(t)$ is the fiber of $\widetilde{\mathcal{R}}$ over $t \in B$, i.e., $\widetilde{R}(t) = \{z : (t,z) \in \widetilde{\mathcal{R}}\}$. We assume $\widetilde{R}(t) \neq \emptyset$ for any $t \in B$, so that $\widetilde{R}(t)$ is Riemann surfaces sheeted over $\mathbb{C}_z$ without ramification points. Consider a subdomain $\mathcal{R}$ in $\widetilde{\mathcal{R}}$ such that, putting $\mathcal{R} = \cup_{t \in B}(t, R(t))$, where $R(t)$ is the fiber of $\mathcal{R}$ over $t \in B$,

1. $\widetilde{R}(t) \Supset R(t) \neq \emptyset,\ t \in B$ and $R(t)$ is a connected Riemann surface of genus $g \geq 0$ such that $\partial R(t)$ in $\widetilde{R}(t)$ consists of a finite number of C^ω smooth contours $C_j(t),\ j = 1, \ldots, \nu$;

2. the boundary $\partial\mathcal{R} = \cup_{t \in B}(t, \partial R(t))$ of $\mathcal{R}$ in $\widetilde{\mathcal{R}}$ is C^ω smooth.

Note that g and ν are independent of $t \in B$. We give the orientation of $C_j(t)$ such that $\partial R(t) = C_1(t) + \cdots + C_\nu(t)$. We regard the two-dimensional unramified domain $\mathcal{R}$ over $B \times \mathbb{C}_z$ as a C^ω smooth variation of Riemann surfaces $R(t)$ (sheeted over $\mathbb{C}_z$ without ramification points and with C^ω smooth boundary $\partial R(t)$) with complex parameter $t\in B$,

$$\mathcal{R}: t\in B \to R(t) \ \Subset \widetilde{R}(t).$$

We denote by $\Gamma(B,\mathcal{R})$ the set of all holomorphic sections of $\mathcal{R}$ over B. Assume that there exist $\Xi_0,\ \Xi_\xi \in \Gamma(B,\mathcal{R})$ such that $\Xi_0 : z=0$ and $\Xi_\xi : z=\xi(t)$ with $\Xi_0\cap\Xi_\xi=\emptyset$. Let $t\in B$ be fixed. It is known (cf: § 3, Chap. III in [1]) that $R(t)$ carries the real-valued functions $p(t,z)$ and $q(t,z)$ such that both functions are continuous on $\overline{R(t)}$ and harmonic on $R(t)\setminus\{0,\xi(t)\}$ with poles $\log\frac{1}{|z|}$ at $z=0$ and $\log|z-\xi(t)|$ at $z=\xi(t)$ normalized $\lim_{z\to 0}(\,p(t,z)-\log\frac{1}{|z|})=\lim_{z\to 0}(\,q(t,z)-\log\frac{1}{|z|}\,)=0$ at $z=0$, and $p(t,z)$ and $q(t,z)$ satisfy the following boundary condition (L_1) and (L_0), respectively: for $j=1,\ldots,\nu$,

$$(L_1)\qquad p(t,z)=\text{const.}\, c_j(t)\ \text{ on } C_j(t) \qquad \text{and} \qquad \int_{C_j(t)}\frac{\partial p(t,z)}{\partial n_z}ds_z=0\,;$$

$$(L_0)\qquad \frac{\partial q(t,z)}{\partial n_z}=0\ \text{ on } C_j(t).$$

We call $p(t,z)$ and $q(t,z)$ the L_1- and the L_0-*principal function* for $(R(t),0,\xi(t))$. We find a neighborhood $U_0(t)$ of $z=0$ such that

$$\begin{aligned} p(t,z)&=\log\tfrac{1}{|z|}+h_0(t,z) \qquad \text{on } U_0(t);\\ q(t,z)&=\log\tfrac{1}{|z|}+\mathfrak{h}_0(t,z) \qquad \text{on } U_0(t), \end{aligned}\tag{2.1}$$

where $h_0(t,z)$, $\mathfrak{h}_0(t,z)$ are harmonic for z on $U_0(t)$ and

$$h_0(t,0),\ \mathfrak{h}_0(t,0)\equiv 0 \qquad \text{on } B.$$

We also find a neighborhood $U_\xi(t)$ of $z=\xi(t)$ such that

$$\begin{aligned} p(t,z)&=\log|z-\xi(t)|+\alpha(t)+h_\xi(t,z) \qquad \text{on } U_\xi(t);\\ q(t,z)&=\log|z-\xi(t)|+\beta(t)+\mathfrak{h}_\xi(t,z) \qquad \text{on } U_\xi(t), \end{aligned}\tag{2.2}$$

where $\alpha(t)$, $\beta(t)$ are the constant terms, and $h_\xi(t,z)$, $\mathfrak{h}_\xi(t,z)$ are harmonic for z on $U_\xi(t)$ and

$$h_\xi(t,\xi(t)),\ \mathfrak{h}_\xi(t,\xi(t))\ \equiv 0 \qquad \text{on } B.\tag{2.3}$$

We call $\alpha(t)$ and $\beta(t)$ the L_1- and the L_0-*constant* for $(R(t),0,\xi(t))$.

The following variation formula of the second order for $\alpha(t)$ is showed:

Lemma 2.1 (Lemma 1.3 in [8])**.**

$$\frac{\partial\alpha(t)}{\partial t}=\frac{1}{\pi}\int_{\partial R(t)}k_1(t,z)\Big|\frac{\partial p(t,z)}{\partial z}\Big|^2ds_z+2\,\frac{\partial h_\xi}{\partial z}\Big|_{(t,\xi(t))}\cdot\,\xi'(t)\ ;$$

$$\frac{\partial^2\alpha(t)}{\partial t\partial\bar t}=\frac{1}{\pi}\int_{\partial R(t)}k_2(t,z)\Big|\frac{\partial p(t,z)}{\partial z}\Big|^2ds_z+\frac{4}{\pi}\iint_{R(t)}\Big|\frac{\partial^2 p(t,z)}{\partial\bar t\partial z}\Big|^2dxdy.$$

Here

$$k_1(t,z)=\frac{\partial\varphi}{\partial t}/\frac{\partial\varphi}{\partial z}\ ;$$

$$k_2(t,z)=\Big(\frac{\partial^2\varphi}{\partial t\partial\bar t}\Big|\frac{\partial\varphi}{\partial z}\Big|^2-2\mathrm{Re}\Big\{\frac{\partial^2\varphi}{\partial\bar t\partial z}\frac{\partial\varphi}{\partial t}\frac{\partial\varphi}{\partial\bar z}\Big\}+\Big|\frac{\partial\varphi}{\partial t}\Big|^2\frac{\partial^2\varphi}{\partial z\partial\bar z}\Big)/\Big|\frac{\partial\varphi}{\partial z}\Big|^3$$

on $\partial\mathcal{R}$, where $\varphi(t,z)$ is a C^2 defining function of $\partial\mathcal{R}$.

Note that $k_i(t,z), i=1,2$ on $\partial\mathcal{R}$ does not depend on the choice of defining functions $\varphi(t,z)$ of $\partial\mathcal{R}$, where $k_1(t,z)$ is due to Hadamard and $k_2(t,z)$ is called the *Levi curvature* for $\partial\mathcal{R}$ ((1.3) in [11] and (7) in [12]). The first formula in the lemma is proved by the similar method to that in Lemma 2.2 below.

We shall give the variation formulas for $\beta(t)$. In case when $R(t)$ is of positive genus $g\geq 1$ we need the following consideration, which was not necessary for the variation formulas for $\alpha(t)$. We draw as usual A,B cycles $\{A_k(t),B_k(t)\}_{1\leq k\leq g}$ on $R(t)$ which vary continuously in $\mathcal{R}$ with $t\in B$ without passing through 0, $\xi(t)$:

$$\begin{aligned}A_k(t)\cap B_l(t)\ &=\emptyset\ \text{for}\ k\neq l;\ A_k\times B_k=1\ \text{for}\ k=1,\dots,g;\\ A_k(t)\cap A_l(t)\ &=B_k(t)\cap B_l(t)=\emptyset\ \text{for}\ k\neq l.\end{aligned}\tag{2.4}$$

Here $A_k(t)\times B_k(t)=1$ means that $A_k(t)$ once crosses $B_k(t)$ from the left-side to the right-side of the direction $B_k(t)$. On $R(t), t\in B$ we put $*dq(t,z)=-\frac{\partial q(t,z)}{\partial y}dx+\frac{\partial q(t,z)}{\partial x}dy$, the conjugate differential of $dq(t,z)$.

Lemma 2.2.

$$\frac{\partial\beta(t)}{\partial t}=-\frac{1}{\pi}\int_{\partial R(t)}k_1(t,z)\Big|\frac{\partial q(t,z)}{\partial z}\Big|^2 ds_z+2\ \frac{\partial\mathfrak{h}_\xi}{\partial z}\Big|_{(t,\xi(t))}\cdot\ \xi'(t)\ ;$$

$$\begin{aligned}\frac{\partial^2\beta(t)}{\partial t\partial\bar t}&=-\frac{1}{\pi}\int_{\partial R(t)}k_2(t,z)\Big|\frac{\partial q(t,z)}{\partial z}\Big|^2 ds_z-\frac{4}{\pi}\iint_{R(t)}\Big|\frac{\partial^2 q(t,z)}{\partial\bar t\partial z}\Big|^2 dxdy\\ &\quad-\frac{2}{\pi}\Im\sum_{k=1}^{g}\Big(\frac{\partial}{\partial t}\int_{A_k(t)}*dq(t,z)\Big)\cdot\Big(\frac{\partial}{\partial\bar t}\int_{B_k(t)}*dq(t,z)\Big).\end{aligned}$$

Proof. It suffices to prove the lemma at $t=0$. If necessary, take a smaller disk B of center 0. Since both $\partial\mathcal{R}$ in $\widetilde{\mathcal{R}}$ and $\partial R(t)$ in $\widetilde{R}(t)$ are C^ω smooth, we find a neighborhood $V=\cup_{j=1}^{\nu}V_j$ (disjoint union) of $\partial R(0)=\cup_{j=1}^{\nu}C_j(0)$ such that $(B\times V)\cap(\Xi_0\cup\Xi_\xi)=\emptyset$; V_j is a thin tubular neighborhood of $C_j(0)$ with $V_j\supset C_j(t)$ for any $t\in B$, and $q(t,z)$ is harmonic on $[R(0)\cup V]\setminus\{0,\xi(t)\}$. We write $\widehat{R}(0):=R(0)\cup V$, so that $q(t,z)$ is defined in the product $B\times\widehat{R}(0)$.

We divide the proof into two steps.

1^{st} *step. Lemma 2.2 is true in the special case when Ξ_ξ is a constant section, say, for example, $\Xi_1: z=1$ on B.*

In fact, formula (2.2) becomes

$$q(t,z) = \log|z-1| + \beta(t) + \mathfrak{h}_1(t,z) \qquad \text{on } U_1(t), \tag{2.5}$$

where $\mathfrak{h}_1(t,1) \equiv 0$ on B. For $t \in B$ we put $u(t,z) := q(t,z) - q(0,z)$ on $\widehat{R}(0) \setminus \{0,1\}$. By putting $u(t,0) = 0$ and $u(t,1) = \beta(t) - \beta(0)$, $u(t,z)$ is harmonic on $\widehat{R}(0)$. Let $\varepsilon : 0 < \varepsilon \ll 1$, $\gamma_\varepsilon(0) = \{|z| < \varepsilon\}$ and $\gamma_\varepsilon(1) = \{|z-1| < \varepsilon\}$. Then Green's formula implies

$$\int_{\partial R(0) - \partial\gamma_\varepsilon(0) - \partial\gamma_\varepsilon(1)} u(t,z)\frac{\partial q(0,z)}{\partial n_z} ds_z - q(0,z)\frac{\partial u(t,z)}{\partial n_z} ds_z = 0.$$

Letting $\varepsilon \to 0$, we have from $\frac{\partial q(0,z)}{\partial n_z} = 0$ on $C_j(0), j = 1, \dots, \nu$,

$$\beta(t) - \beta(0) = \frac{-1}{2\pi}\sum_{j=1}^{\nu}\int_{C_j(0)} q(0,z)\frac{\partial q(t,z)}{\partial n_z} ds_z =: \frac{-1}{2\pi}\sum_{j=1}^{\nu} I_j(t). \tag{2.6}$$

We take a point $z_j^0(t)$ on each $C_j(t), t \in B$ such that $z_j^0(t)$ continuously moves in $\partial\mathcal{R}$ with $t \in B$, and choose a harmonic conjugate function $q_j^*(t,z)$ of $q(t,z)$ in V_j such that $q_j^*(t, z_j^0(t)) = 0$. Since $\frac{\partial q(t,z)}{\partial n_z} = 0$ on $C_j(t)$, $q_j^*(t,z)$ is single-valued in V_j and

$$q_j^*(t,z) = 0 \quad \text{ for } z \in C_j(t),\ t \in B. \tag{2.7}$$

Since $dq_j^*(t,z) = \frac{\partial q(t,z)}{\partial n_z} ds_z$, $dq(0,z) = -\frac{\partial q_j^*(0,z)}{\partial n_z} ds_z$ along $C_j(0)$, we have

$$\begin{aligned} I_j(t) &= \int_{C_j(0)} q(0,z) dq_j^*(t,z) \\ &= \int_{C_j(0)} d[q(0,z)q_j^*(t,z)] - q_j^*(t,z) dq(0,z) \\ &= \int_{C_j(0)} q_j^*(t,z)\frac{\partial q_j^*(0,z)}{\partial n_z} ds_z. \end{aligned}$$

Differentiating both sides by t and $\overline{t}$ at $t = 0$, we have

$$\frac{\partial I_j}{\partial t}(0) = \int_{C_j(0)} \frac{\partial q_j^*}{\partial t}(0,z)\frac{\partial q_j^*(0,z)}{\partial n_z} ds_z; \tag{2.8}$$

$$\frac{\partial^2 I_j}{\partial t \partial \overline{t}}(0) = \int_{C_j(0)} \frac{\partial^2 q_j^*}{\partial t \partial \overline{t}}(0,z)\frac{\partial q_j^*(0,z)}{\partial n_z} ds_z. \tag{2.9}$$

We recall the following

Proposition 2.1 ((1.2) in [8])**.** *Let $u(t,z)$ be a C^2 function for (t,z) in a neighborhood $\mathcal{V}_j = \cup_{t\in B}(t, V_j(t))$ of $\mathcal{C}_j = \cup_{t\in B}(t, C_j(t))$ over $B \times \mathbb{C}_z$ such that each $u(t,z), t \in B$ is harmonic for z in $V_j(t)$ and $u(t,z) =$*

a certain const. $c_j(t)$ *on* $C_j(t)$*. Then*

$$(i)\ \frac{\partial u}{\partial t}\frac{\partial u}{\partial n_z}ds_z = 2\,k_1(t,z)\Big|\frac{\partial u}{\partial z}\Big|^2 ds_z \qquad \textit{along } C_j(t);$$

$$(ii)\ \frac{\partial^2 u}{\partial t\partial\bar t}\frac{\partial u}{\partial n_z}ds_z = 2\,k_2(t,z)\Big|\frac{\partial u}{\partial z}\Big|^2 ds_z + \frac{\partial^2 c_j(t)}{\partial t\partial\bar t}\frac{\partial u}{\partial n_z}ds_z$$
$$+\,4\,\Im\Big\{\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial\bar t\partial z}dz\Big\} - 4\Im\Big\{\frac{\partial c_j(t)}{\partial t}\frac{\partial^2 u}{\partial\bar t\partial z}dz\Big\} \quad \textit{along } C_j(t).$$

We apply (i) for $u(t,z)=q_j^*(t,z)$ with (2.7) to (2.8) and obtain

$$\frac{\partial I_j}{\partial t}(0) = 2\int_{C_j(0)} k_1(0,z)\Big|\frac{\partial q_j^*(0,z)}{\partial z}\Big|^2 ds_z.$$
$$\therefore\quad \frac{\partial\beta}{\partial t}(0) = -\frac{1}{\pi}\int_{\partial R(0)} k_1(0,z)\Big|\frac{\partial q(0,z)}{\partial z}\Big|^2 ds_z \quad \text{by (2.6)},$$

which proves the first formula in Lemma 2.2 in the 1st step.

To prove the second one, we apply (ii) for $u(t,z)=q_j^*(t,z)$ with (2.7) to (2.9) and obtain

$$\frac{\partial^2 I_j}{\partial t\partial\bar t}(0) = 2\int_{C_j(0)} k_2(0,z)\Big|\frac{\partial q_j^*(0,z)}{\partial z}\Big|^2 ds_z + 4\,\Im\int_{C_j(0)}\frac{\partial q_j^*}{\partial t}(0,z)\frac{\partial^2 q_j^*}{\partial\bar t\partial z}(0,z)dz.$$

We put

$$\mathbf{a}_k(t) = \textstyle\int_{A_k(t)} *dq(t,z), \qquad \mathbf{b}_k(t) = \int_{B_k(t)} *dq(t,z).$$

We fix a point z^0 ($\neq 0,1$) such that $B\times\{z^0\}\subset\mathcal{R}$. On each $R(t), t\in B$ we choose a branch $q^*(t,z)$ of harmonic conjugate function of $q(t,z)$ on $\widehat{R}(0)\setminus\{0,1\}$ such that $q^*(t,z^0)=0$. Since $\int_{C_j(0)} *dq(t,z)=0$, we have

$$q^*(t,z') = q^*(t,z'') \quad \mathrm{mod}\ \{2\pi,\ \mathbf{a}_k(t),\ \mathbf{b}_k(t)\ (k=1,\dots,g)\}$$

for any z', z'' over the same point $z\in\widehat{R}(0)\setminus\{0,1\}$. We also have

$$q_j^*(t,z) - q^*(t,z) = c_j(t) \qquad \text{on } V_j,$$

where $c_j(t)$ is a certain constant for $z\in V_j$. It follows that

$$\int_{C_j(0)}\frac{\partial q_j^*}{\partial t}(0,z)\frac{\partial^2 q_j^*}{\partial\bar t\partial z}(0,z)dz$$
$$= \int_{C_j(0)}\frac{\partial q^*}{\partial t}(0,z)\frac{\partial^2 q^*}{\partial\bar t\partial z}(0,z)dz + \frac{\partial c_j}{\partial t}(0)\int_{C_j(0)}\frac{\partial^2 q^*}{\partial\bar t\partial z}(0,z)dz.$$

If we put $f(t,z) := q^*(t,z) - iq(t,z)$ for $(t,z)\in B\times V_j$, then $f\in C^\omega(B\times V_j)$ and each $f(t,z), t\in B$ is single-valued and holomorphic for z in V_j, so that

$$\int_{C_j(0)} \frac{\partial^2 q^*}{\partial \bar{t} \partial z}(0,z)dz = \frac{1}{2}\Big[\frac{\partial}{\partial \bar{t}}\Big(\int_{C_j(0)} f'_z(t,z)dz\Big)\Big]_{t=0} = 0.$$

$$\therefore \ \frac{\partial^2 I_j}{\partial t \partial \bar{t}}(0) = 2\int_{C_j(0)} k_2(0,z)\Big|\frac{\partial q^*(0,z)}{\partial z}\Big|^2 ds_z + 4\,\Im\Big\{\int_{C_j(0)} \frac{\partial q^*}{\partial t}(0,z)\frac{\partial^2 q^*}{\partial \bar{t}\partial z}(0,z)dz\Big\}.$$

It follows from (2.6) that

(2.10)

$$\frac{\partial^2 \beta}{\partial t \partial \bar{t}}(0) = -\frac{1}{\pi}\int_{\partial R(0)} k_2(0,z)\Big|\frac{\partial q^*(0,z)}{\partial z}\Big|^2 ds_z - \frac{2}{\pi}\,\Im\Big\{\int_{\partial R(0)} \frac{\partial q^*}{\partial t}(0,z)\frac{\partial^2 q^*}{\partial \bar{t}\partial z}(0,z)dz\Big\}.$$

We shall divide the proof into two cases.

Case when $R(t)$ is planar, i.e., $g=0$. In this case, each $q^*(t,z), t \in B$ is determined up to additive constants mod 2π. It follows from (2.1) and (2.5) that, for any fixed $t \in B$, $\frac{\partial q^*(t,z)}{\partial t}$ is a single-valued harmonic function on $\widehat{R}(0)$, and $\frac{\partial^2 q^*(t,z)}{\partial \bar{t}\partial z}$ is a single-valued holomorphic function on $\widehat{R}(0)$. We have by Green's formula

$$\int_{\partial R(0)} \frac{\partial q^*}{\partial t}(0,z)\frac{\partial^2 q^*}{\partial \bar{t}\partial z}(0,z)dz = 2i\iint_{R(0)} \Big|\frac{\partial^2 q^*}{\partial \bar{t}\partial z}(0,z)\Big|^2 dx\,dy.$$

$$\therefore \ \frac{\partial^2 \beta}{\partial t \partial \bar{t}}(0) = -\frac{1}{\pi}\int_{\partial R(0)} k_2(0,z)\Big|\frac{\partial q(0,z)}{\partial z}\Big|^2 ds_z - \frac{4}{\pi}\iint_{R(0)} \Big|\frac{\partial^2 q}{\partial \bar{t}\partial z}(0,z)\Big|^2 dx\,dy,$$

which is desired.

Case when $R(t)$ is of genus $g \geq 1$. We put $R'(0) = R(0)\setminus \cup_{k=1}^g (A_k(0) \cup B_k(0))$ and $\widehat{R}'(0) = R'(0) \cup V$, so that $R'(0)$ and $\widehat{R}'(0)$ are planar Riemann surfaces such that

$$\partial R'(0) = \partial R(0) + \sum_{k=1}^g (A_k^+(0) + A_k^-(0)) + \sum_{k=1}^g (B_k^+(0) + B_k^-(0)).$$

Here $A_k^+(0)(A_k^-(0))$ is the same(opposite) direction of $A_k(0)$, and $B_k^+(0)$ $(B_k^-(0))$ is similar. For $t \in B$, if we restrict the branch $q^*(t,z)$ (with $q^*(t,z^0)=0$) to $R'(0)\setminus\{0,1\}$, then $q^*(t,z') = q^*(t,z'')$ mod 2π for z', z'' over the same point $z \in \widehat{R}'(0)$. Hence $\frac{\partial q^*}{\partial t}(0,z)$, $\frac{\partial^2 q^*}{\partial \bar{t}\partial z}(0,z)$ are single-valued harmonic functions on $\widehat{R}'(0)$, so that

$$\begin{aligned}
&\int_{\partial R(0)} \frac{\partial q^*}{\partial t}(0,z)\frac{\partial^2 q^*}{\partial \bar{t}\partial z}(0,z)dz \\
&= \iint_{R'(0)} d\Big(\frac{\partial q^*}{\partial t}(0,z)\frac{\partial^2 q^*}{\partial \bar{t}\partial z}(0,z)dz\Big) - \sum_{k=1}^g \int_{A_k^{\pm}(0)+B_k^{\pm}(0)} \frac{\partial q^*}{\partial t}(0,z)\frac{\partial^2 q^*}{\partial \bar{t}\partial z}(0,z)dz \\
&=: J_1 - J_2.
\end{aligned}$$

Since $\frac{\partial q^*}{\partial \bar{t}\partial z}(0,z)$ is holomorphic on $R'(0)$, we have by Green's formula

$$J_1 = 2i \iint_{R(0)} \big|\frac{\partial^2 q}{\partial t \partial \overline{z}}(0,z)\big|^2 dx\, dy;$$

$$J_2(A_k) := \int_{A_k^{\pm}(0)} \frac{\partial q^*}{\partial t}(0,z) \frac{\partial^2 q^*}{\partial \bar{t}\partial z}(0,z) dz$$

$$= \int_{A_k(0)} \big(\frac{\partial q^*}{\partial t}(0,z^+) - \frac{\partial q^*}{\partial t}(0,z^-) \big) \frac{\partial^2 q^*}{\partial \bar{t}\partial z}(0,z) dz.$$

By (2.4) and $\int_{C_j(0)} *dq(t,z) = 0, j = 1,\dots,q$, we have, for $z^{\pm}$ over any $z \in A_k(0)$,

$$q^*(t,z^+) - q^*(t,z^-) = \int_{B_k(0)} *dq(t,\zeta) \quad \mod 2\pi.$$

$$\therefore \quad \frac{\partial q^*}{\partial t}(t,z^+) - \frac{\partial q^*}{\partial t}(t,z^-) = \frac{\partial}{\partial t}\int_{B_k(0)} *dq(t,\zeta),$$

which is independent of $z \in A_k(0)$. It follows from $\frac{\partial q^*(t,z)}{\partial z} dz = \frac{1}{2}(*dq(t,z) - i\, dq(t,z))$ that

$$J_2(A_k) = \big[\frac{\partial}{\partial t}\big(\int_{B_k(0)} *dq(t,\zeta)\big)\big]_{t=0} \cdot \big[\frac{\partial}{\partial \bar{t}}\big(\int_{A_k(0)} \frac{\partial q^*(t,z)}{\partial z} dz\big)\big]_{t=0}$$

$$= \frac{1}{2}\, \frac{\partial \mathbf{b}_k}{\partial t}(0) \cdot \frac{\partial \mathbf{a}_k}{\partial \bar{t}}(0).$$

By $B_k(0) \times A_k(0) = -1$, it similarly holds $J_2(B_k) = -\frac{1}{2}\frac{\partial \mathbf{a}_k}{\partial t}(0) \cdot \frac{\partial \mathbf{b}_k}{\partial \bar{t}}(0)$, so that $J_2(A_k) + J_2(B_k) = -i\, \Im \left\{ \frac{\partial \mathbf{a}_k}{\partial t}(0) \cdot \frac{\partial \mathbf{b}_k}{\partial \bar{t}}(0) \right\}$. We thus have

$$\Im \big\{ \int_{\partial R(0)} \frac{\partial q^*}{\partial t}(0,z) \frac{\partial^2 q^*}{\partial \bar{t}\partial z}(0,z) dz \big\}$$

$$= \Im \; \{ \; J_1 - \sum_{k=1}^{g} (J_2(A_k) + J_2(B_k)) \; \}$$

$$= 2 \iint_{R(0)} \big|\frac{\partial^2 q}{\partial \bar{t} \partial z}(0,z)\big|^2 dx\, dy + \Im \big\{ \sum_{k=1}^{g} \frac{\partial \mathbf{a}_k}{\partial t}(0) \cdot \frac{\partial \mathbf{b}_k}{\partial \bar{t}}(0) \big\}.$$

This with (2.10) completes the second formula in the 1st step.

2^{nd} *step. Lemma 2.2 is true in the general case .*

In fact, it suffices to prove Lemma 2.2 at $t = 0$. If necessary, take a smaller disk B of center 0. Then we find a linear transformation $T : (t,z) \in B \times \mathbb{P}_z \mapsto (t,w) = (t, f(t,z)) \in B \times \mathbb{P}_w$ such that $f(t,0) = 0$; $\frac{\partial f}{\partial z}(t,0) = 1$; $f(t,\xi(t)) =$ const. c for $t \in B$, and $\mathcal{D} := T(\mathcal{R})$ is an unramified domain over $B \times \mathbb{C}_w$. We write $D(t) = f(t, R(t))$, $t \in B$, so that $\mathcal{D} = \cup_{t \in B}(t, D(t))$ and $\mathcal{D}$ has two constant sections $\Theta_0 : w = 0$ and $\Theta_c : \;\; w = c$ (the pull backs of Ξ_0 and Ξ_ξ by T), hence the variation $\mathcal{D} : t \in B \to D(t)$ is a case in the 1st step. For $t \in B$, we consider the L_0-principal function $\widetilde{q}(t,w)$ and the L_0-constant $\widetilde{\beta}(t)$ for $(D(t),0,c)$, so that

$$\begin{aligned} \widetilde{q}(t,w) &= \log \tfrac{1}{|w|} + \widetilde{\mathfrak{h}}_0(t,w) && \text{in } U_0(t); \\ \widetilde{q}(t,w) &= \log|w-c| + \widetilde{\beta}(t) + \widetilde{\mathfrak{h}}_c(t,w) && \text{in } U_c(t), \end{aligned}$$

where $\widetilde{\mathfrak{h}}_0(t,0)$, $\widetilde{\mathfrak{h}}_c(t,c) \equiv 0$ on B. We put $\widetilde{A}_k(t) = f(t,A_k(t))$ and $\widetilde{B}_k(t) = f(t,B_k(t))$ on $D(t)$ which continuously vary in $\mathcal{D}$ with $t \in B$ without passing through $w=0,c$. Since

$$w = f(t,z) = \begin{cases} z + b_2(t)z^2 + \cdots & \text{at } z=0; \\ c + a_1(t)(z-\xi(t)) + a_2(t)(z-\xi(t))^2 + \cdots & \text{at } z=\xi(t), \end{cases}$$

where $a_1(t) \neq 0, a_2(t), \ldots; b_2(t), \ldots$ are holomorphic on B, we have $q(t,z) = \widetilde{q}(t,f(t,z))$ in $\mathcal{R}$, i.e.,

$$q(t,z) = \log|f(t,z)-c| + \widetilde{\beta}(t) + \widetilde{\mathfrak{h}}_c(t,f(t,z)) \qquad \text{at } z=\xi(t),$$

so that

$$\begin{aligned} \text{(2.11)} \qquad \beta(t) &= \widetilde{\beta}(t) + \log|a_1(t)|; \\ \mathfrak{h}_\xi(t,z) &= \widetilde{\mathfrak{h}}_c(t,f(t,z)) + \log\Big|\ 1 + \frac{a_2(t)}{a_1(t)}(z-\xi(t)) + \ldots\ \Big|. \end{aligned}$$

Let $\psi(t,w)$ be a C^ω defining function of $\partial\mathcal{D}$. Then $\varphi(t,z) := \psi(t,f(t,z))$ is that of $\partial\mathcal{R}$, so that we have for $w=f(t,z)$

$$k_1(t,z) = \frac{\frac{\partial\varphi(t,z)}{\partial t}}{|\frac{\partial\varphi(t,z)}{\partial z}|} = \frac{\widetilde{k}_1(t,w)}{|\frac{\partial f(t,z)}{\partial z}|} + \frac{\frac{\partial f(t,z)}{\partial t}}{|\frac{\partial f(t,z)}{\partial z}|} \cdot \frac{\frac{\partial\psi}{\partial w}(t,w)}{|\frac{\partial\psi}{\partial w}(t,w)|}, \qquad (t,z) \in \partial\mathcal{R}.$$

$$\begin{aligned} \therefore \quad & \int_{\partial R(0)} k_1(0,z)\Big|\frac{\partial q(0,z)}{\partial z}\Big|^2 ds_z \\ &= \int_{\partial R(0)} \frac{\widetilde{k}_1(0,w)}{|\frac{\partial f(0,z)}{\partial z}|}\ \Big|\frac{\partial q(0,z)}{\partial z}\Big|^2\ ds_z + \int_{\partial R(0)} \frac{\frac{\partial f}{\partial t}(0,z)}{|\frac{\partial f(0,z)}{\partial z}|} \cdot \frac{\frac{\partial\psi}{\partial w}(0,w)}{|\frac{\partial\psi}{\partial w}(0,w)|}\ \Big|\frac{\partial q(0,z)}{\partial z}\Big|^2\ ds_z \\ &=: J_1 + J_2. \end{aligned}$$

Since $\frac{\partial\widetilde{q}(0,w)}{\partial w}\frac{f(0,z)}{dz} = \frac{\partial q(0,z)}{\partial z}dz$, we have by the 1st step and (2.11)

$$J_1 = \int_{\partial D(0)} \widetilde{k}_1(0,w)\Big|\frac{\partial\widetilde{q}(0,w)}{\partial w}\Big|^2\ ds_w = -\pi\frac{\partial\widetilde{\beta}}{\partial t}(0) = -\pi\,(\frac{\partial\beta}{\partial t}(0) - \frac{1}{2}\frac{a_1'(0)}{a_1(0)}).$$

For a fixed $t \in B$, we write $z = g(t,w) := f^{-1}(t,w)$. We put $\widetilde{C}_j(0) = f(0,C_j(0))$ and $\widetilde{V}_j = f(0,V_j), j = 1,\ldots,\nu$, and consider the single-valued conjugate harmonic function $\widetilde{q}_j^*(0,w)$ of $\widetilde{q}(0,w)$ in $\widetilde{V}_j$ which vanishes on $\widetilde{C}_j(0)$. Then we find a function $k(w) \in C^\omega(V_j)$ such that

$\widetilde{q}_j^*(0,w) = k(w)\psi(0,w)$ in $\widetilde{V}_j$. This and the residue theorem imply

$$\begin{aligned}
J_2 &= -\int_{\partial D(0)} \frac{\frac{\partial g}{\partial t}(0,w)}{\frac{\partial g(0,w)}{\partial w}} \frac{\frac{\partial \psi(0,w)}{\partial w}}{|\frac{\partial \psi(0,w)}{\partial w}|} \Big|\frac{\partial \widetilde{q}^*(0,w)}{\partial w}\Big|^2 \, ds_w \\
&= i \int_{\partial D(0)} \frac{\frac{\partial g}{\partial t}(0,w)}{\frac{\partial g(0,w)}{\partial w}} \Big(\frac{\partial \widetilde{q}_j^*(0,w)}{\partial w}\Big)^2 \, dw \\
&= 2\pi \, \mathrm{Res}_{\,w=0,\,c} \Big\{ \frac{\frac{\partial g}{\partial t}(0,w)}{\frac{\partial g(0,w)}{\partial w}} \Big(\frac{\partial \widetilde{q}(0,w)}{\partial w}\Big)^2 \Big\} \\
&= 2\pi \Big(\frac{\partial \mathfrak{h}_\xi}{\partial z}(0,\xi(0))\, \xi'(0) - \frac{1}{4}\frac{a_1'(0)}{a_1(0)} \Big).
\end{aligned}$$

$$\therefore \quad J_1 + J_2 = -\pi \Big(\frac{\partial \beta}{\partial t}(0) - 2\, \frac{\partial \mathfrak{h}_\xi}{\partial z}(0,\xi(0))\; \xi'(0) \Big),$$

which is identical with the first formula in the 2nd step.

To prove the second one, we have from the 1st step

$$\begin{aligned}
\frac{\partial^2 \widetilde{\beta}}{\partial t \partial \bar{t}}(0) = &-\frac{1}{\pi} \int_{\partial D(0)} \widetilde{k}_2(0,w) \Big|\frac{\partial \widetilde{q}(0,w)}{\partial w}\Big|^2 ds_w - \frac{4}{\pi} \iint_{D(0)} \Big|\frac{\partial^2 \widetilde{q}}{\partial \bar{t} \partial w}(0,w)\Big|^2 \, du\, dv \\
&- \frac{2}{\pi}\, \Im \sum_{k=1}^{g} \Big[\frac{\partial}{\partial t}\int_{\widetilde{A}_k(t)} * d\widetilde{q}(t,w)\Big]_{t=0} \cdot \Big[\frac{\partial}{\partial \bar{t}}\int_{\widetilde{B}_k(t)} * d\widetilde{q}(t,w)\Big]_{t=0},
\end{aligned}$$

where $\widetilde{k}_2(t,w)$ is the Levi curvature of $\partial \mathcal{D}$. It suffices to show that each term of the above formula is invariant for $T : (t,z) \in \mathcal{R} \to (t,w) = (t, f(t,z)) \in \mathcal{D}$, i.e., it holds for $t \in B$

i. $\displaystyle \frac{\partial^2 \widetilde{\beta}(t)}{\partial t \partial \bar{t}} = \frac{\partial^2 \beta(t)}{\partial t \partial \bar{t}};$

ii. $\displaystyle \int_{\partial D(t)} \widetilde{k}_2(t,w) \Big|\frac{\partial \widetilde{q}(t,w)}{\partial w}\Big|^2 ds_w = \int_{\partial R(t)} k_2(t,w) \Big|\frac{\partial q(t,z)}{\partial z}\Big|^2 ds_z;$

iii. $\displaystyle \iint_{D(t)} \Big|\frac{\partial^2 \widetilde{q}}{\partial \bar{t} \partial w}(t,w)\Big|^2 \, du\, dv = \iint_{R(t)} \Big|\frac{\partial^2 q}{\partial \bar{t} \partial z}(t,z)\Big|^2 \, dx\, dy;$

iv. $\displaystyle \frac{\partial}{\partial t}\int_{\widetilde{A}_k(t)} * d\widetilde{q}(t,w) = \frac{\partial}{\partial t}\int_{A_k(t)} * dq(t,z),$ and similar for $\widetilde{B}_k(t)$ and $B_k(t)$.

In fact, *i.* is clear from (2.11). Since $\widetilde{q}(t,w) = q(t,z)$ (where $w = f(t,z)$ for $(t,z) \in \mathcal{R}$) is harmonic on each $R(t)$, $t \in B$, we have *iii.* and *iv.* Further, by the simple calculation, we see, in general, that $k_2(t,z)\,\frac{1}{|dz|}$ *on* $\partial \mathcal{R}$ *is invariant under all holomorphic transformations* T *of the form* $T : (t,z) \in \mathcal{R} \cup \partial \mathcal{R} \mapsto (t,w) = (t, f(t,z)) \in \widetilde{\mathcal{R}} \cup \partial \widetilde{\mathcal{R}}$, i. e., $\widetilde{k}_2(t,w) = k_2(t,z)\big|\frac{\partial f(t,z)}{\partial z}\big|$. It follows that

$$\widetilde{k}_2(t,w) \Big|\frac{\partial \widetilde{q}(t,w)}{\partial w}\Big|^2 |dw| = k_2(t,z) \Big|\frac{\partial q(t,z)}{\partial z}\Big|^2 |dz|$$

for $z \in \partial R(t)$ and $w = f(t,z)$. This implies *ii*. We complete the proof of Lemma 2.2. □

As noted in [8], since $\mathcal{R}$ is pseudoconvex in $\widetilde{\mathcal{R}}$ iff $k_2(t,z) \geq 0$ on $\partial\mathcal{R}$, Lemma 2.1 implies that, if $\mathcal{R}$ is pseudoconvex in $\widetilde{\mathcal{R}}$, then the L_1-constant $\alpha(t)$ for $(R(t), 0, \xi(t))$ is C^ω subharmonic on B, while Lemma 2.2 makes the following contrast with it:

Theorem 2.1. *If $\mathcal{R}$ is pseudoconvex in $\widetilde{\mathcal{R}}$ and $R(t), t \in B$ is planar, then the L_0-constant $\beta(t)$ for $(R(t), 0, \xi(t))$ is C^ω superharmonic on B.*

3. Harmonic span and its geometric meaning

We a little recall the slit mapping theory in one complex variable. Let R be a planar Riemann surface sheeted over $\mathbb{C}_z$ bounded by a finite number of smooth contours $C_j, j = 1, \dots, \nu$.

Let $R \ni 0$ and let $\mathcal{U}(R)$ denote the set of all univalent functions f on R such that $f(z) - \frac{1}{z}$ is regular at 0. For $w = f(z) \in \mathcal{U}(R)$ we consider the Euclidean area $E(f)$ of $\mathbb{C}_w \setminus f(R)$ and put

$$\mathcal{E}(R) = \sup\{E(f) : f \in \mathcal{U}(R)\}.$$

Due to P. Koebe (see Chap. X in [5]), we have two special ones $w = f_i(z), i = 1, 0$ in $\mathcal{U}(R)$ such that $f_1(R)(f_0(R))$ is a vertical (horizontal) slit univalent domain in $\mathbb{P}_w$. In his pioneering work [6], H. Grunsky showed in p. 139-140: if we consider

$$g := \frac{1}{2}(f_1 + f_0) \quad \text{on } R$$

and $K_j = -g(C_j), j = 1, \dots, \nu$, then each K_j bounds an unramified domain G_j over $\mathbb{C}_w$ such that, if we denote by $E_j(g)$ the Euclidean (multivalent) area of G_j and put $E(g) = \sum_{j=1}^{\nu} E_j(g)$, then $E(g) \geq \mathcal{E}(R)$. Then, in his substantial work [16], M. Schiffer in p. 209 introduced the following quantity $S(R)$, which he named the *span* for R,

$$S(R) := \Re\{a_1 - b_1\},$$

where a_1 and b_1 are the coefficients of z (the first degree) of the Taylor expansions of $f_1(z) - \frac{1}{z}$ and $f_0(z) - \frac{1}{z}$ at 0, respectively, and showed the following beautiful results (p. 216 in [16]): $g \in \mathcal{U}(R)$; *each $G_j, j = 1, \dots, \nu$ is a convex domain in $\mathbb{C}_w$, and*

$$E(g) = \mathcal{E}(R) = \frac{\pi}{2} S(R).$$

His proofs were rather intuitive and short. The precise proofs are found in §12, Chap. III in [1].

Let $\xi \in R$, $\xi \neq 0$ and let $\mathcal{S}(R)$ denote the set of all univalent functions f on R such that $f(z) - \frac{1}{z}$ is regular at 0 and $f(\xi) = 0$, say

$$f(z) = c_1(z - \xi) + c_2(z - \xi)^2 + \dots \quad \text{at } \xi.$$

We then put $c(f) = c_1 (\neq 0)$. We draw a simple curve l on R from ξ to 0. Let $f \in \mathcal{S}(R)$ and $w = f(z)$ on R. Then $f(l)$ is a simple curve from 0 to ∞ in $\mathbb{P}_w$, and each branch of $\log f(z)$ on $R \setminus l$ is single-valued *univalent*. Fix one of them, say $\tau = \log f(z)$. Consider the Euclidean area $E_{\log}(f)\,(\geq 0)$ of the complement of $\log f(R \setminus l)$ in $\mathbb{C}_\tau$ and put

$$\mathcal{E}_{\log}(R) = \sup\{E_{\log}(f) : f \in \mathcal{S}(R)\}.$$

Now let $p(z)\,(\,q(z)\,)$ and $\alpha\,(\beta)$ be the L_1-(L_0-)principal function and L_1-(L_0-)constant for $(R, 0, \xi)$. We choose the harmonic conjugate $p^*(z)$ $(q^*(z))$ on R such that, if we put $P(z) = e^{p(z)+ip^*(z)}(\,Q(z) = e^{q(z)+iq^*(z)})$ on R, then $P(z) - \frac{1}{z}\,(\,Q(z) - \frac{1}{z})$ is regular at 0. Then $P, Q \in \mathcal{S}(R)$ and $w = P(z)\,(\,Q(z))$ is a circular (radial) slit mapping with $\log|c(P)| = \alpha$ $(\,\log|c(Q)| = \beta)$ and $E_{\log}(P) = E_{\log}(Q) = 0$. We see in §13, Chap. III in [1] that *P maximizes $2\pi \log|c(f)| + E_{\log}(f)$, while Q minimizes $2\pi \log|c(f)| - E_{\log}(f)$ among $\mathcal{S}(R)$.*

On the other hand, M. Nakai expected that the quantity

$$s(R) := \alpha - \beta \tag{3.1}$$

will have some gemetric meaning. In [15] he named $s(R)$ the *harmonic span* for $(R, 0, \xi)$. Hereafter in this paper we shall show that $s(R)$ has some remarkable properties not only in one complex variable but also in several complex variables.

We precisely write

$$\begin{aligned} P(z) &= e^{\alpha + i\theta_1}\ (z-\xi) + \textstyle\sum_{n=2}^{\infty} a_n (z-\xi)^n \quad \text{ at }\ \xi; \\ Q(z) &= e^{\beta + i\theta_0}\ (z-\xi) + \textstyle\sum_{n=2}^{\infty} b_n (z-\xi)^n \quad \text{ at }\ \xi, \end{aligned} \tag{3.2}$$

where θ_1, θ_0 are certain constants. We put

$$D_1 := P(R) = \mathbb{P}_w \setminus \cup_{j=1}^{\nu} P(C_j) = \mathbb{P}_w \setminus \cup_{j=1}^{\nu}\ \mathrm{arc}\{A_j^{(1)}, A_j^{(2)}\};$$

$$D_0 := Q(R) = \mathbb{P}_w \setminus \cup_{j=1} Q(C_j) = \mathbb{P}_w \setminus \cup_{j=1}^{\nu}\ \mathrm{segment}\{B_j^{(1)}, B_j^{(2)}\}.$$

Here

$$\begin{aligned} \mathrm{arc}\{A_j^{(1)}, A_j^{(2)}\} \quad &= \{r_j e^{i\theta} : \theta_j^{(1)} \leq \theta \leq \theta_j^{(2)}\}; \\ \mathrm{segment}\{B_j^{(1)}, B_j^{(2)}\} \quad &= \{r e^{i\theta_j} : 0 < r_j^{(1)} \leq r \leq r_j^{(2)} < \infty\}, \end{aligned} \tag{3.3}$$

where $0 < \theta_j^{(2)} - \theta_j^{(1)} < 2\pi$ and $r_j, \theta_j^{(k)}, \theta_j, r_j^{(k)}\,(j = 1, \ldots, \nu; k = 1, 2)$ are constants. We take the points $a_j^{(k)}, b_j^{(k)} \in C_j$ such that

$$P(a_j^{(k)}) = A_j^{(k)}, \qquad Q(b_j^{(k)}) = B_j^{(k)}. \tag{3.4}$$

By conditions (L_1) and (L_0) for $p(z)$ and $q(z)$, $\sqrt{P(z)Q(z)}$ consists of two single-valued branches $H(z)$ and $-H(z)$ on R where $H(z)$ has only one pole at $z = 0$ such that $H(z) - \frac{1}{z}$ is regular at 0, and $H(z)$ has 0 only at $z = \xi$. We write

$$H(z) = \sqrt{P(z)Q(z)} \quad \text{ on } R.$$

Each branch of $\log P(z)(\log Q(z))$ is also single-valued univalent on $R \setminus l$, while $\log H(z)$ is single-valued but not univalent so far. We choose three branches in $R \setminus l$ such that

$$\tau = \log H(z) = \frac{1}{2}(\log P(z) + \log Q(z)).$$

We fix a tubular neighborhood V_j of each contour C_j with $V_i \cap V_j = \emptyset$ $(i \neq j)$ and $V_j \not\ni 0, \xi$, where $\log H(z)$ on V_j is single-valued. Then we have the following geometric meaning of the harmonic span $s(R)$:

Theorem 3.1.

1. *Each* $-(\log H)(C_j)$, $j = 1, \dots, \nu$ *is a* C^ω *convex curve in* $\mathbb{C}_\tau$, *and* $-H(C_j)$ *is a* C^ω *simple closed curve in* $\mathbb{C}_w$.
2. $H \in \mathcal{S}(R)$ *and* $E_{\log}(H) = \mathcal{E}_{\log}(R) = \dfrac{\pi}{2}\, s(R)$.
3. *Assume that* R *is simply connected, and let* $d(0, \xi)$ *denote the Poincaré distance between* 0 *and* ξ *on* R. *Then*

$$s(R) = 4 \log \cosh d(0, \xi).$$

The method in the proofs in Chp. III in [1] of M. Schiffer's results seems to have some gaps to prove 1. and 2. in Theorem 3.1. We get over them by the idea of using the Schottky double (compact) Riemann surface $\widehat{R}$ of R. We also apply this idea to prove Corollary 4.1 for the variations of Riemann surfaces.

Proof of Theorem 3.1. Similarly to $F := \frac{df_1}{df_0}$ used in p. 182 in [1] (cf: (25) in [16]), we consider the following function

$$W = F(z) := \frac{d \log Q}{d \log P}, \qquad z \in R \cup \partial R, \tag{3.5}$$

which is a single-valued meromorphic function on R such that $\Re F = 0$ on ∂R, since $\log P(C_j)$ is a vertical segment and $\log Q(C_j)$ is a horizontal segment in $\mathbb{C}_\tau$. It follows from the Schwarz reflexion principle that F is meromorphically extended to the Schottoky double Riemann surface $\widehat{R} = R \cup \partial R \cup R^*$ of R such that $F(z^*) = -\overline{F(z)}$, where $z^* \in R^*$ is the reflexion point of $z \in R$. Fix $C_j, j = 1, \dots, \nu$. Since $\Re \log P(z) = p(z)$ and $\Re \log Q(z) = q(z)$ on R, we have

$$\log P(z) = u_1(z) + i v_1(z), \quad \log Q(z) = u_0(z) + i v_0(z), \quad z \in V_j, \tag{3.6}$$

where $u_1(z)(v_0(z)) =$ const. c_1 (c_0) on C_j. Then $\mathfrak{C}_j := \log H(C_j)$ is a closed (not necessarily simple so far) curve in $\mathbb{C}_\tau$:

$$\tau = \frac{1}{2}(\, c_1 + u_0(z)) + \frac{i}{2}(\, c_0 + v_1(z)), \quad z \in C_j. \tag{3.7}$$

Using notation (3.4), we shall show:

i) $\{a_j^{(k)}, b_j^{(k)}\}_{k=1,2}$ are 4 distinct points, which necessarily line cyclically, for example, $(a_j^{(1)}, b_j^{(1)}, a_j^{(2)}, b_j^{(2)})$ on C_j;

ii) the zeros of $F(z)$ are $\{b_j^{(k)}\}_{j=1,\ldots,\nu;\,k=1,2}$ of order one, and the poles are $\{a_j^{(k)}\}_{j=1,\ldots,\nu;\,k=1,2}$ of order one;

iii) the curve $\mathfrak{C}_j$ is locally non-singular in $\mathbb{C}_\tau$;

iv) $\Re F(z) > 0$ on R;

v) at any $\tau \in \mathfrak{C}_j$, the curvature $\frac{1}{\rho_j(\tau)}$ of $\mathfrak{C}_j$ is negative.

We divide the proof into two steps.

1^{st} *step. If we admit i), then ii) $\sim$ v) hold.*

In fact, i) clearly implies iii). Since $P(z)(Q(z))$ is univalent on R with the circular (radial) slit boundary condition, we have $F(z) \neq 0, \infty$ on $R \cup R^*$ and $F(z)$ has zeros at most $b_j^{(k)}$ and poles at most $a_j^{(k)}$, of order one. Thus, i) implies ii). Further, i) implies that $W = F(z)$ is locally one-to-one in a neighborhood of at any $z \in C_j$ even at $a_j^{(k)}, b_j^{(k)} (k = 1, 2)$, so that F is a meromorphic function on $\widehat{R}$ of degree 2ν. Hence, for each fixed $j = 1, \ldots, \nu$, if z travels C_j all once, then $F(z)$ travels the imaginary axis all just *twice*. It follows that $F(\widehat{R})$ is a 2ν sheeted compact Riemann surface over $\mathbb{P}_W$ with $2(2\nu + g - 1)$ branch points lying on $\mathbb{P}_W \setminus \{\Re W = 0\}$, and is divided by ν closed curves $F(C_j), j = 1, \ldots, \nu$ into two connected parts over $\Re W > 0$ and $\Re W < 0$. Since $F(0) = 1$, we have $\Re F(z) > 0$ on R and $\Re F(z) < 0$ on R^*, which is iv). To prove v), fix $p_0 \in C_j$ and take a local parameter $z = x + iy$ of a neighborhood V of p_0 such that p_0 corresponds to $z = 0$ and the oriented arc $C_j \cap V$ corresponds to $I := (-\rho, \rho)$ on the x-axis. Then using this parameter, we see from $\Re F(z) > 0$ on R that

$$\Im F'(x) = \Im \frac{\partial F(x)}{\partial x} < 0 \quad \text{on } I. \tag{3.8}$$

By (3.7) the subarc $\Gamma_j := \log H(I)$ of $\mathfrak{C}_j$ in $\mathbb{C}_\tau$ is of the form:

$$\tau = u(x) + iv(x) = \frac{1}{2}[(c_1 + u_0(x)) + i(c_0 + v_1(x))], \quad x \in I.$$

Since the arc Γ_j is locally non-singular by iii), we calculate the curvature $1/\rho_j(x)$ at the point $(u(x), v(x))$ of Γ_j:

$$\frac{1}{\rho_j(x)} = \frac{v''(x)u'(x) - v'(x)u''(x)}{(v'(x)^2 + u'(x)^2)^{3/2}} = \frac{v_1''(x)u_0'(x) - v_1'(x)u_0''(x)}{(v_1'(x)^2 + u_0'(x)^2)^{3/2}}.$$

On the other hand, by (3.6) we have, for $x \in I \subset C_j$,

$$\Im F'(x) = \Im \left\{ \frac{d}{dx} \left(\frac{\frac{du_0(x)}{dx} + i\frac{dc_0}{dx}}{\frac{dc_1}{dx} + i\frac{dv_1(x)}{dx}} \right) \right\} = \frac{v_1''(x)u_0'(x) - v_1'(x)u_0''(x)}{v_1'(x)^2}.$$

$$\therefore \quad \frac{1}{\rho_j(x)} = \frac{v_1'(x)^2}{(v_1'(x)^2 + u_0'(x)^2)^{3/2}} \cdot \Im F'(x).$$

Since $v_1'(0) = 0$ iff $x = a_j^{(k)}$, (3.8) proves v) for $p_0 \neq a_j^{(k)}$. For $p_0 = a_j^{(k)}$, since $v_1'(0) = 0$, $v_1''(0)$, $u_0'(0) \neq 0$ under i), $v_1'(x)^2 \cdot \Im F'(x)$ is regular and $\neq 0$. Hence $\frac{1}{\rho_j(p_0)} < 0$, which proves v).

2^{nd} *step. i) is true.*

In fact, assume that R does not satisfy i). Clearly it does not occur $\{a_j^{(1)}, a_j^{(2)}\} = \{b_j^{(1)}, b_j^{(2)}\}$ for any j, so that $\{a_j^{(1)}, a_j^{(2)}\} \cap \{b_j^{(1)}, b_j^{(2)}\}$ consisits of one point for some j, say $j = 1, \dots, \nu' (\leq \nu)$. We denote by o_j such one point on C_j. Hence each $\mathfrak{C}_j := \log H(C_j), j = 1, \dots, \nu'$ is a closed curve in $\mathbb{C}_\tau$ with only one singular point at $\mathfrak{o}_j := \log H(o_j)$ and F is a meromorphic function of degree $2\nu - \nu'$ on $\widehat{R}$. By the same reasoning as in the 1st step, if z travels $C_j, j = 1, \dots, \nu'$ all once, then $F(z)$ travels the imaginary axis all just *once* in $\mathbb{C}_\tau$, and $\Re F(z) > 0$ on R and $\Re F(z) < 0$ on R^*. This fact implied that $\frac{1}{\rho_j(\tau)} < 0$ for $\tau \in \mathfrak{C}_j \setminus \{\mathfrak{o}_j\}$. To reach a contradiction, we focus to C_1. We may assume $\mathfrak{o}_1 = 0$ of $\mathfrak{C}_1 (\subset \mathbb{C}_\tau)$ and $a_1^{(1)} = b_1^{(1)} = o_1$ on $C_1 (\subset \mathbb{C}_z)$. If we take a small subarc C_1' centered at o_1 of C_1 and identify C_1' with $I = (-r, r)$ on the x-axis such that o_1 corresponds to $0 \in I$, then the subarc $\Gamma := \log H(C_1')$ of $\mathfrak{C}_1$ is written

$$\tau = \frac{1}{2}[(a_2 x^2 + a_3 x^3 + \dots) + i(b_2 x^2 + b_3 x^3 + \dots)], \quad x \in I,$$

where all a_k, b_k are real and $a_2, b_2 \neq 0$. The other cases being similar, we assume $a_2, b_2 > 0$. We put $\Gamma'(\Gamma'') = \{\log H(x) \in \Gamma : x$ travels from 0 to $r\,(-r\,)\}$, so that $\Gamma = -\Gamma'' + \Gamma'$. Since $1/\rho_1(\tau) < 0$ for $\tau \in \mathfrak{C}_1 \setminus \{\mathfrak{o}_1\}$, $\mathfrak{C}_1$ has a cusp singularity at $\mathfrak{o}_1$ such that Γ' (Γ'') starts at $\mathfrak{o}_1$ whose tangent decreases from $b_2/a_2 > 0$ as x increases (decreases) from 0 to r $(-r)$. We put $\mathfrak{a} = \log H(a_1^{(2)})$ and $\mathfrak{b} = \log H(b_1^{(2)})$. Since the tangent $T(\tau)$ of $\mathfrak{C}_1$ at $\tau = \log H(z)$ is $T(\tau) = v_1'(z)/u_0'(z)$, we have $T(\mathfrak{a}) = 0$, $|T(\mathfrak{b})| = \infty$ and *vise versa*. This contradicts that $\mathfrak{C}_1$ is a closed curve with $1/\rho_1(\tau) < 0$ for any $\tau \in \mathfrak{C}_1 \setminus \{\mathfrak{o}_1\}$, which proves i).

The first assertion 1. in Theorem 3.1 follows v). Using notation (3.3), we have for $j = 1, \dots, \nu$,

$$\mathrm{Max}_{z \in C_j}\{\Im \log H(z)\} - \mathrm{Min}_{z \in C_j}\{\Im \log H(z)\} \leq \frac{1}{2}(\theta_j^{(2)} - \theta_j^{(1)}) < \pi,$$

so that $-H(C_j)$ in $\mathbb{C}_w$ as well as $-\log H(C_j)$ in $\mathbb{C}_\tau$ is a simple closed curve which bounds a bounded domain in $\mathbb{C}_w$. The second assertion in 1. is proved. To prove 2., given $w' \in \mathbb{C}_w \setminus \cup_{j=1}^{\nu} H(C_j)$, we write $N(w')$ for the number of z in R such that $H(z) = w'$. If we denote by $W_j(w')$ the winding number of $H(C_j)$ about w', then we have $W_j(w') \leq 0$ by the second assertion in 1. Since $H(z)$ has only one pole at $z = 0$ of order one on R, we have by the argument principle

$$N(w') - 1 = \sum_{j=1}^{\nu} W_j(w') \ \leq 0,$$

so that $N(w') = 0$ or 1. Hence, $H(z)$ is univalent on R, which is the first assertion in 2. To prive the other ones in 2., let $f \in \mathcal{S}(R)$. We put $u(z) := \log|f(z)|$ and $h(z) := \log|H(z)| = \frac{1}{2}(p(z)+q(z))$. Then $u(z)-h(z)$ is harmonic on the whole R. Consider the Dirichlet integral of $u-h$ on R, $D_R(u-h) := \iint_R \big[(\frac{\partial(u-h)}{\partial x})^2 + (\frac{\partial(u-h)}{\partial y})^2\big]dxdy \geq 0$. By Green's formula we have

$$D_R(u-h) = \int_{\partial R} udu^* - \int_{\partial R} udh^* - \int_{\partial R} hdu^* + \int_{\partial R} hdh^*.$$

By $\int_{C_j} du^* = 0, j = 1, \ldots, \nu$ and condition $(L_1)((L_0))$ for $p(z)(q(z))$, we have

$$\int_{\partial R} udh^* = \frac{1}{2}\int_{\partial R} udp^* - pdu^* = \pi(\log|c(f)| - \alpha);$$

$$\int_{\partial R} hdu^* = \frac{1}{2}\int_{\partial R} qdu^* - udq^* = \pi(\beta - \log|c(f)|).$$

$$\therefore \quad D_R(u-h) = \int_{\partial R} udu^* + \pi(\alpha-\beta) + \int_{\partial R} hdh^*.$$

We put $u = h$, in particular, to obtain

$$E_{\log}(H) = -\int_{\partial R} hdh^* = \frac{\pi}{2}(\alpha-\beta), \quad E_{\log}(H) - E_{\log}(f) = D_R(u-h) \geq 0,$$

which are desired.

To prove 3., we calculate the harmonic span for the disk $D = \{|z| < r\}$ in $\mathbb{C}_z$. For $\xi \in D$, we denote by $p(z)(q(z))$ the L_1-(L_0-)principal function and by $\alpha\,(\beta)$ the L_1-(L_0-)constant for $(D, 0, \xi)$. We write $P(z)\,(Q(z))$ the corresponding circular (radial) slit mapping on D, where $p(z)(q(z)) = \log|P(z)|\,(\log|Q(z)|)$. We have in § 5 in [8]

$$\begin{aligned} (3.9) \qquad P(z) &= \frac{-1}{\xi}\cdot\frac{z-\xi}{z}\cdot\big(1-\frac{z}{r}\frac{\overline{\xi}}{r}\big)^{-1}, \quad z \in D; \\ \alpha &= \log|\frac{dP}{dz}(\xi)| = -2\log|\xi| - \log\big[1-(\frac{|\xi|}{r})^2\big]. \end{aligned}$$

Putting $\theta_\xi = \arg\xi$, we also have

$$\begin{aligned} (3.10) \qquad Q(z) &= \frac{1}{re^{\theta_\xi}}\,\big[(\frac{z}{re^{i\theta_\xi}} + \frac{re^{i\theta_\xi}}{z}) - (\frac{|\xi|}{r} + \frac{r}{|\xi|})\big] \\ &= \frac{-1}{\xi}\cdot\frac{z-\xi}{z}\cdot\big(1-\frac{z}{r}\frac{\overline{\xi}}{r}\big), \quad z \in D; \\ \beta &= \log|\frac{dQ}{dz}(\xi)| = -2\log|\xi| + \log\big[1-(\frac{|\xi|}{r})^2\big]. \end{aligned}$$

Hence, the harmonic span $s(D) = \alpha - \beta$ for $(D, 0, \xi)$ is

$$(3.11) \qquad s(D) = 2\log\frac{1}{1-\big(\frac{|\xi|}{r}\big)^2}.$$

Now let R be any simply connected domain over $\mathbb{C}_z$ with $R \ni 0, \xi$. We consider the Riemann's conformal mapping $w = \varphi(z)$ from R onto a disk $\widetilde{D} := \{|w| < r\}$ in $\mathbb{C}_w$ such that $\varphi(0) = 0$ and $\varphi'(0) = 1$. We put $\Xi := \varphi(\xi) \in \widetilde{D}$. We consider the circular (radial) slit mapping $\widetilde{P}(w)$ ($\widetilde{Q}(w)$) of $\widetilde{D}$ such that $\widetilde{P}(w) - \frac{1}{w}$ ($\widetilde{Q}(w) - \frac{1}{w}$) is regular at 0 and $\widetilde{P}(\Xi)(\widetilde{Q}(\Xi)) = 0$; the L_1-(L_0-)constant $\widetilde{\alpha}$ ($\widetilde{\beta}$), and the harmonic span $s(\widetilde{D}) = \widetilde{\alpha} - \widetilde{\beta}$ for $(\widetilde{D}, 0, \Xi)$. By (3.11), we have $s(\widetilde{D}) = -2\log[1 - \big(\frac{|\Xi|}{r}\big)^2]$. Since $P(z) := \widetilde{P}(\varphi(z))$ ($Q(z) := \widetilde{Q}(\varphi(z))$) becomes a circular (radial) slit mapping on R such that $P(z) - \frac{1}{z}$ ($Q(z) - \frac{1}{z}$) is regular at 0 and $P(\xi)(Q(\xi)) = 0$. Thus, $\log|P(z)|$ ($\log|Q(z)|$) is the L_1- (L_0-)principal function for $(R, 0, \xi)$, so that the L_1(L_0-)constant α (β) for $(R, 0, \xi)$ is

$$\alpha = \log|\frac{dP}{dz}(\xi)| = \log\big(|\frac{d\widetilde{P}}{dw}(\Xi)| \cdot |\frac{d\varphi}{dz}(\xi)|\big) = \widetilde{\alpha} + \log|\frac{d\varphi}{dz}(\xi)|;$$

$$\beta = \log|\frac{dQ}{dz}(\xi)| = \log\big(|\frac{d\widetilde{Q}}{dw}(\Xi)| \cdot |\frac{d\varphi}{dz}(\xi)|\big) = \widetilde{\beta} + \log|\frac{d\varphi}{dz}(\xi)|.$$

Hence, the harmonic span $s(R) = \alpha - \beta$ for $(R, 0, \xi)$ is

$$s(R) = \widetilde{\alpha} - \widetilde{\beta} = s(\widetilde{D}) = 2\log\frac{1}{1 - \big(\frac{|\Xi|}{r}\big)^2}.$$

Since the Poincaré distance $d(0, \xi)$ between 0 and ξ in R is equal to $\frac{1}{2}\log\frac{1+\frac{|\Xi|}{r}}{1-\frac{|\Xi|}{r}}$, we have $s(R) = 4\log\cosh d(0, \xi)$, which proves 3. $\square$

Example 3.1. We certify 1. and 2. in Theorem 3.1 for the case $D = \{|z| < r\}$ and $\xi \in D$. By (3.9) and (3.10) we have

$$H(z) = \sqrt{P(z)Q(z)} = \frac{1}{z} - \frac{1}{\xi}, \quad z \in D.$$

Thus $H(z)$ is univalent on D. Since $C := \partial D = \{re^{i\theta} : 0 \le \theta \le 2\pi\}$, the closed curve $-H(C) = \{\frac{e^{i\theta}}{r} - \frac{1}{\xi} : 0 \le \theta \le 2\pi\}$ is simple and $-\log H(C)$ is a convex curve. Further, we have $E_{\log}(H) = \pi\log\frac{1}{1-|\xi/r|^2}$.

In fact, we prove it in case $r = 1$ and $|\xi| < 1$ for simplicity. Since each branch of $\log(\frac{1}{z} - \frac{1}{\xi})$ is holomorphic in $\mathbb{C}_z \setminus D$, we have

$$\begin{aligned}
E_{\log}(H) &= \frac{i}{2}\int_{-C}\log(\frac{1}{z} - \frac{1}{\xi})\ d\,\overline{\log(\frac{1}{z} - \frac{1}{\xi})} \\
&= \frac{-i}{2}\int_{C}\log(\frac{1}{z} - \frac{1}{\xi})\ \frac{dz}{z - 1/\overline{\xi}} \qquad \text{since } z\overline{z} = 1 \text{ on } C \\
&= \frac{-i}{2}\cdot 2\pi i\big[(-\log(\frac{1}{1/\overline{\xi}} - \frac{1}{\xi}) + \log(\frac{-1}{\xi})\big] \quad \text{by Cauchy theorem} \\
&= \pi\log\frac{1}{1 - |\xi|^2},
\end{aligned}$$

which is desired. By (3.11) we thus have $E_{\log}(H) = \pi s(D)/2$.

Remark 3.1. (1) Let $R_i, i = 1, 2$ be a planar Riemann surface such that $R_i \ni 0, \xi$. If we denote by s_i the harmonic span for $(R_i, 0, \xi)$, then we have by 2. in Theorem 3.1 that $R_1 \subset R_2$ induces $s_1 \geq s_2$, even when R_1 and R_2 are not homeomorphic to each other.

(2) Let R be a planar Riemann surface. By the similar proof of 3., the harmonic span $s_R(0, \xi)$ for $(R, 0, \xi)$ is invariant under the holomorphic transformations. Thus the harmonic span $s_R(\xi, \eta)$ for (R, ξ, η) is a C^ω positive function for $(\xi, \eta) \in (R \times R) \setminus \cup_{\xi \in R}(\xi, \xi)$. It is clear that $s_R(\xi, \eta) = s_R(\xi, \eta)$ and, for a fixed $\xi_0 \in R$, $\lim_{\eta \to \partial R} s_R(\xi_0, \eta) = +\infty$. If we put $s_R(\xi, \xi) = 0$ for $\xi \in R$, then $s_R(\xi, \zeta)$ is C^2 function on $R \times R$ which satisfies, for a fixed $\xi_0 \in R$, there exist $K > 0$ and $\delta > 0$ such that

$$\frac{1}{K}|\eta - \xi_0|^2 \leq s(\xi_0, \eta) \leq K|\eta - \xi_0|^2 \quad \text{for } |\eta - \xi_0| < \delta. \tag{3.12}$$

In fact, we may assume R is a bounded domain in $\mathbb{C}_z$ and $\xi_0 = 0 \in R$. We take $D_a := \{|z| < a\} \Subset R \Subset \{|z| < b\} := D_b$ in $\mathbb{C}_z$. By (1) and (3.11) we have, for $\eta \in D_a$,

$$2 \log \frac{1}{1 - |\eta/b|^2} = s_{D_b}(0, \eta) \leq s_R(0, \eta) \leq s_{D_a}(0, \eta) = 2 \log \frac{1}{1 - |\eta/a|^2},$$

which implies (3.12).

We call the function $s_R(\xi, \eta)$ on $R \times R$ the *S-function for R.*

4. Variation formulas for the harmonic spans

We return to the variation of Riemann surfaces. In this section, as in section 2., we assume that $\widetilde{\mathcal{R}} = \cup_{t \in B}(t, \widetilde{R}(t))$ is an unramified domain over $B \times \mathbb{C}_z$ and $\mathcal{R} = \cup_{t \in B}(t, R(t))$ satisfies conditions **1.** and **2.** in the beginning of section 2. For a fixed $t \in B$, let $p(t, z)$ $(q(t, z))$; $\alpha(t)$ $(\beta(t))$ and $s(t)$ denote the L_1-(L_0-)principal function; the L_1-(L_0-)constant and the harmonic span, for $(R(t), 0, \xi(t))$. Then Lemmas 2.1 and 2.2 imply the following variation formulas:

Lemma 4.1.

$$\frac{\partial s(t)}{\partial t} = \frac{1}{\pi} \int_{\partial R(t)} k_1(t, z) \ \Big(\ |\frac{\partial p(t, z)}{\partial z}|^2 + |\frac{\partial q(t, z)}{\partial z}|^2 \ \Big) \, ds_z;$$

$$\begin{aligned} \frac{\partial^2 s(t)}{\partial t \partial \bar{t}} &= \frac{1}{\pi} \int_{\partial R(t)} k_2(t, z) \ \Big(\ |\frac{\partial p(t, z)}{\partial z}|^2 + |\frac{\partial q(t, z)}{\partial z}|^2 \ \Big) \, ds_z \\ &\quad + \frac{4}{\pi} \iint_{R(t)} \Big(\ |\frac{\partial^2 p(t, z)}{\partial \bar{t} \partial z}|^2 + |\frac{\partial^2 q(t, z)}{\partial \bar{t} \partial z}|^2 \ \Big) \, dxdy \\ &\quad + \frac{2}{\pi} \Im \sum_{k=1}^{g} \Big(\frac{\partial}{\partial t} \int_{A_k(t)} *dq(t, z) \Big) \cdot \Big(\frac{\partial}{\partial \bar{t}} \int_{B_k(t)} *dq(t, z) \Big). \end{aligned}$$

We say, in general, that $\mathcal{R} : t \in B \to R(t)$ is *equivalent to a trivial variation*, if there exists a biholmorphic transformation T from the total space $\mathcal{R}$ onto a product space $B \times D$ (where D is a Riemann surface) of the form $T : (t, z) \in \mathcal{R} \mapsto (t, w) = (t, f(t, z)) \in B \times D$.

In case $R(t)$ is planar, following (3.2), on each $R(t), t \in B$ we construct the circular and radial slit mappings:

$$P(t, z) = e^{p(t,z)+ip(t,z)^*} \quad \text{and} \quad Q(t, z) = e^{q(t,z)+iq(t,z)^*}$$

such that $P(t, z) - \frac{1}{z}$ and $Q(t, z) - \frac{1}{z}$ are regular at $z = 0$. We put $D_1(t) = P(t, R(t))$ and $D_0(t) = Q(t, R(t))$, so that

$$D_1(t) = \mathbb{P}_w \setminus \cup_{j=1}^{\nu} P(t, C_j(t)) = \mathbb{P}_w \setminus \cup_{j=1}^{\nu} \text{ arc}\{A_j^{(1)}(t), A_j^{(2)}(t)\};$$

$$D_0(t) = \mathbb{P}_w \setminus \cup_{j=1} Q(t, C_j(t)) = \mathbb{P}_w \setminus \cup_{j=1}^{\nu} \text{ segment}\{B_j^{(1)}(t), B_j^{(2)}(t)\}.$$

Theorem 4.1. *Assume that $\mathcal{R} = \cup_{t\in B}(t, R(t))$ is pseudoconvex in $\widetilde{\mathcal{R}}$ and each $R(t)$, $t \in B$ is planar. Then*

1. *$s(t)$ is C^ω subharmonic on B;*
2. *if $s(t)$ is harmonic on B, then*
   - (i) *$s(t)$ is constant on B;*
   - (ii) *$\mathcal{R} : t \in B \to R(t)$ is equivalent to a trivial variation. More concretely,*
     - ($\diamond$) *$\mathcal{R}$ is biholomorphic to the product domain $B \times \widetilde{D}_1$, where $\widetilde{D}_1$ is a circular slit domain in $\mathbb{P}_w$ such that $\widetilde{D}_1 = \mathbb{P}_w \setminus \cup_{j=1}^{\nu}\{\widetilde{A}_j e^{i\theta} : 0 \le \theta \le \Theta_j\}$, where $\widetilde{A}_1 = 1$ and each $\widetilde{A}_j(\neq 0), j = 2, \dots, \nu$ is constant, by the holomorphic transformation $T_0 :\ (t, z) \in \mathcal{R} \mapsto (t, w) = (t, \widetilde{P}(t, z)) \in B \times \widetilde{D}_1$, where $\widetilde{P}(t, z) = P(t, z)/A_1^{(1)}(t)$.*

The concrete ($\diamond$) will be used in the proof of Corollary 5.1.

Proof. Lemma 4.1 implies 1. To prove 2., we may assume that $\mathcal{R} = \cup_{t\in B}(t, R(t))$ is an unramified domain over $B \times \mathbb{C}_z$ such that each $R(t)$, $t \in B$ is contained in an unramified planar domain $\widetilde{R}$ over $\mathbb{C}_z$ and the holomorphic section ξ is constant: $t \in B \to \xi(t) = 1 \in R(t)$. Assume that $s(t)$ is harmonic on B. By Lemma 4.1, we have

a) $k_2(t, z) \equiv 0$ on $\partial\mathcal{R}$, i.e., $\partial\mathcal{R}$ is a Levi flat surface over $B \times \mathbb{C}_z$;
b) both $\frac{\partial p(t,z)}{\partial z}$ and $\frac{\partial q(t,z)}{\partial z}$ are holomorphic for $t \in B$.

By b) and the normalization at $z = 0$, both $w = P(t, z)$ and $w = Q(t, z)$ are holomorphic for two complex variables (t, z) in $\mathcal{R}$ except $B \times \{0\}$. We put $D_1(t) = P(t, R(t)) \subset \mathbb{P}_w$ for $t \in B$, and $\mathcal{D}_1 = \cup_{t\in B}(t, D_1(t))$. Since $\mathcal{D}_1$ as well as $\mathcal{R}$ over $B \times \mathbb{C}_z$ is a pseudoconvex (univalent) domain in $B \times \mathbb{P}_w$, it follows from Kanten Satz (p. 352 in [3]) that each edge point $A_j^{(k)}(t)$ is holomorphic for $t \in B$, and $A_j^{(2)}(t) = A_j^{(1)}(t)e^{i\Theta_j}$, where Θ_j is constant for $t \in B$. We consider the map $(t, w) \in \mathcal{D}_1 \mapsto (t, \widetilde{w}) = (t, L(t, w)) \in B \times \mathbb{P}_{\widetilde{w}}$, where $L(t, w) =$

$w/A_1^{(1)}(t)$, and put $\widetilde{\mathcal{D}}_1 = \cup_{t\in B}(t, \widetilde{D}_1(t))$ where $\widetilde{D}_1(t) = L(t, D_1(t))$. Each $\widetilde{D}_1(t)$, $t \in B$ is circular slit domain $\mathbb{P}_{\widetilde{w}} \setminus \cup_{j=1}^{\nu} \widetilde{C}_j(t)$ such that the first circular slit $\widetilde{C}_1(t) = \{e^{i\theta} : 0 \le \theta \le \Theta_1\}$ is independent of $t \in B$, say $\widetilde{C}_1 := \widetilde{C}_1(t)$. Since $\mathcal{R}$ is biholomorphic to $\widetilde{\mathcal{D}}_1$, and each $\widetilde{D}_1(t), t \in B$ has no ramification points, it suffices for ($\diamond$) in 2. (ii) to prove that the edge point $\widetilde{A}_j^{(1)}(t) := A_j^{(1)}(t)/A_1^{(1)}(t)$ of each arc $\widetilde{C}_j^{(1)}(t), j = 2, \ldots, \nu$ does not depend on $t \in B$.

In fact, we see from b) that the function $F(t,z)$ defined in (3.5):

$$W = F(t,z) = \frac{d_z \log Q(t,z)}{d_z \log P(t,z)}, \qquad z \in R(t) \cup \partial R(t)$$

is holomorphic for $t \in B$ such that $F(t,0) = 1$ and $\Re F(t,z) = 0$ on $\partial R(t)$, i.e., $F(t,z)$ is meromorphic function for two complex variables $(t,z) \in \mathcal{R}$ such that $\Re F(t,z) = 0$ on $\partial \mathcal{R}$. We put $K_j(t) = F(t, C_j(t)), j = 1, \ldots, \nu$ in $\mathbb{P}_W$. In the 1st step of the proof of 1. in Theorem 3.1 we proved that $K_j(t)$ rounds just twice on the imaginary axis in $\mathbb{P}_W$. We put $W(t) = F(t, R(t))$ and $\mathcal{W} = \cup_{t\in B}(t, W(t))$, so that $\partial\mathcal{W} = \cup_{t\in B}(t, \cup_{j=1}^{\nu} K_j(t))$, and $\mathcal{R} \approx \mathcal{W}$ (biholomorphic) by $T : (t,z) \in \mathcal{R} \mapsto (t, W) = (t, F(t,z)) \in \mathcal{W}$. Thus, $W(t)$ has $2\nu + g - 1$ ramification points. Consider the following biholomorphic mapping $(t, W) \in \mathcal{W} \to (t, \widetilde{w}) = (t, \widetilde{G}(t,W)) \in \widetilde{\mathcal{D}}_1$, where $\widetilde{G}(t,W) := L(t, P(t, F^{-1}(t,W)))$. We use the following elementary fact:

($*$) *Let $B = \{|t| < \rho\}$ in $\mathbb{C}_t$ and $E = \{|z| < r\} \cap \{\Re z \ge 0\}$ in $\mathbb{C}_z$. If $f(t,z)$ is a holomorphic function for two complex variables (t,z) on $B \times E$ such that $|f(t,z)| = 1$ on $B \times (E \cap \{\Re z = 0\})$, then $f(t,z)$ does not depend on $t \in B$.*

We choose a point W_0 on $\partial K_1(0) \subset \partial\mathcal{W}$ such that $\widetilde{G}(0, W_0) = e^{i\theta_0} \in \widetilde{C}_1$ with $0 < \theta_0 < \Theta_1$ and the direction of $\widetilde{C}_1$ at $e^{i\theta_0}$ follows as θ_0 increases. Then we have a small disk $B_0 \subset B$ of center 0 and a small half-disk $E = \{|W - W_0| < r\} \cap \{\Re W \ge 0\}$ in $\mathbb{C}_W$ such that $|\widetilde{G}(t,W)| \le 1$ $(= 1)$ on $B_0 \times E$ $(B_0 \times (E \cap \{\Re W = 0\}))$. By ($*$), $\widetilde{G}(t,W)$ for $W \in E \cap \{\Re W \ge 0\}$ does not depend on $t \in B_0$. By the analytic continuation, $\widetilde{G}(t,W)$ on $\mathcal{W} \cup \partial\mathcal{W}$ does not depend on $t \in B$.

Now assume that some $\widetilde{A}_j^{(1)}(t)$, $2 \le \exists\, j \le \nu$ is not constant for $t \in B$. We take a point $W_0 \in \mathbb{C}_W$ with $\Re W_0 = 0$. Since the component $K_j(t)$ of $\partial W(t)$ winds twice around the imaginary axis in $\mathbb{P}_W$, for each $t \in B$ we find 4 points of $K_j(t)$ over W_0. We fix one of them, say $W_0(t) \in K_j(t)$, to whom the corresponding point $z_j(t) \in C_j(t)$ continuously varies in $\partial\mathcal{R}$ with $t \in B$. Since $\widetilde{C}_j(t) = \widetilde{G}(t, K_j(t)) = \{\widetilde{A}_j^{(1)}(t)e^{i\theta} : 0 \le \theta \le \Theta_j\}$, where Θ_j is constant for $t \in B$, we have $\widetilde{G}(t, W_0) = \widetilde{A}_j^{(1)}(t)e^{i\theta(t)}$, where $\theta(t)$ $(0 < \theta(t) < \Theta_j)$, continuously varies with $t \in B$.

Since $|\widetilde{A}_j^{(1)}(t)|$ as well as $\widetilde{A}_j^{(1)}(t)$ is not constant for $t \in B$, $\widetilde{G}(t, W_0)$ *does* depend on $t \in B$, a contradiction, and 2. (ii) is proved.

From (2) in Remark 3.1 the harmonic span $s(t)$ for $(R(t), 0, 1)$ is equal to that for $(\widetilde{D}_1(t), \infty, 0)$. Since $\widetilde{D}_1(t) = \widetilde{D}_1(0)$ for any $t \in B$, $s(t)$ is constant on B, which proves 2. (i). □

For 2. (ii) in Theorem 4.1 we cannot replace the condition of the harmonicity of $s(t)$ on B by that of $\alpha(t)$ or $\beta(t)$ on B, in general. However, when $R(t), t \in B$ is simply connected, such replacement is possible by the same idea of the proof of 2. (ii).

Corollary 4.1. *Assume that $\mathcal{R} = \cup_{t\in B}(t, R(t))$ is pseudoconvex over $B \times \mathbb{C}_z$ and each $R(t)$, $t \in B$ is planar. Then the S-function $s(t, \xi, \eta)$ for $R(t), t \in B$ is C^2 plurisubharmonic on $\mathcal{R}^2 := \cup_{t\in B}(t, R(t) \times R(t))$. In particular, for a fixed $t_0 \in B$, we simply put $R(t_0) = R$ and $s(t_0, \xi, \eta) = s(\xi, \eta)$. Then $s(\xi, \eta)$ is C^2 plurisubharmonic on $R \times R$ such that, for any complex line l except $\xi = \eta$ in $R \times R$, the restriction of $s(\xi, \eta)$ on $l \cap (R \times R)$ is strictly subharmonic.*

Proof. Let $t \in B \to (\xi(t), \eta(t)) \in R(t) \times R(t)$ be any holomorphic mapping from B into $\mathcal{R}^2$. We put $s(t) := s(t, \xi(t), \eta(t))$ for $t \in B$, and $B' = B \setminus \{t \in B : \xi(t) = \eta(t)\}$. Consider the translation $T : (t, z) \in \mathcal{R} \mapsto (t, w) = (t, z - \eta(t))$ for $t \in B'$, and put $\widetilde{\mathcal{R}} := T(\mathcal{R})$ and $\widetilde{\xi} = T\xi$. Then $\widetilde{\mathcal{R}}$ is pseudoconvex over $B' \times \mathbb{C}_w$ and $\widetilde{\xi} \in \Gamma(B', \widetilde{\mathcal{R}})$. By Theorem 4.1, the harmonic span $\widetilde{s}(t)$ for $(\widetilde{R}(t), 0, \widetilde{\xi}(t))$ is C^ω subharmonic on B', and so is $s(t)$ on B'. It follows from (3.12) that $s(t)$ is C^2 subharmonic on B, which proves the former part in the corollary. Further, by the same argument we can prove the latter part under the second variation formula in Lemma 4.1 and (3.12). □

Theorem 4.1 with 3. in Theorem 3.1 directly implies

Corollary 4.2. *Assume that $\mathcal{R} = \cup_{t\in B}(t, R(t))$ is pseudoconvex over $B \times \mathbb{C}_z$ and $R(t)$, $t \in B$ is simply connected. Let $\xi_i \in \Gamma(B, \mathcal{R})$, $i = 1, 2$ and let $d(t)$ denote the Poincaré distance between $\xi_1(t)$ and $\xi_2(t)$ on $R(t)$. Then $\delta(t) := \log\cosh d(t)$ is subharmonic on B. Moreover, $\delta(t)$ is harmonic on B if and only if $\mathcal{R}$ is equivalent to the trivial variation.*

Prof. M. Brunella said to us that he could prove the stronger fact: "$\log d(t)$ *is subharmonic on B*" than "$\delta(t)$ *is subharmonic on B*" by the same idea in p. 139 in [4] which is based on [2], (though there was not its exact statement).

Remark 4.1. (1) In §2 and §3, $\mathcal{R} = \cup_{t\in B}(t, R(t))$ is assumed to be a subdomain of an unramified domain $\widetilde{\mathcal{R}} = \cup_{t\in B}(t, \widetilde{R}(t))$ over $B \times \mathbb{C}_z$ which satisfies conditions **1.** and **2.** stated in §2. By the standard use of the immersion theorem for open Riemann surfaces in [7] or [13], the results in §2 and §3 hold for the following $\mathcal{R}$: let $B = \{t \in \mathbb{C} : |t| < \rho\}$ and let $\pi : \widetilde{\mathcal{R}} \to B$ be a two-dimensional holomorphic family (namely,

$\widetilde{\mathcal{R}}$ is a complex two-dimensional manifold and π is a holomorphic projection from $\widetilde{\mathcal{R}}$ onto B) such that each fiber $\widetilde{R}(t) = \pi^{-1}(t)$, $t \in B$ is irreducible and non-singular in $\widetilde{\mathcal{R}}$. Putting $\widetilde{\mathcal{R}} = \cup_{t\in B}(t, \widetilde{R}(t))$, our $\mathcal{R}$ is a subdomain of $\widetilde{\mathcal{R}}$ defined by $\mathcal{R} = \cup_{t\in B}(t, R(t)) \subset \widetilde{\mathcal{R}}$ which satisfies the corresponding conditions **1**. and **2**.

(2) In conditions **1.** and **2.**, if we replace C^ω smooth by C^∞ smooth, i.e., $\mathcal{R} : t \in B \to R(t) \Subset \widetilde{R}(t)$ is a variation such that $\partial R(t), t \in B$ is C^∞ smooth in $\widetilde{R}(t)$ and $\partial\mathcal{R}$ is C^∞ smooth in $\widetilde{\mathcal{R}}$, then the results in §2 and §3 hold by replacing C^ω by C^∞. In fact, Lemmas 2.1 and 2.2, on which all results are based, hold for the C^∞ category by a little not essentially change of the proofs for the C^ω category (cf: §2 in [11]).

5. Approximation theorem for general variations of planar Riemann surfaces

We consider the general variation of Riemann surfaces $\mathcal{R} : t \in \Delta \to R(t)$ defined as follows: let Δ be an open or a compact Riemann surface and $\pi : \mathcal{R} \to \Delta$ be a two-dimensional holomorphic family such that each fiber $R(t) = \pi^{-1}(t),\ t \in \Delta$ is irreducible and non-singular in $\mathcal{R}$ and is *planar*. In case Δ is open, we assume that $\mathcal{R}$ is Stein. We call such $\mathcal{R}$ the variation *of type* (**A**). In case Δ is compact, we assume that, for any disk $B \subset \Delta$, $\mathcal{R}|_B$ is of type (**A**), i.e., $\pi^{-1}(B) = \cup_{t\in B}(t, R(t))$ is Stein. We call such $\mathcal{R}$ the variation *of type* (**B**). In general, $R(t)$ might be infinite ideal boundary components and $\mathcal{R} : t \in \Delta \to R(t)$ might not be topologically trivial. To state the approximation theorem for these variations $\mathcal{R}$ we make the following

Preparation. Let Δ and $\pi : \mathcal{R} \to \Delta$ be of type (**A**). Due to Oka-Grauert (cf: Theorem 8.22 in [14]), $\mathcal{R}$ admits a C^ω strtictly plurisubharmonic exhaustion function $\psi(t, z)$. Let $\xi,\ \eta \in \Gamma(\Delta, \mathcal{R})$ such that $\xi \cap \eta = \emptyset$. Let $B \Subset \Delta$ be a small disk such that we find a continuous curve $g(t)$ connecting $\xi(t)$ and $\eta(t)$ on $R(t), t \in B$ which continuously varies in $\mathcal{R}$ with $t \in B$. We put $\mathcal{R}|_B = \cup_{t\in B}(t, R(t))$; $\xi|_B = \cup_{t\in B}(t, \xi(t))$; $\eta|_B = \cup_{t\in B}(t, \eta(t))$, and $g|_B = \cup_{t\in B}(t, g(t))$. We take so large $a \gg 1$ that $\mathcal{R}(a)|_B := \{(t, z) \in \mathcal{R}|_B : \psi(t, z) < a\} \supset g|_B$. Then we find an increasing sequence $\{a_n\}_n$ such that $\lim_{n\to\infty} a_n = \infty$ and if we put

$$\mathcal{R}_n = \text{the conn. comp. of } \mathcal{R}(a_n)|_B \text{ which contains } g|_B, \tag{5.1}$$

then 1) each $\mathcal{R}_n, n = 1, 2, \ldots$ is a connected domain with real three-dimensional C^ω surfaces $\partial\mathcal{R}_n$ in $\mathcal{R}|_B$ (but each $R_n(t), t \in B$ is not always connected);

2) if we consider the set $\mathcal{L}$ of points $t \in B$ such that there exists a point $(t, z(t)) \in \partial\mathcal{R}_n$ with $\frac{\partial\psi}{\partial z}(t, z(t)) = 0$, then $\mathcal{L}$ consists of two kind of families $\mathcal{L}',\ \mathcal{L}''$ of finite C^ω arcs in B:

$$\mathcal{L}' = \{l'_1, \ldots, l'_m\}, \qquad \mathcal{L}'' = \{l''_1, \ldots, l''_\mu\},$$

which have the following property:

<u>for $\mathcal{L}'$</u>: for any $t_0 \in \mathcal{L}'$ (except a finite set at which some l'_i and l'_j or l'_i itself intersect transversally), say $t_0 \in l'_i$, $\partial R_n(t_0)$ (consisting of a finite number of closed curves) has only one singular point at $z(t_0)$, and we find a small bi-disk $B_0 \times V$ centered at $(t_0, z(t_0))$ in $\mathcal{R}_{n+1}$ such that $B_0 \Subset B$ and $l'_i \cap B_0$ divides B_0 into two connected domains B'_0 and B''_0 in the manner that

i) each $\partial R_n(t), t \in B'_0 \cup B''_0$ has no singular points;

ii) each $\partial R_n(t), t \in l'_i \cap B_0$ has one singular point $z(t)$ at which two subarcs of $\partial R_n(t)$ transversally intersect;

iii) each $R_n(t) \cap V, t \in B'_0 \cup (l'_i \cap B_0)$ consists of two (connected) domains, while each $R_n(t) \cap V, t \in B''_0$ consists of one domain;

<u>for $\mathcal{L}''$:</u> for any $t_0 \in \mathcal{L}''$ (except a finite point set), say $t_0 \in l''_i$, we find a unique point $(t_0, z(t_0)) \in \partial \mathcal{R}_n$ with $\frac{\partial \psi}{\partial z}(t_0, z(t_0)) = 0$, and a small bi-disk $B_0 \times V$ centered at $(t_0, z(t_0))$ in $\mathcal{R}_{n+1}$ such that $B_0 \Subset B$ and $l''_i \cap B_0$ divides B_0 into two connected domains B'_0 and B''_0; a C^ω mapping $\mathfrak{z}$: $t \in l''_i \cap B_0 \to z(t)$ such that $(t, z(t)) \in \partial \mathcal{R}_n$ with $\frac{\partial \psi}{\partial z}(t, z(t)) = 0$ in the manner that

i) $[R_n(t) \cup \partial R_n(t)] \cap V = \emptyset$ for $t \in B'_0 \cup (l''_i \cap B_0)$;

ii) $R_n(t) \cap V$ for $t \in B''_0$ is a simply connected domain $\delta_n(t)$ such that, for a given $t^0 \in l''_i \cap B_0$, $\delta_n(t)$ shrinkingly approaches the point $z(t^0)$ as $t \in B''_0 \to t^0$.

For the singular point $z(t), t \in l'_i \subset \mathcal{L}'$, we have the connected component $C(t)$ of $\partial R_n(t)$ passing through $z(t)$. Then $C(t)$ consists of one closed curve, or two closed curves $C_i(t), i = 1, 2$ such that $C(t) = C_1(t) \cup C_2(t)$ and $C_1(t) \cap C_2(t) = z(t)$. For example, in figure (FIII) below, $C(t)$ consists of one closed curve, and in figures (FI), (FII), $C(t)$ consists of two closed curves.

For the singular point $z(t), t \in l''_i \subset \mathcal{L}''$, we have $(t, z(t)) \in \partial \mathcal{R}_n$ but $z(t) \notin \partial R_n(t)$.

Fix $t \in B$ and $n \geq 1$ and consider the connected component $R'_n(t)$ of $R_n(t)$ which contains $g(t)$. We put $\mathcal{R}'_n = \cup_{t \in B}(t, R'_n(t))$ and $\partial \mathcal{R}'_n = \cup_{t \in B}(t, \partial R'_n(t))$. The variation

$$\mathcal{R}'_n : t \in B \to R'_n(t)$$

is no longer smooth variation of $R'_n(t)$ with $t \in B$, i.e., $\mathcal{R}'_n$ satisfies neither corresponding condition **1.** nor **2.** of $\mathcal{R}$ in §2. Since $R(t)$ is irreducible in $\mathcal{R}$, we have $R'_n(t) \Subset R'_{n+1}(t)$; $\mathcal{R}'_n \to \mathcal{R}|_B$ $(n \to \infty)$, and $R'_n(t) \to R(t)$ $(n \to \infty)$ for $t \in B$.

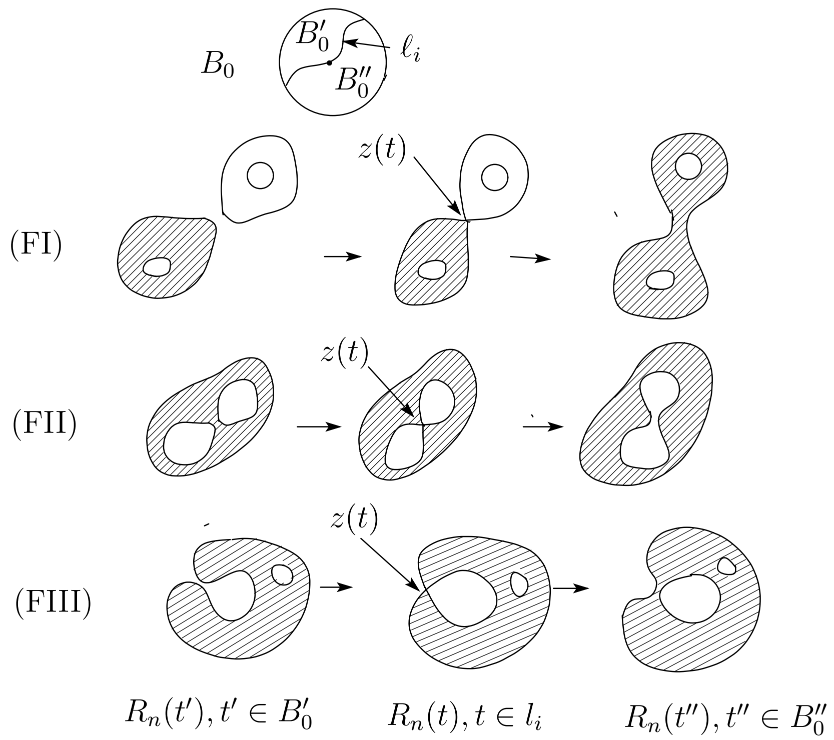
By i), ii) for $\mathcal{L}''$, there exists a neighborhood $\mathcal{V}$ of $\cup_{t \in \mathcal{L}''}(t, z(t))$ in $\mathcal{R}_{n+1}$ such that $[\mathcal{R}'_n \cup \partial \mathcal{R}'_n] \cap \mathcal{V} = \emptyset$, so that $\mathcal{L}''$ does not give any influence for the variation $\mathcal{R}'_n$ (contrary to for $\mathcal{R}_n$).

Each $R(t), t \in \Delta$ is assumed *planar*. We separate the singular point $z(t)$ of $\partial R_n(t), t \in l_i' \subset \mathcal{L}'$ such that $z(t) \in \partial R_n'(t)$ into the following two cases (**c1**) and (**c2**): let $C(t)$ denote the connected component of $\partial R_n(t)$ passing through $z(t)$. Then

(**c1**) $C(t)$ consists of two closed curves $C_i(t), i = 1, 2$, and one of them, say $C_1(t)$, is one of boundary components of $R_n'(t)$, so that $[C_2(t) \setminus \{z(t)\}] \cap \partial R_n'(t) = \emptyset$;

(**c2**) $C(t)$ is one of the boundary components of $R_n'(t)$.

For example, if we take the shadowed part in figure (FI) (resp. (FII), (FIII)) as $R_n'(t)$, then the singular point $z(t)$ is of case (**c1**) (resp. (**c2**)).



For $t \in B$ we consider the L_1-(L_0-)principal function $p_n(t, z)(q_n(t, z))$ and the harmonic span $s_n(t)$ for $(R_n'(t), \xi(t), \eta(t))$. Then we have

Lemma 5.1. *(S. Hamano* [9]*) Let $\mathcal{R}$ be a Stein manifold and each $R(t), t \in \Delta$ is planar. Then*

1. *$p_n(t, z)$ and $q_n(t, z)$ are continuous for (t, z) in $\mathcal{R}_n'$, and $s_n(t)$ is continuous on B;*
2. *assume that at each singular point $z(t)$ of $\partial R_n(t)$, $t \in l_i' \subset \mathcal{L}'$ such that $z(t) \in R_n'(t)$, case (**c1**) only occurs. Then*
   (i) *$p_n(t, z)$ and $q_n(t, z)$ are of class C^1 for (t, z) on $\mathcal{R}_n' \setminus \{\xi|_B, \eta|_B\}$;*
   (ii) *$s_n(t)$ is C^1 subharmonic on B.*
3. *there exist counter-examples for case (**c2**) such that $p_n(t, z)$ or $q_n(t, z)$ is not of class C^1 on $\mathcal{R}_n' \setminus \{\xi|_B, \eta|_B\}$, and $s_n(t)$ is neither of class C^1 on B nor subharmonic on B.*

As an example of figure (F I), let $B=\{|t|<1/10\}$; $D=\{|z|<2\}$; $\psi_1=(e^{-100+|t|^2}/|z-1|^2)-1; \psi_2=|z^2-1|-(1-2\Re t-|t|^2); \psi_3=(e^{-100+|t|^2}/|z+1|^2)-1$ and $\mathcal{R}=\{(t,z)\in B\times D:\psi_1<0,\psi_2<0,\psi_3<0\}$, so that $\partial\mathcal{R}$ is C^ω strictly pseudoconvex in $B\times D$. Then the arc $l'=\{t\in B: 2\Re t\ +|t|^2=0\}$ divides B into two domains $B'\cup B''$ such that the connected components of $\partial R(t), t\in l'$ consists of two circles $\psi_1(t,z)=0$, $\psi_3(t,z)=0$ and the leminiscate $C:|z^2-1|=1$ which is singular at $z(t)=0$.

As an example of $\mathcal{L}''$. Let $B,\ D$ be the same as above. Let $\psi(t,z):=|z-t|^2+|t|^2+2\Re t$ and put $\mathcal{R}=\{(t,z)\in B\times D:\psi(t,z)<0\}$. Then the arc $l''=\{t\in B:\phi(t)=0\}$, where $\phi(t)=-|t|^2-2\Re t$, divides B into two domains $B'(B'')=\{t\in B:\phi(t)<0(>0)\}$ such that $R(t)=\emptyset$ for $t\in B'\cup l''$ and $R(t)=\{|z-t|^2<\phi(t)\}$ for $t\in B''$. The mapping $\mathfrak{z}: t\in l''\to z(t)=t$ so that $(t,t)\in\partial\mathcal{R}$ but $t\notin\partial R(t)$, and each $R(t), t\in B''$ is a disk $\{|z-t|<\phi(t)\}$ which schrinkingly approaches the point $z=t^0$ as $t\to t^0\in l''$.

Since the Stein manifold admits a C^ω strictly plurisubharmonic exhaustion function, we immediately have

Lemma 5.2. *Let $\mathcal{R}:\Delta\to R(t)$ be of type* (**A**) *or* (**B**)*, and let $\xi,\eta\in\Gamma(\Delta,\mathcal{R})$ such that $\xi\cap\eta=\emptyset$. Assume*

- ($\star$) *$R(t), t\in\Delta$ is homeomorphic to a domain D in $\mathbb{C}_w$ bounded by ν boundary component such that $1\le\nu<\infty$ and ν is independent of $t\in\Delta$.*

Then, for any $t_0\in\Delta$, there exists a small disk $B\Subset\Delta$ of center t_0 such that we find an increasing sequence $\{\mathcal{R}'_n\}_n$ of case (**c1**) *such that $\lim_{n\to\infty}\mathcal{R}'_n=\mathcal{R}|_B$.*

Now we consider the variation $\mathcal{R}:t\in\Delta\to R(t)$ of type (**A**). Let $\xi,\ \eta\in\Gamma(\Delta,\mathcal{R})$ such that $\xi\cap\eta=\emptyset$. We fix so small disk $B\Subset\Delta$ that we can fix local parameters (t,z) of $\xi|_B$ and $\eta|_B$ in $\mathcal{R}|_B$ and $\{\mathcal{R}_n\}_n$ satisfies conditions in *Preparation* to these Δ and B, precisely saying, similar to (5.1) we define

$$\mathcal{R}_n=\text{the conn. comp. of }\mathcal{R}(a_n)|_B\text{ which contains }g|_B, \tag{5.2}$$

which satisfies conditions 1) and 2) in *Preparation.* We put $\mathcal{R}_n=\cup_{t\in B}(t,R_n(t))$, and for $t\in B$ we denote by $R'_n(t)$ the connected component of $R_n(t)$ which contains $g|_B(\supset\xi(t),\eta(t))$ and put $\mathcal{R}'_n=\cup_{t\in B}(t,R'_n(t))$. We then have the L_1-(L_0-)principal function $p_n(t,z)(q_n(t,z))$; the L_1-(L_0-)constant α_n (β_n) and the harmonic span $s_n(t)$ for $(R'_n(t),\xi(t),\eta(t))$. In one complex variable it is known (cf: § 8, Chap. III in [1]) that, for a fixed $t\in B$, $p_n(t,z)(q_n(t,z))$ uniformly converges to a certain function $p(t,z)(q(t,z))$ on any compact set in $R(t)\setminus\{\xi(t),\eta(t)\}$. Thus $p(t,z)\,(q(t,z))$ is harmonic on $R(t)\setminus\{\xi(t),\eta(t)\}$ with the same pole as $p_n(t,z)(q_n(t,z))$ at $\xi(t)$ and $\eta(t)$. Putting $\alpha(t)(\beta(t))=\lim_{z\to\eta(t)}(p(t,z)-\log|z-\eta(t)|)$ $(\lim_{z\to\eta(t)}(q(t,z)-\log|z-\eta(t)|))$, we have $\alpha_n(t)\to$

$\alpha(t)$ ($\beta_n(t) \to \beta(t)$) as $n \to \infty$. We call $p(t,z)(q(t,z))$ the L_1-(L_0-)*principal function* and $s(t) := \alpha(t) - \beta(t)$ the *harmonic span* for $(R(t), \xi(t), \eta(t))$. Since $R(t)$ is planar, we have

$$s_n(t) \searrow s(t) \quad \text{as } n \to \infty, \qquad t \in B. \tag{5.3}$$

Their proofs in [1] also imply that, given $K \Subset \mathcal{R}|_B \setminus \{\xi|_B, \eta|_B\}$, for sufficiently large n,

$$p_n(t,z), q_n(t,z), p(t,z), q(t,z) \text{ are uniformly bounded on } K. \tag{5.4}$$

Note that $p(t,z), q(t,z), \alpha(t), \beta(t)$ depend on the choice of local parameters of $\xi|_B$ and $\eta|_B$ but $s(t)$ does not depend on them, so that $s(t)$ is a non-negative function on Δ.

Using these notations we have the following approximation

Theorem 5.1. *Let $\mathcal{R} : t \in \Delta \to R(t)$ be of type* (**A**) *and let $\xi, \eta \in \Gamma(\Delta, \mathcal{R})$ such that $\xi \cap \eta = \emptyset$. Let $s(t)$ denote the harmonic span for $(R(t, \xi(t), \eta(t)), t \in \Delta$. Assume*

(∗) *for any $t_0 \in \Delta$, there exists a small disk $B \Subset \Delta$ of center t_0 such that we find an increasing sequence $\{\mathcal{R}'_n\}_n$ of case* (**c1**) *such that $\lim_{n\to\infty} \mathcal{R}'_n = \mathcal{R}|_B$.*

Then

1. *$s(t)$ is subharmonic on Δ;*
2. *(Simultaneous uniformization) if $s(t)$ is harmonic on Δ, then $\mathcal{R}$ is a biholomprphic to a univalent domain in $\Delta \times \mathbb{P}$.*

Proof. To show 1. let $t_0 \in \Delta$. Then we have a disk $B \subset \Delta$ which satisfies condition (∗). By 2.(ii) in Lemma 5.1, $s_n(t)$ is C^1 subharmonic on B, hence $s(t)$ is suharmonic on B, and on Δ. To prove 2., we cover Δ by small disks $\{B_i\}_{i=1,2,\dots}$ with condition (∗), i..e, for fixed B_i, we find an increasing sequences $\{\mathcal{R}'_n\}_n$ (depending on B_i) such that each $\mathcal{R}'_n, n = 1, 2, \dots$ is of case (**c1**) and $\lim_{n\to\infty} \mathcal{R}'_n = \mathcal{R}|_{B_i}$. We divide the proof of 2. into two steps.

1^{st} *step. Each $\mathcal{R}|_{B_i}$, $i = 1, 2, \dots$ is biholmorphic to a univalent domain $\mathcal{D}_i$ in $B \times \mathbb{P}$.*

In fact, we simply write $B = B_i$. We put $\mathcal{R}'_n = \cup_{t\in B}(t, R'_n(t))$, $n = 1, 2, \dots$ and consider $p_n(t,z), q_n(t,z)$ and $s_n(t)$ for each $(R'_n(t), \xi(t), \eta(t))$, $t \in B$ as above. We put

$$\begin{aligned} P_n(t,z)(P(t,z)) &= e^{p_n(t,z)+ip_n(t,z)^*} \ (e^{p(t,z)+ip(t,z)^*}); \\ Q_n(t,z)(Q(t,z)) &= e^{q_n(t,z)+iq_n(t,z)^*} \ (e^{q(t,z)+iq(t,z)^*}), \end{aligned} \tag{5.5}$$

which are normalized

$$\frac{1}{z-\xi(t)} + a_0(t) + a_1(t)(z - \xi(t)) + \dots \qquad \text{at } z = \xi(t). \tag{5.6}$$

For a fixed $t \in B$, $P_n(t,z)(Q_n(t,z))$ uniformly converges to $P(t,z)(Q(t,z))$ on any compact set in $R(t)$; $w = P_n(t,z)(Q_n(t,z))$ is a circular (radial) slit mapping on $R'_n(t)$, and hence $P(t,z)(Q(t,z))$ is an univalent function on $R(t)$. We call such $P(t,z)(Q(t,z))$ the circular (radial) slit mapping for $(R(t),\xi(t),\eta(t))$. For the 1st step it suffices to show

(a) the harmonicity of $s(t)$ on B implies that $P(t,z)$ is holomorphic for two complex variables (t,z) in $\mathcal{R}|_B\setminus\{\xi|_B\}$.

In fact, fix a point (t_0,z_0) in $\mathcal{R}|_B\setminus\{\xi|_B,\eta|_B\}$ and let $B_0\times V\Subset\mathcal{R}|_B\setminus\{\xi|_B,\eta|_B\}$ be a bi-disk centered at (t_0,z_0), a local coordinate of a neighborhood of (t_0,z_0). We put $f(t,z):=\frac{\partial p(t,z)}{\partial z}$ for $(t,z)\in B_0\times V$. From (5.6) it suffices for (a) to prove that $f(t,z)$ is holomorphic for (t,z) in $B_0\times V$. Since each $f(t,z), t\in B_0$ is holomorphic for $z\in V$ and $f(t,z)$ is uniformly bounded in $B_0\times V$ by (5.4), it thus suffices for (a) to show that, for any fixed $z'\in V$, it holds $\frac{\partial f(t,z')}{\partial \bar t}=0$ on B_0 in the sense of distribution, i.e., it holds, for any $\varphi(t)=\varphi(t_1+it_2)\in C_0^\infty(B_0)$,

$$I:=\int_{B_0} f(t,z')\frac{\partial\varphi(t)}{\partial\bar t}dt_1dt_2=0. \tag{5.7}$$

To prove this by contradiction, assume $I\neq 0$. We fix a small disk $V_0=\{|z-z'|<r_0\}\Subset V$ of center z', so that we have $R'_n(t)\Supset V_0$ for any $t\in B_0$ and $n\geq\exists\, n_0$. We see from the mean-value theorem for holomorphic functions for z that

$$I=\frac{1}{\pi r_0^2}\iint_{B_0\times V_0} f(t,z)\frac{\partial\varphi(t)}{\partial\bar t}dt_1dt_2dxdy.$$

Since $f_n(t,z):=\frac{\partial p_n(t,z)}{\partial z}\to f(t,z)$ $(n\to\infty)$ uniformly on V_0 for a fixed $t\in B_0$ and since $f_n(t,z), f(t,z)$ are uniformly bounded in $B_0\times V_0$ by (5.4), it follows from Lebesgue bounded theorem that

$$I=\frac{1}{\pi r_0^2}\lim_{n\to\infty}\iint_{B_0\times V_0} f_n(t,z)\frac{\partial\varphi(t)}{\partial\bar t}dt_1dt_2dxdy.$$

$$\therefore\quad \Big|\frac{1}{\pi r_0^2}\iint_{B_0\times V_0} f_n(t,z)\frac{\partial\varphi(t)}{\partial\bar t}dt_1dt_2dxdy\Big|\geq\frac{|I|}{2}>0\quad\text{for any } n\geq\exists\, N.$$

On the other hand, using 2.(ii) in Lemma 5.1 under condition $(*)$ in Theorem 5.1, we see that, for a fixed $z\in V_0$, $p_n(t,z)$, and hence $f_n(t,z)$ is of class C^1 for $t\in B_0$. It follows that

$$\int_{B_0} f_n(t,z)\frac{\partial\varphi(t)}{\partial\bar t}dt_1dt_2=-\int_{B_0}\varphi(t)\frac{\partial f_n(t,z)}{\partial\bar t}dt_1dt_2.$$

Hence, putting $I_0=\frac{\pi r_0^2|I|}{2}>0$, we have from Schwarz inequality

$$\begin{aligned}I_0^2 &\leq \Big(\iint_{B_0\times V_0}|\varphi(t)|^2dt_1dt_2dxdy\Big)\ \Big(\iint_{B_0\times V_0}\Big|\frac{\partial f_n(t,z)}{\partial\bar t}\Big|^2dt_1dt_2dxdy\Big)\\ &=: C\iint_{B_0\times V_0}\Big|\frac{\partial f_n(t,z)}{\partial\bar t}\Big|^2dt_1dt_2dxdy,\end{aligned}$$

where $C>0$ is independent of n. Lemma 4.1 and ni) for $\mathcal{L}'$ in *Preparation* for the pseudoconvex domain $\mathcal{R}_n$ imply

$$0 \le \frac{4}{\pi} \int_{R'_n(t)} \Big|\frac{\partial f_n(t,z)}{\partial \bar t}\Big|^2 dxdy \le \frac{\partial^2 s_n(t)}{\partial t \partial \bar t} \quad \text{for any } t \in B \setminus \mathcal{L}'.$$

Since $\mathcal{L}'$ (depending on n) consists of a finite number of C^ω arcs in B; $R'_n(t) \supset V_0$ for $n \ge n_0$, and $f_n \in C^1(B_0 \times V_0)$, it follows that

$$I_0^2 \le C \iint_{(B_0\setminus\mathcal{L}')\times V_0} \Big|\frac{\partial f_n(t,z)}{\partial \bar t}\Big|^2 dt_1 dt_2 dxdy \le \frac{C\pi}{4} \int_{B_0\setminus\mathcal{L}'} \frac{\partial^2 s_n(t)}{\partial t\partial \bar t} dt_1 dt_2.$$

We fix a disk $B_1 : B_0 \Subset B_1 \Subset B$ and a C_0^∞ function $\varphi_1(t) \ge 0$ on B_1 such that $\varphi_1(t) \equiv 1$ on B_0. Since $\frac{\partial^2 s_n(t)}{\partial t\partial \bar t} \ge 0$ on $B_1 \setminus \mathcal{L}'$, we have

$$\int_{B_0\setminus\mathcal{L}'} \frac{\partial^2 s_n(t)}{\partial t\partial \bar t} dt_1 dt_2 \le \int_{B_1\setminus\mathcal{L}'} \varphi_1(t) \frac{\partial^2 s_n(t)}{\partial t\partial \bar t} dt_1 dt_2.$$

Since $s_n(t)$ is of class C^1 on B and $\varphi(t) \equiv 0$ on ∂B_1, we have

$$\int_{B_1\setminus\mathcal{L}'} \varphi_1(t) \frac{\partial^2 s_n(t)}{\partial t\partial \bar t} dt_1 dt_2 = \int_{B_1} s_n(t) \frac{\partial^2 \varphi_1(t)}{\partial t\partial \bar t} dt_1 dt_2,$$

both being equal to $-\frac{1}{4}\int_{B_1} (\frac{\partial \varphi_1}{\partial t_1}\frac{\partial s_n}{\partial t_1} + \frac{\partial \varphi_1}{\partial t_2}\frac{\partial s_n}{\partial t_2}) dt_1 dt_2$. We have by (5.3)

$$\begin{aligned} 0 < I_0^2 &\le \frac{C\pi}{4} \int_{B_1} s_n(t) \frac{\partial^2 \varphi_1(t)}{\partial t\partial \bar t} dt_1 dt_2 \\ &\to \frac{C\pi}{4} \int_{B_1} s(t) \frac{\partial^2 \varphi_1(t)}{\partial t\partial \bar t} dt_1 dt_2 \quad \text{as } n \to \infty \\ &= 0 \quad \text{by the harmonicity of } s(t) \text{ on } B, \end{aligned}$$

which is a contradiction, and the 1st step is proved.

2^{nd} *step. Assertion 2. is true.*

In fact, fix $B_i, i = 1,2,\dots$ and let $P_i(t,z)$ denote the circular slit mapping for $(R(t),\xi(t),\eta(t))$ used in (a) in the 1st step for $\mathcal{R}|_{B_i}$. From the theory of one complex variable, for a fixed $t \in B_i \cap B_j$, there exists $a_{ij}(t) \neq 0$ such that $P_i(t,z) = a_{ij}(t) P_j(t,z)$ on $R(t)$. Since $a_{ij}(t)$ is holomorphic on $B_i \cap B_j$ and since Δ is an open Riemann surface, we have nonvanishing holomorphic function $a_i(t)$ on B_i such that $a_{ij}(t) = a_j(t)/a_i(t)$ on $B_i \cap B_j$. Thus, $a_i(t) P_i(t,z)$ on $B_i, i = 1,2,\dots$ defines a holomorphic function $\mathcal{P}(t,z)$ on $\mathcal{R}$, so that $T : (t,z) \in \mathcal{R} \to (t,w) = (t, \mathcal{P}(t,z)) \in B \times \mathbb{P}_w$ proves the 2nd step. $\square$

Corollary 5.1. *(Rigidity) Let $\mathcal{R} : t \in \Delta \to R(t)$ be a variation of type* (**A**) *or* (**B**) *and let $\xi, \eta \in \Gamma(\Delta, \mathcal{R})$ such that $\xi \neq \eta$. Let $s(t)$ denote the harmonic span for each $(R(t),\xi(t),\eta(t)), t \in \Delta$. Assume that $R(t), t \in \Delta$ satisfies condition $(\star)$ in Lemma 5.2. Then*

1. *in case when $\mathcal{R}$ is of type* ($\mathbf{A}$), *$s(t)$ is subharmonic on Δ. Moreover, if $s(t)$ is harmonic on Δ, then $\mathcal{R}$ is equivalent to a trivial variation;*
2. *in case when $\mathcal{R}$ is of type* ($\mathbf{B}$), *$s(t)$ is constant on Δ. Moreover, if there exists at least one ideal boundary component $C(t)$ of $R(t), t \in \Delta$ such that*
   (i) *$C(t)$ moves homotopically with $t \in \Delta$ in $\mathcal{R}$;*
   (ii) *each $C(t), t \in \Delta$ is positive harmonic measure on $R(t)$,*
   then $\mathcal{R}$ is equivalent to a trivial variation.

Proof. The proofs of assertions 1. and 2. are essentially same, we give the proof of 2. By Lemma 5.2 and 1. in Theorem 5.1, $s(t)$ is subharmonic on the compact Δ, so that $s(t)$ is constant on Δ. By Lemma 5.2 we cover Δ by small disks $\{B_i\}_{i=1,2,\dots}$ which satisfies condition $(*)$ in Theorem 5.1. Since $s(t) \equiv$ const. on B_i, it follows by the proof of 2. in Theorem 5.1 that the circular slit mapping $P_i(t, z)$ for $(R(t), \xi(t), \eta(t))$ for $t \in B_i$ is holomorphic for $t \in B_i$. Since $D_i(t) := P_i(t, R(t))$ is a circular slit domain with ν circular arcs $\{A_j^{(1)}(t), A_j^{(2)}(t)\}$ (which might reduce to a point, i.e., $A_j^{(1)}(t) = A_j^{(2)}(t))$ for some j), it follows from Kanten Satz that $A_j^{(k)}(t), k = 1, 2; j = 1, \dots, \nu$ is holomorphic on B_i. For each $i = 1, 2, \dots$ we conventionally rename arc $\{A_1^{(1)}(t), A_1^{(2)}(t)\} = P_i(t, C(t))$, where $C(t)$ is stated in 2. By the homotpy condition (i), $A_1(t)$ is single-valued on Δ. By (ii), the arc $\{A_1^{(1)}, A_1^{(2)}\}$ does not reduce to a point. By the same argument in 3. (ii) in Theorem 4.1, we see that that $\widetilde{P}_i(t, z) := P_i(t, z)/A_1^{(1)}(t)$ is independent of $t \in B_i$, hence so is of $i = 1, 2, \dots$. We denote it by $\widetilde{\mathcal{P}}(t, z) = \widetilde{\mathcal{P}}(z)$ on $\mathcal{R}$. Then $T_0 : (t, z) \in \mathcal{R} \to (t, w) = (t, \widetilde{\mathcal{P}}(z))$ biholomorphically maps $\mathcal{R}$ onto $\Delta \times \widetilde{D}_1$ where $\widetilde{D}_1$ is the same form $(\diamond)$ in (ii) in Theorem 4.1 (but some boundary components of $\widetilde{D}_1$ might be points $\widetilde{A}_j$). □

Applying Corollary 5.1 to the case when each $R(t), t \in \Delta$ is simply connected, we have

Corollary 5.2. *Corollary 4.2 holds under the weaker condition that $\mathcal{R} = \cup_{t \in B}(t, R(t))$ is a Stein manifold.*

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